Rowmotion on 321-avoiding permutations

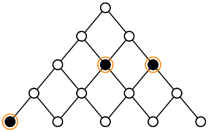
Ben Adenbaum
joint work with Sergi Elizalde

Dartmouth College

July 5, 2023

Rowmotion on antichains and order ideals

antichains



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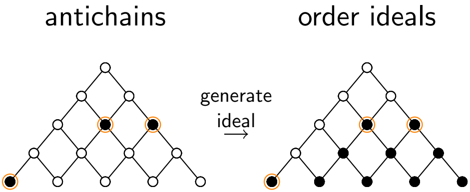

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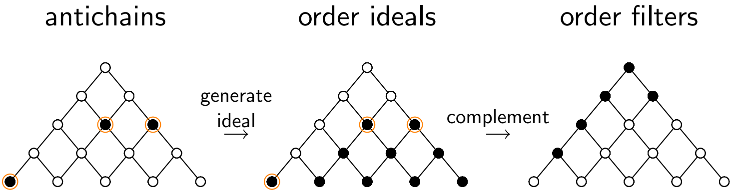

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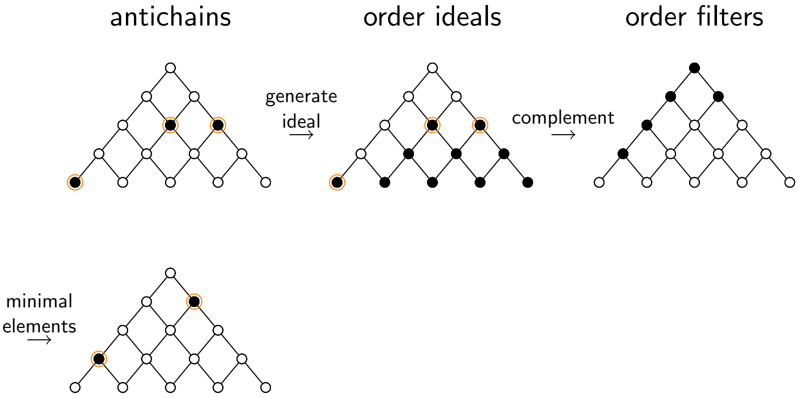

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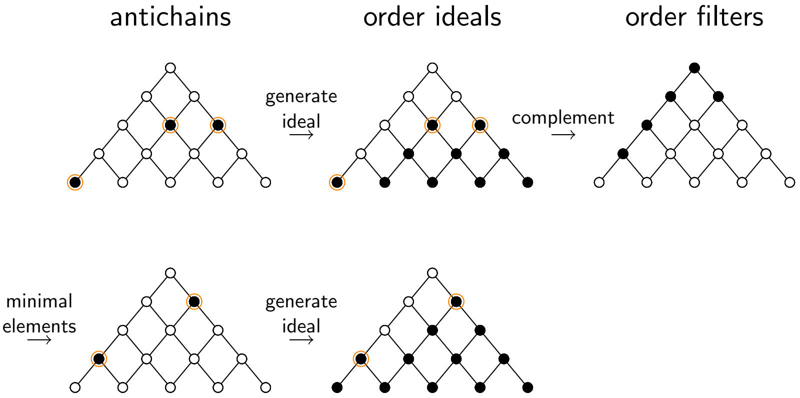

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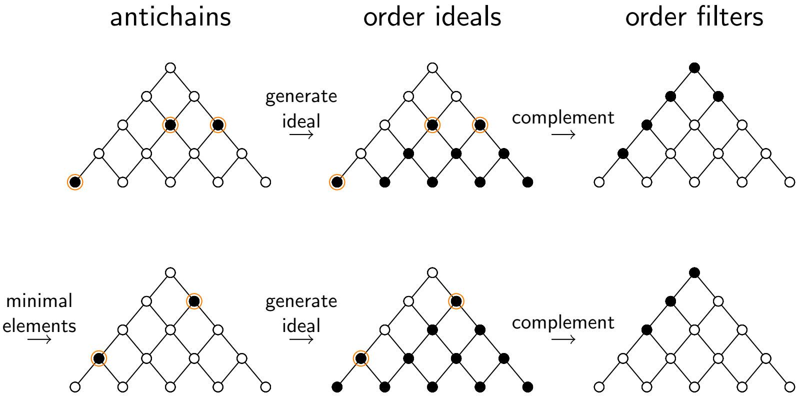

