GUILLOTINE RECTANGULATIONS AND MESH PATTERNS

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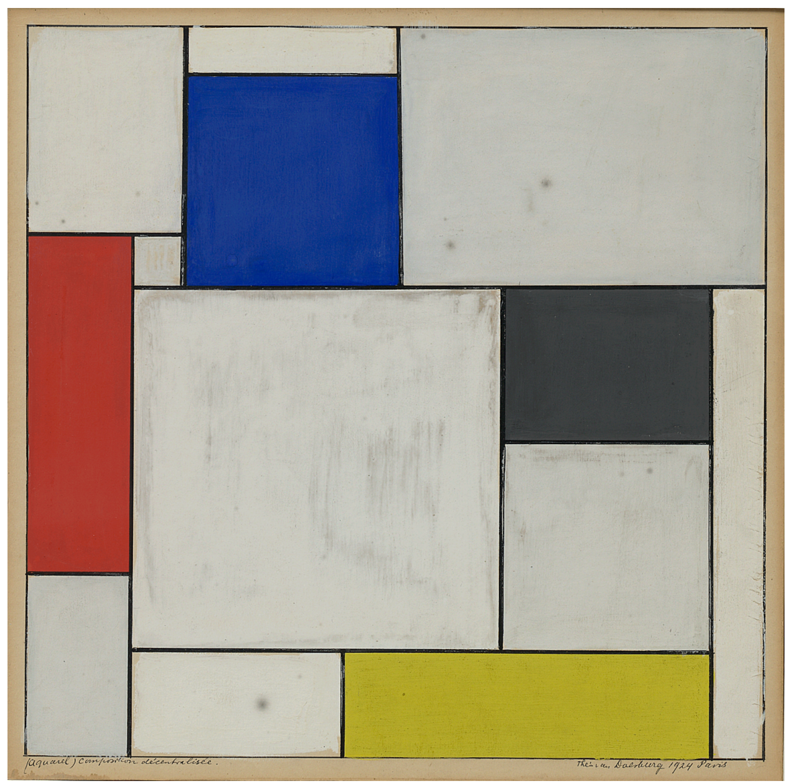
Permutation Patterns 2023
Dijon, July 2023

* Supported by FWF – The Austrian Science Fund

A **rectangulation** (also: a **floorplan**) is a partition of a rectangle into rectangles.
Convention: No point is common to four rectangles.

The **size** of a rectangulation is the number of rectangles.



