An analougue of chain-permutational poset

Amrita Acharyya
Department of Mathematics and Statistics, University of Toledo

Introduction

It is known from section 6 in [1], when we call a poset P, a $\mathcal{P}-\mathsf{chain}-\mathsf{permutational}$ given a subset of permutations $\mathcal{P}$ of S_n. In this work, we use the same idea to study subset of words that are not necessarily permutations for example especially when they are certain classes of restricted growth functions (r.g.f.).

Definition

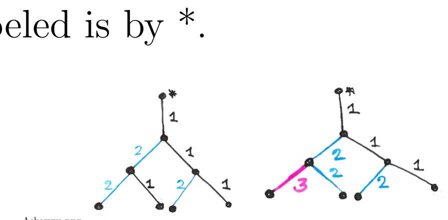
A poset P will be called $\mathcal{P}-\mathsf{chain}-\mathsf{word}$, where $\mathcal{P}$ is a subset of the set of all words of length n, if it is possible to label the covering relations of P (i.e. the edges of the Hasse diagram of P) with numbers from $\{1, 2, ..., n\}$ in such a way that along different maximal chains of P (whose length is necessarily n) the labels form different words from $\mathcal{P}$, and every word $w \in \mathcal{P}$ arises in this manner. For example, it is well known that the Boolean lattice B_n is a chain word poset where the set of words is S_n.

Denote the set of all r.g.f.'s of length n as R_n and it's subset avoiding certain pattern ν as $R_n(\nu)$.

Lemma

The hass diagrams of the chain word posets of $R_n(121), R_n(122), R_n(123)$ *are rooted binary tree for any* $n \geqslant 3$*, and the respective vertex degrees except the leaves and the root are two. For* $n \geqslant 3$, $R_n(112)$ *is an* n*-ary tree and only one vertex has* n*-many children.*

Following are the Hass diagrams of the posets $R_3(123)$ (the left one) and $R_3(121)$. The vertices are not labeled. The root is labeled is by *.



Since r.g.f.'s have the first digit "1" these posets from $R_n(\nu)$ are rooted tree with the top edge leveled by 1. With each leaf at the bottom level we atach an additional edge ϵ calling the resulting lattice as $R_n(L)(\nu)$.

Distributivity and R.G.F.'s

Theorem

For $n \geqslant 3$ *the Hass diagrams of* $R_n(L)(121), R_n(L)(122), R_n(L)(123)$ *are distributive lattice, whereas that of* $R_n(L)(111)$ *and* $R_n(L)(112)$ *are not in general.*

We consider these r.g.f.'s of length n with maximal letter k which correspond to the number of blocks in the set partitions inducing the r.g.f.'s. We denote the resulting lattice (ommiting ϵs' they can be considered as posets only) as $R_{nk}(L)(\nu)$.

Theorem

For $n \geqslant 3$, $2 \leqslant k \leqslant n$*, the Hass diagrams of* $R_{nk}(L)(121), R_{nk}(L)(122)$*, give distributive lattice, whereas that of* $R_{nk}(L)(112)$ *are so iff* $k \leqslant 2, n$ *and* $R_{nk}(L)(123)$ *(For the pattern 123 this lattice is defined only when* $k = 1$ *or* $k = 2$*) is distributive if and only if* $k = 1, 2$. $R_{nk}(L)(111)$ *is defined iff* $\lceil \frac{n}{2} \rceil \leqslant k \leqslant n$*. And this lattice is distributive iff* $k = \lceil \frac{n}{2} \rceil, n-1, n$.

We construct the rank enerating functions (R.G.F.'s), $R_{nk}(\nu)$ for some specifically given k. Attaching the ϵs', the same can be found for the corresponding $R_{nk}(L)(\nu)$.

Theorem

i *The rank generating function of* $R_{n2}(121)$ *is* $1 + x + 2x^2 + 3x^3 + \cdots + (n-2)x^{(n-2)} + (n-1)x^{(n-1)} + (n-1)x^n$.

ii *The vertex cardinality of* $R_{n2}(121)$ *is sum of the coefficients in the above rank generating functions which is* $\frac{n(n+1)}{2}$*. And the edge cardinality is 1 less than that.*

iii *The rank generating function of* $R_{n3}(122)$ *is* $1 + x + 2x^2 + 4x^3 + \sum_{r=4}^{n} \alpha_r x^r$*, where* $\forall r, 4 \leqslant r \leqslant n-2, \alpha_r = \frac{r^2-r+2}{2}$ *and* $\alpha_n = \alpha_{n-1} = \frac{n^2-3n+2}{2}$.

iv *For* $k \geqslant 2$ *in the rank generating function of* $R_{nk}(121), R_{nk}(122)$*, the coefficient of* x^n *is same with that of* x^{n-1}.