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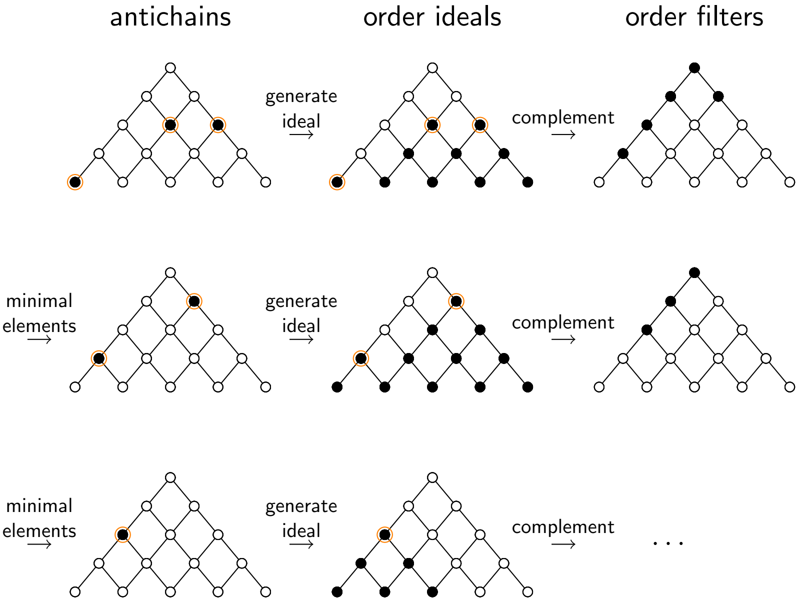

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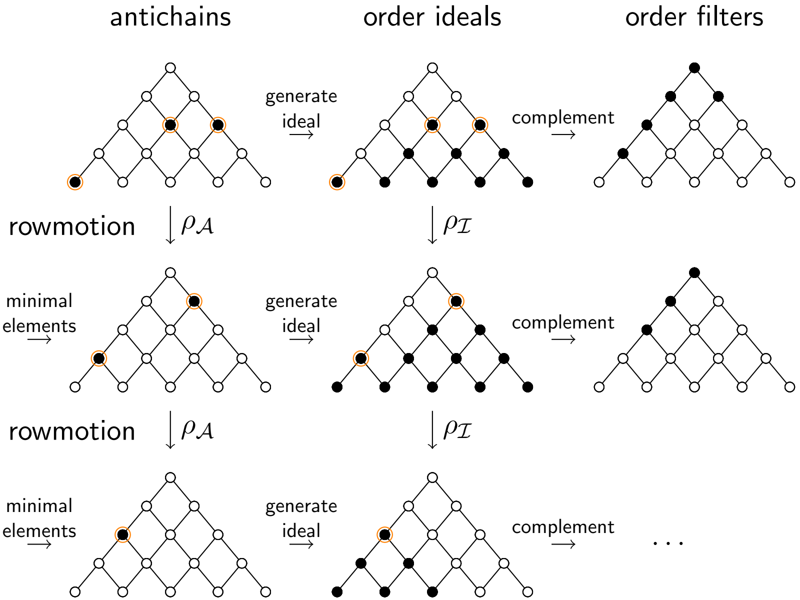
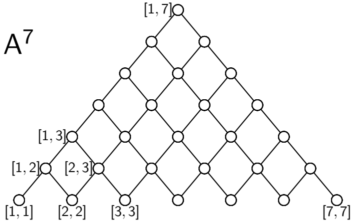


Type A root poset and Dyck paths

Let A^{n-1} denote the positive root poset of type A_{n-1}; equivalently, the set of intervals $\{[i,j] : 1 \leq i \leq j \leq n-1\}$ ordered by inclusion.



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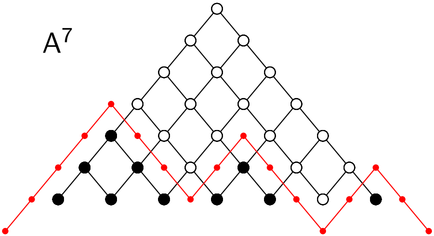

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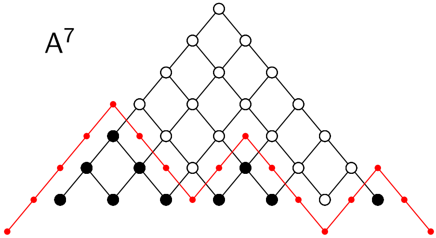