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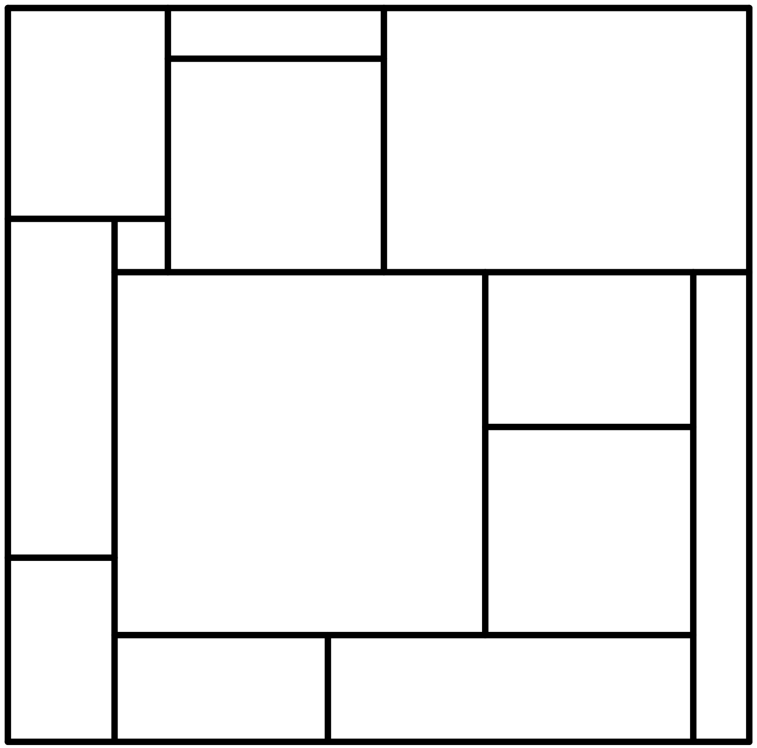
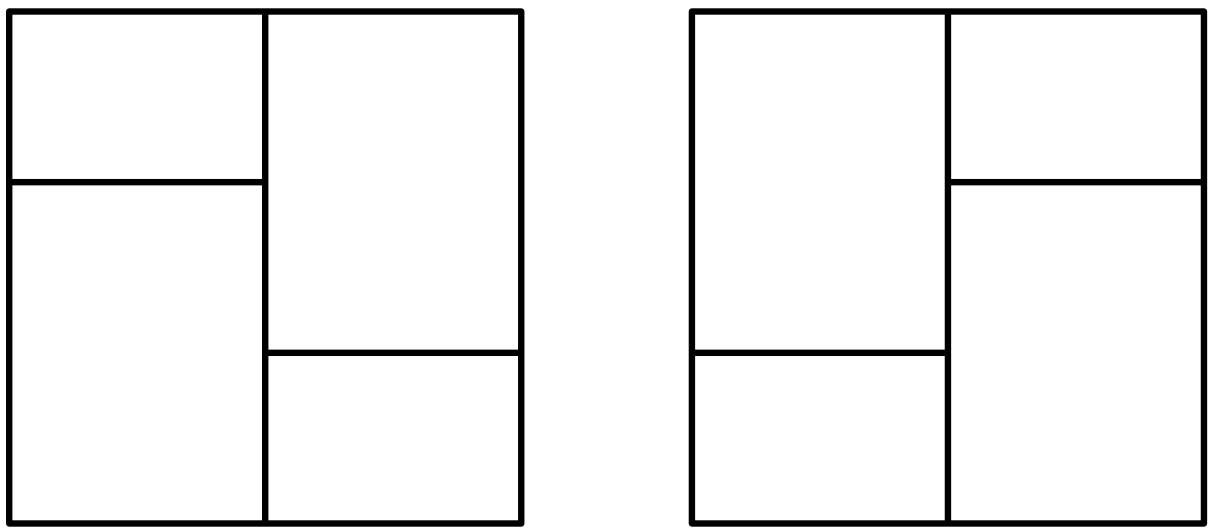
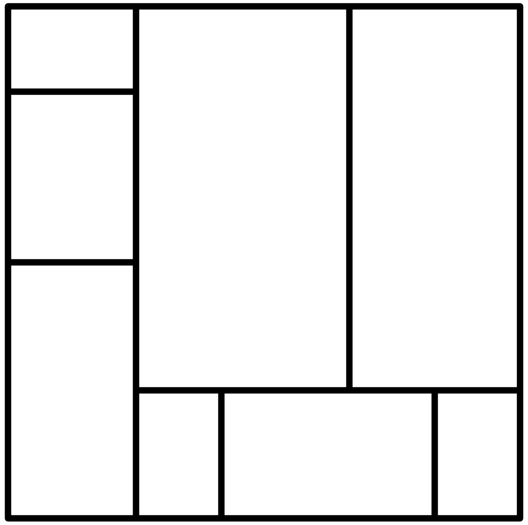


Two kinds of equivalence:

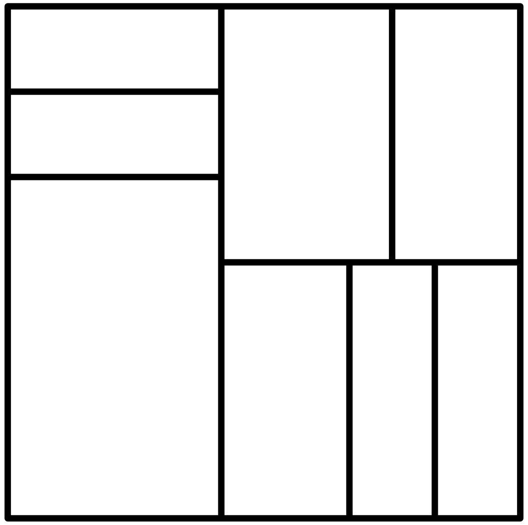
- weak (preserves rectangle-segment contacts),
- strong (preserves rectangle-rectangle contacts).



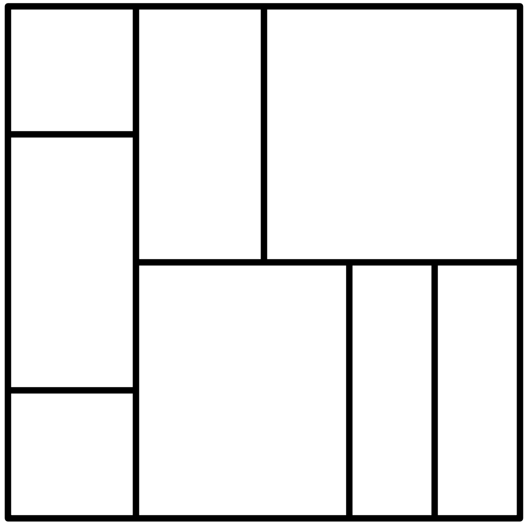
These two rectangulations are weakly equivalent but not strongly equivalent.