v *In the rank generating function of* $R_{nk}(121), R_{nk}(122)$*, the coefficients eventually stabilizes as* n *grows.*

vi *The Hass diagram of* $R_{nk}(L)(121)$ *(and of* $R_{nk}(121)$*) are symmetric on both sides of the vertical line drawn through the top edge leveled by "1" iff* $n = 2k - 1$.

vii *The rank generating function of* $R_{n(n-1)}(122)$ *is the same as that of* $R_{n2}(121)$ *as in* i.

viii *The rank generating function of* $R_{n(n-2)}(112)$ *is* $\sum_{i=0}^{n-2} x^i + (n-2)x^{n-1} + \frac{n^2-3n+2}{2}x^n$.

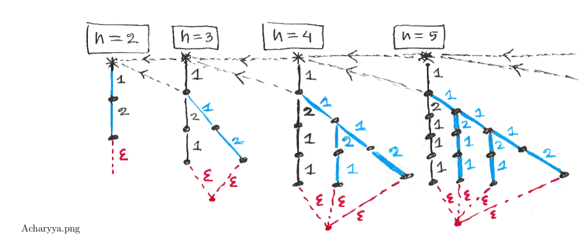
Projective system of lattices (and trees)

For any pattern ν, $R_n(L)(\nu) \subseteq R_{n+1}(L)(\nu) \subseteq R_{n+2}(L)(\nu) \cdots$ and the same follows if we specify the number of blocks k and/or the ϵs' are ommitted. So, from any pattern avoidence class we can construct a countable family of projective system of Lattices (and trees). Each map of the countable projective system being surjective this construction resembles in many ways to Projective Frasse limit of graphs and trees as in [3]. To get a projective system of maps of graphs we considered non rigid map of graphs as in [4], where an edge can be mapped onto a vertex.

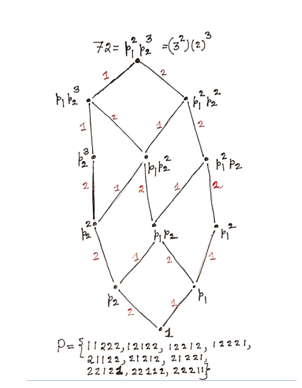
Theorem

Let the pattern $\nu \in \{112, 121, 122, 123\}$*. Then for any k (k=1 or 2 for the pattern 123) we can define a projective system of maps of graphs (possibly non rigid).* $\{\phi_{ij} : R_{jk}(L)(\nu) \to R_{ik}(L)(\nu)\}_{i,j \in \mathbb{N}, j \geqslant i}$ *in an algorithmic way.(*$\mathbb{N}$ *is the set of all positive integers). The corresponding projective limit is a connected (as each quotient is finite discrete path connected graph) second countable profinite graph* Γ *having a subgraph* T*, where* T *is a projective limit of the underlying trees inside each* $R_{ik}(L)(\nu)$ *ommitting all* ϵ*s but one. And vertex set of* Γ *is same as the vertex set of* T.

A picture of part of the projective system for the pattern 122 and $k = 2$.



A chainword poset where the words may not be r.g.f.'s: For any possitive integer n, D_n, the set of all positive integer divisors of n form a poset. Using the unique prime factorization of a possitive divisor of n we get a maximal chain whose edge level form a word denoting that divisor.



Q i. Is this possible to find rank generating functions, of the chainword posets of any $R_{nk}(L)(\nu)$ for any pattern ν, in terms of n and k?

Q ii. Is this possible to find algorithms to define the maps of graphs of those projective systems for any pattern ν in terms of n only (when the number of blocks is not specified), and in terms of n and k as well?

Reference

- Rodica Simion, Frank W. Schmidts, *RESTRICTED PERMUTATIONS*, European J. Combin. 6 (1985), no. 4, 383-406.
- Lindsey R. Campbell, Samantha Dahlberg, Robert Dorward, Jonathan Gerhard,Thomas Grubb, Carlin Purcell, Bruce E Sagan (2021) published by Elsevier, *RESTRICTED GROWTH FUNCTION PATTERNS AND STATISTICS*
- Wlodzimierz J. Charatonik AND Robert P. Roe *PROJECTIVE FRAISSE LIMITS OF TREES*
- Amrita Acharyya, Jon M. Corson, AND Bikash C. Das (2019) published by Communications in Algebr, Volume 47-Issue 10 *VARIETIES OF PROFINITE GRAPHS*
- L. Ribes Ergebnisse der Mathematik und ihrer Grenzgebiete, vol. 66, Springer International Publishing, 2017 *PROFINITE GRAPHS and GROUPS*.

Email: amrita.acharyya@utoledo.edu