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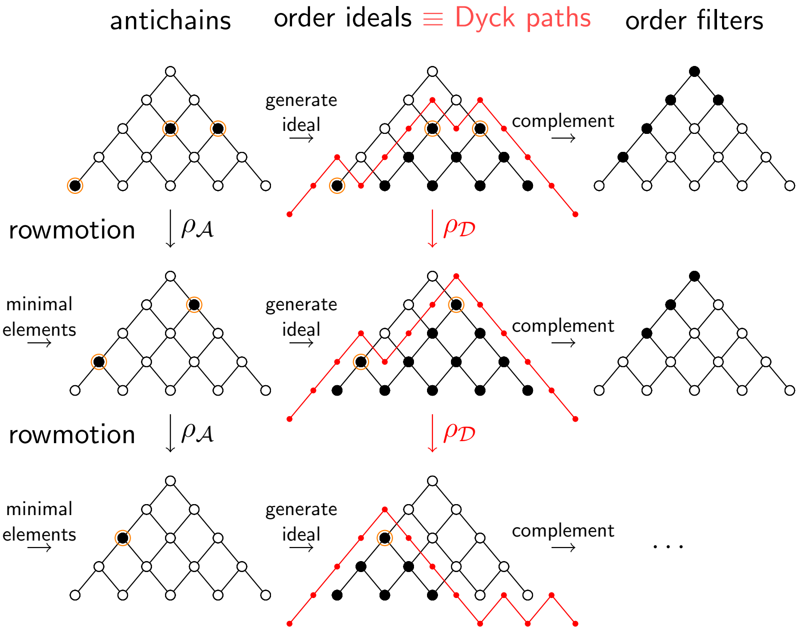
We can view rowmotion on ideals of A^{n-1} as an operation $\rho_{\mathcal{D}} : \mathcal{D}_n \to \mathcal{D}_n$.

Rowmotion on Dyck paths



321-avoiding permutations

$\mathcal{S}_n(321) = \{\pi \in \mathcal{S}_n : \pi \text{ is 321-avoiding}\}$

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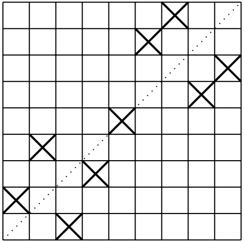
$\pi = 241358967$



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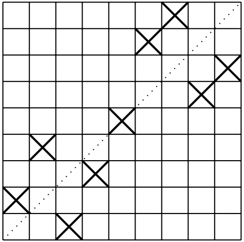

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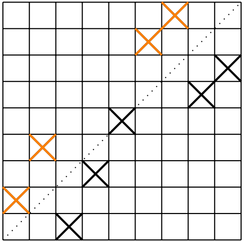

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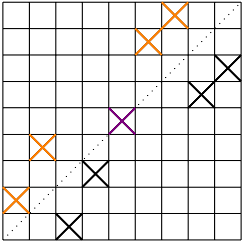

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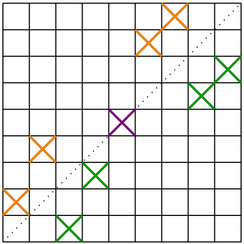

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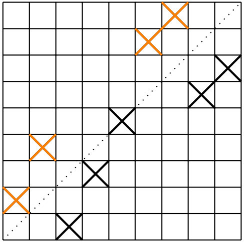

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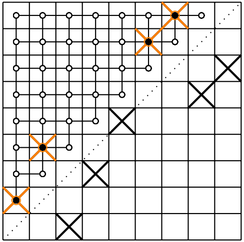

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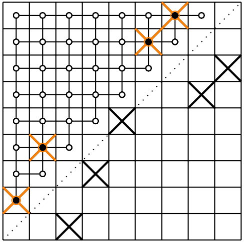

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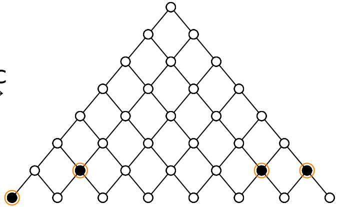

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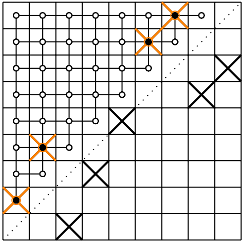