A



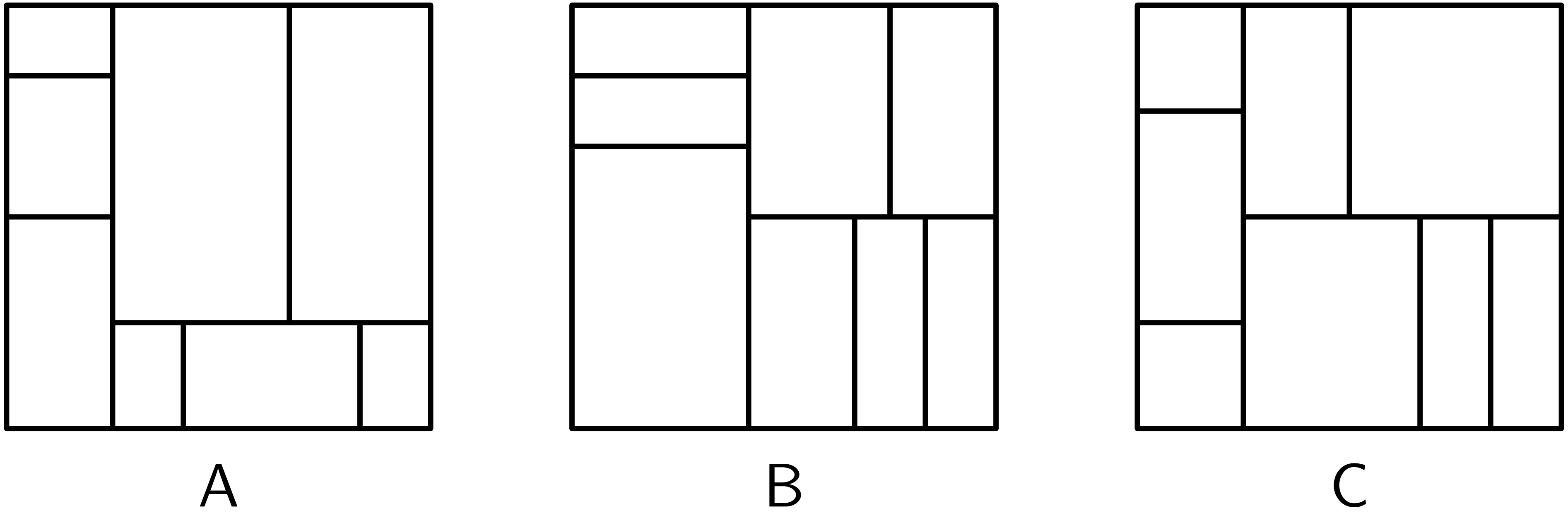
B



C

A, B, and C are weakly equivalent.

A and B are strongly equivalent,
C is not strongly equivalent to them.