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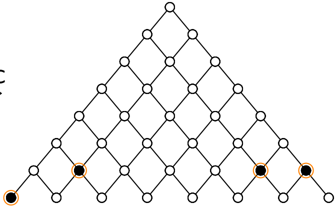

$$\mathrm{Exc}(\pi) = \{[i, \pi(i) - 1] : (i, \pi(i)) \text{ is an excedance of } \pi\}$$

Rowmotion on 321-avoiding permutations

We define a rowmotion operation $\rho_{\mathcal{S}} : \mathcal{S}_n(321) \to \mathcal{S}_n(321)$ by

$$\rho_{\mathcal{S}} = \mathsf{Exc}^{-1} \circ \rho_{\mathcal{A}} \circ \mathsf{Exc}\,.$$

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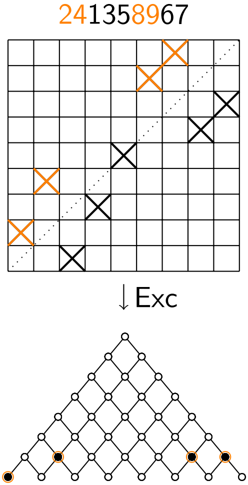

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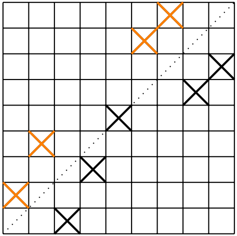
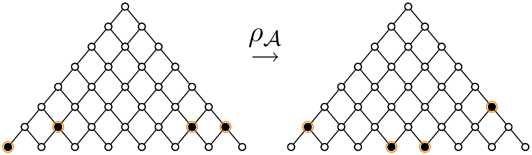


$\downarrow$ Exc



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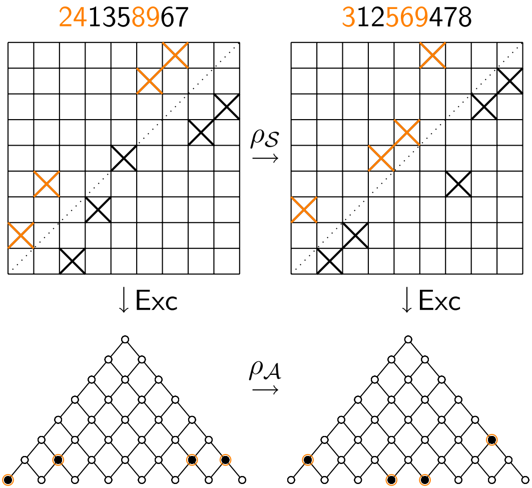

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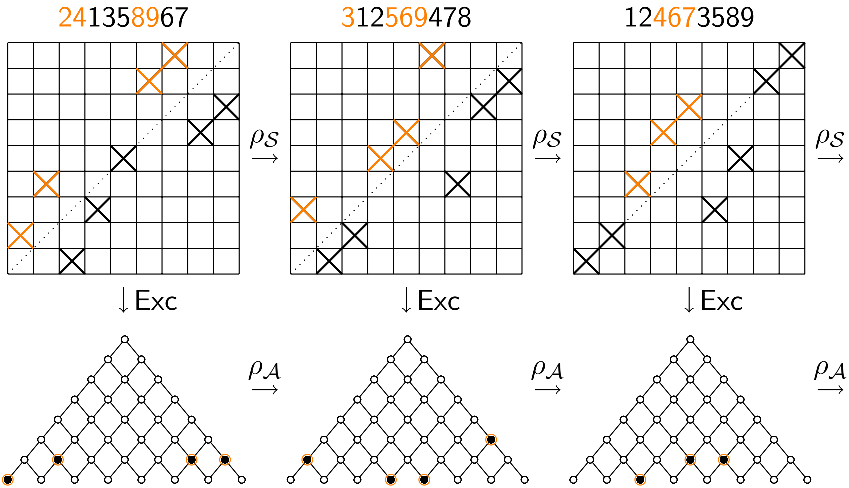

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If we consider the antichains of A^{n-1} given by the deficiencies of π instead, $\mathsf{Def}(\pi) := \mathsf{Exc}(\pi^{-1})$, then $\rho_{\mathcal{S}}$ is equivalent to *inverse rowmotion* of these antichains:

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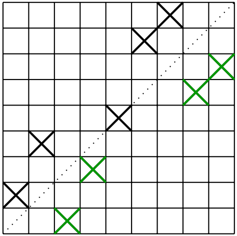

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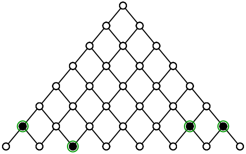

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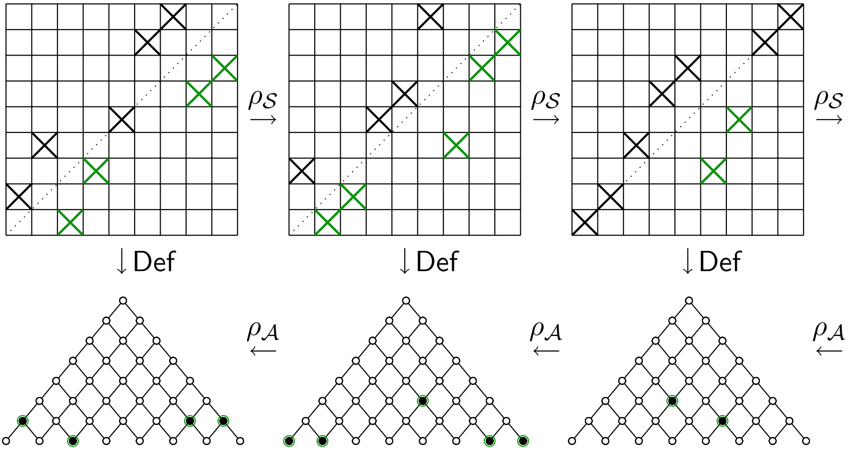

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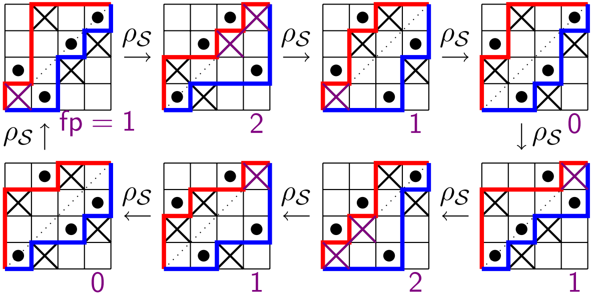
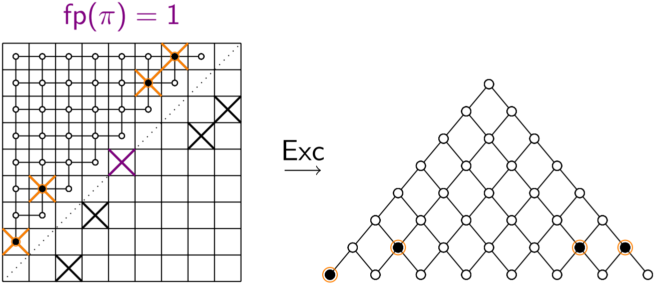


$$\frac{1+2+1+0+1+2+1+0}{8} = 1$$

Homomesy of fixed points

The statistic fp does not correspond to a natural statistic on antichains.



The statistics h_i

Hopkins and Joseph define the following family of statistics on antichains A of A^{n-1}:

$$h_i(A) = \sum_{j=1}^{i} \mathbb{1}_{[j,i]} + \sum_{j=i}^{n-1} \mathbb{1}_{[i,j]}, \qquad \text{where } \mathbb{1}_x = \begin{cases} 1 & \text{if } x \in A, \\ 0 & \text{if } x \notin A. \end{cases}$$

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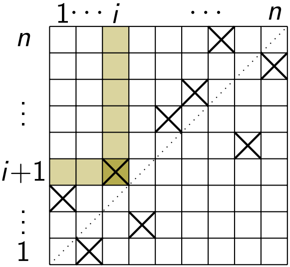
In terms of the permutation $\pi \in \mathcal{S}_n(321)$ such that $A = \mathrm{Exc}(\pi)$, this statistic is the number of crosses in the shaded region:



$h_3(314267958) = 2$

where the darker square in the corner is counted twice.

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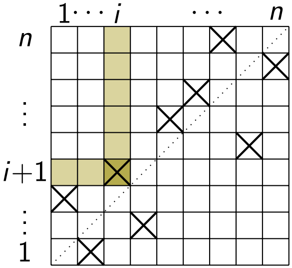

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Formally,

$$h_i(\pi) = \chi_{\pi^{-1}(i+1)<i+1} + \chi_{\pi(i)>i},$$

where χ is the indicator function.