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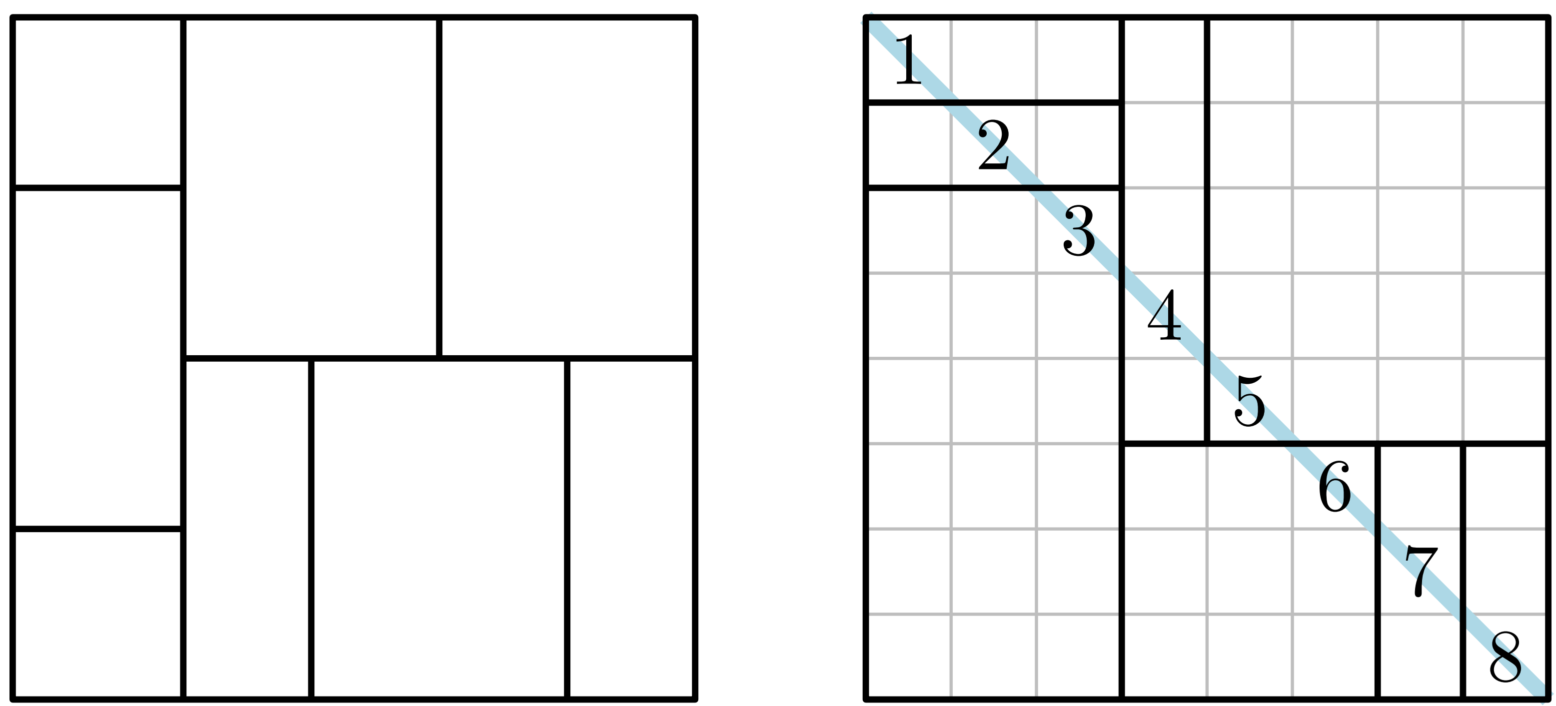
Equivalence classes up to the weak equivalence are called **mosaic rectangulations** (I will say: weak rectangulations).

Equivalence classes up to the strong equivalence are called **generic rectangulations** (I will say: strong rectangulations).

Every rectangulation of size n is weakly equivalent to precisely one **diagonal rectangulation** which uses the grid $[n] \times [n]$, and the NW–SE diagonal intersects every rectangle.
It defines the **NW–SE labeling** of rectangles.