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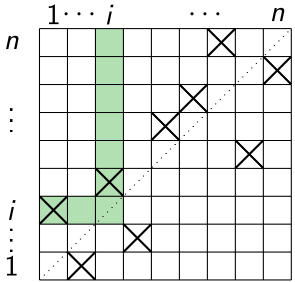

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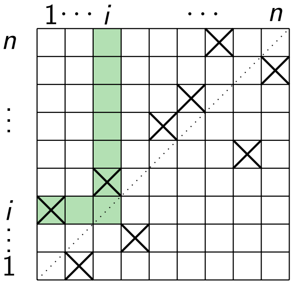

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Theorem

The statistics ℓ_i are 1-mesic under the action of $\rho_{\mathcal{S}}$ on $\mathcal{S}_n(321)$.

Using that h_i and ℓ_i are 1-mesic, we get another proof that fp is 1-mesic as well, since

$$\mathrm{fp}(\pi) = \sum_{i=1}^{n} \ell_i(\pi) - \sum_{i=1}^{n-1} h_i(\pi).$$

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Corollary

For even n, the statistic sgn *is* 0*-mesic under $\rho_{\mathcal{S}}$ on $\mathcal{S}_n(321)$.*

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For even n, the statistic sgn *is* 0*-mesic under $\rho_{\mathcal{S}}$ on $\mathcal{S}_n(321)$.*

Simion–Schmidt '85 showed that $\mathcal{S}_n(321)$ contains the same number of odd and even permutations, for even n.

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In fact, the map $\pi \mapsto \rho_{\mathcal{S}}(\pi^{-1})$ gives a sign-reversing involution.

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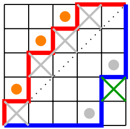

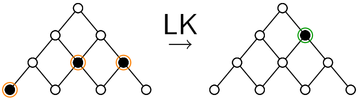

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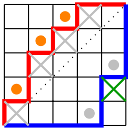

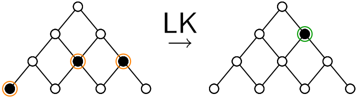

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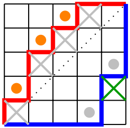

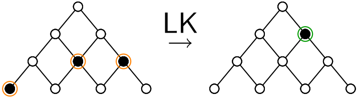

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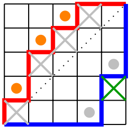

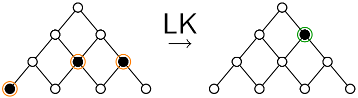

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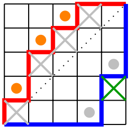

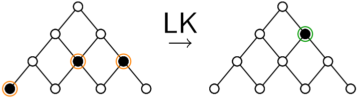

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using a classical result of Simion–Schmidt '85.

Promotion

Recall Schützenberger's promotion on standard Young tableaux:

$$T = \begin{array}{|c|c|c|c|c|}\hline 1 & 2 & 4 & 7 & 8 \\ \hline 3 & 5 & 6 & 9 & 10 \\ \hline \end{array} \xrightarrow{\text{delete } 10} \begin{array}{|c|c|c|c|c|}\hline 1 & 2 & 4 & 7 & 8 \\ \hline 3 & 5 & 6 & 9 & \\ \hline \end{array} \rightarrow \begin{array}{|c|c|c|c|c|}\hline 1 & 2 & 4 & 7 & 8 \\ \hline 3 & 5 & 6 & & 9 \\ \hline \end{array}$$

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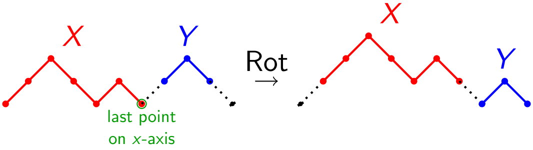
Define a rotation operation on Dyck paths:



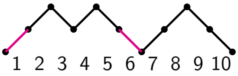
Promotion and rotation

Via the standard bijections, promotion translates to rotation on Dyck paths and on non-crossing matchings:

1	2	4	7	8
3	5	6	9	10

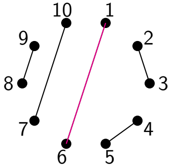
→

$\mathcal{D}_n$



→

$\mathcal{N}_n$



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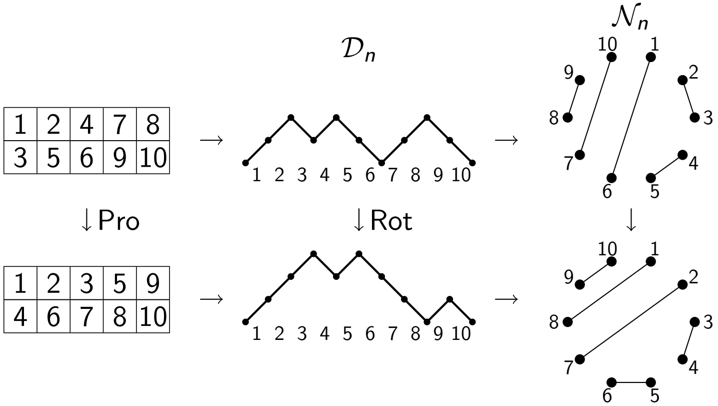

The Armstrong–Stump–Thomas bijection

Theorem (Armstrong–Stump–Thomas '13)

There is an equivariant bijection AST *between* $\mathcal{A}(\mathsf{A}^{n-1})$ *under rowmotion, and* $\mathcal{N}_n$ *(equivalently,* $\mathcal{D}_n$*) under rotation.*

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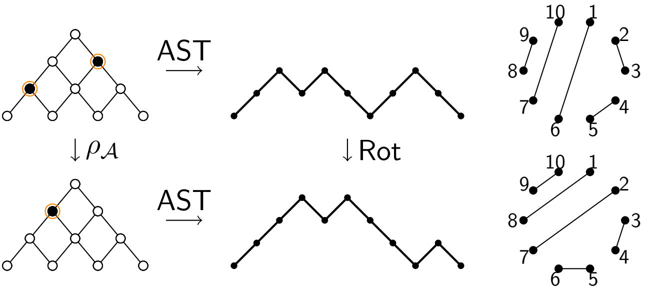

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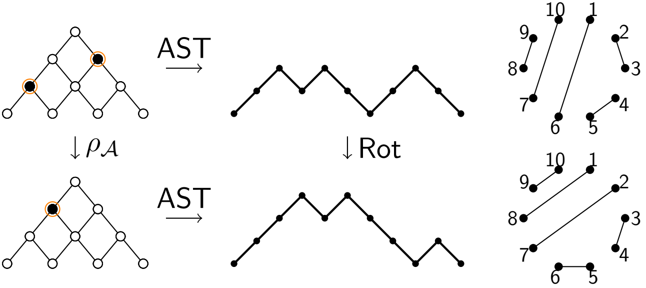
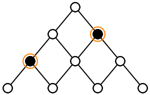


The bijection AST has a complicated description, and it generalizes to other Weyl groups, with the correct formulation.

A simpler description of AST

We can use 321-avoiding permutations to give a simple description of the AST bijection (in type A):



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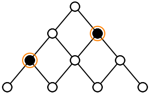
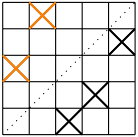


$\downarrow \mathrm{Exc}^{-1}$



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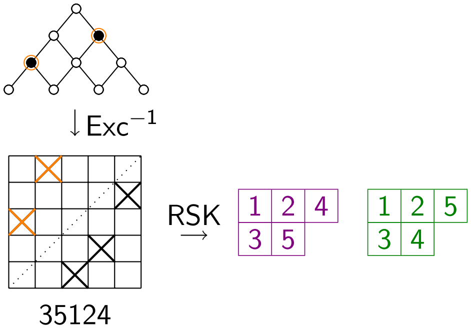

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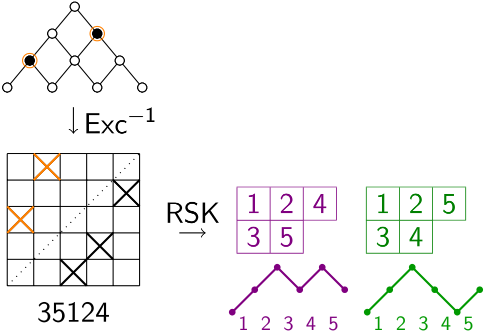

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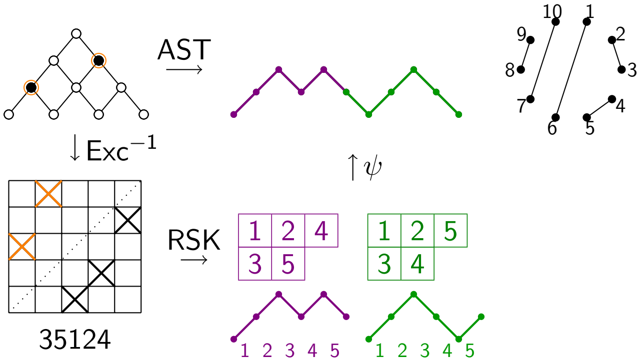

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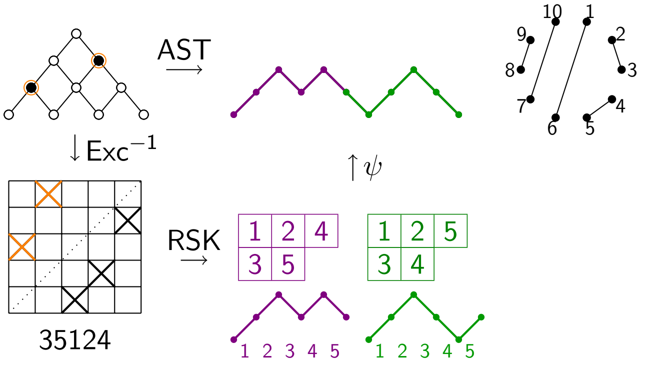

Theorem

AST $= \psi \circ$ RSK $\circ$ Exc^{-1}

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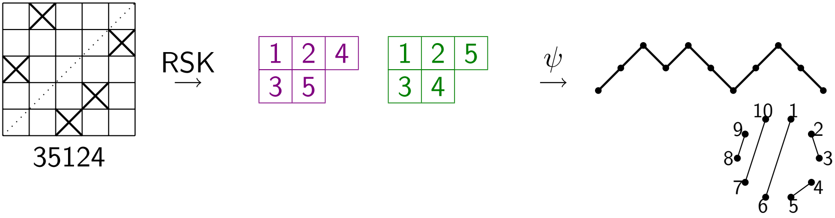

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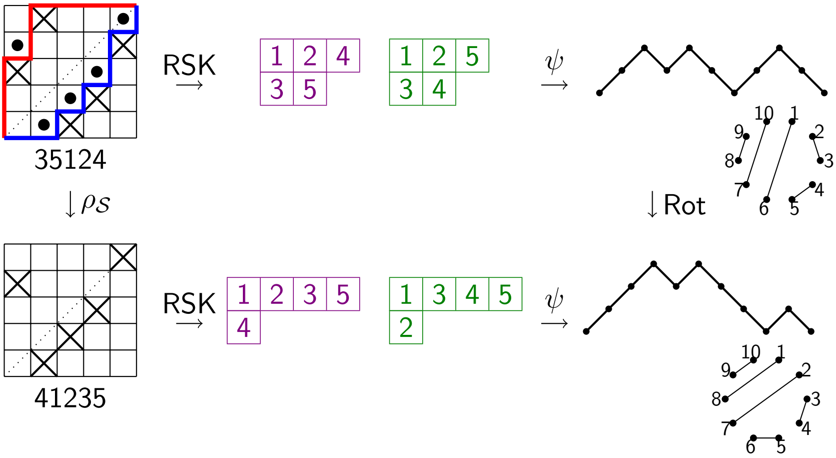

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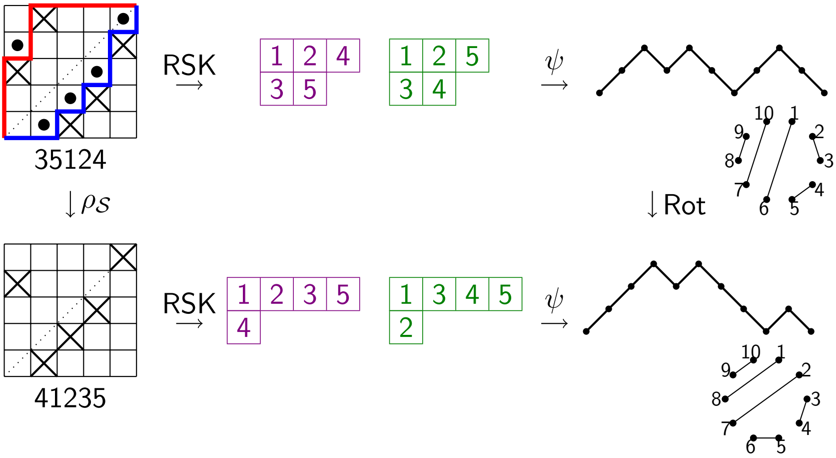

THANK YOU!