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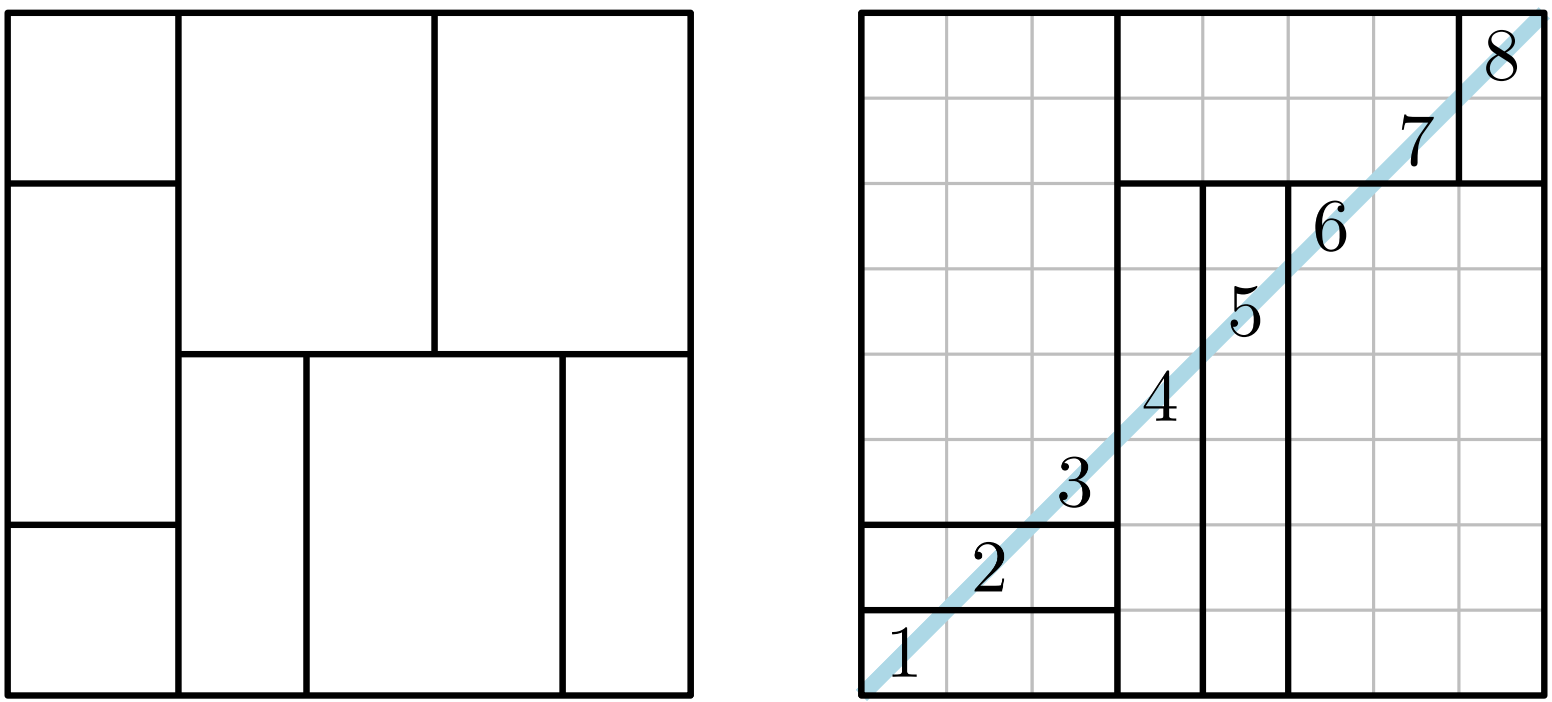


The **anti-diagonal** rectangulation $\longrightarrow$ the **SW–NW** labeling.

Permutation classes to appear in this talk:

Baxter permutations: Av$(2[41]3, 3[14]2)$.

Twisted Baxter permutations: Av$(2[41]3, 3[41]2)$.

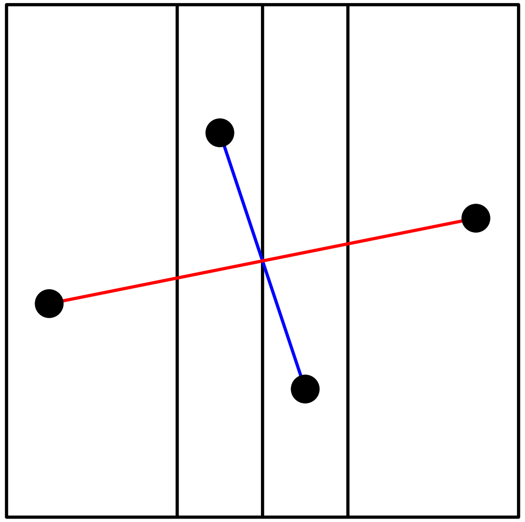
Co-twisted Baxter permutations: Av$(2[14]3, 3[14]2)$.

These three classes are enumerated by **Baxter numbers**.

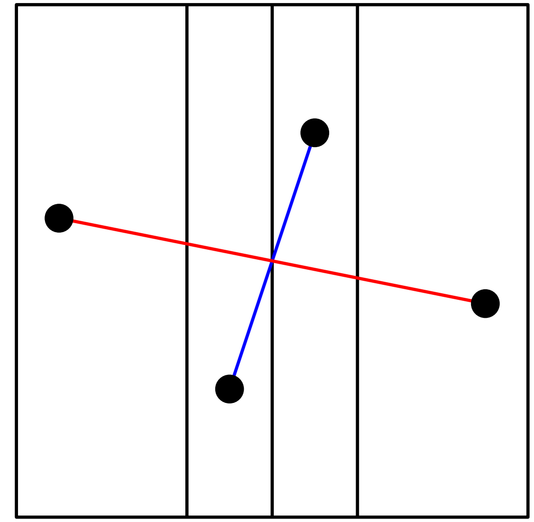
Separable permutations: Av$(2413, 3142)$.

Two-clumped permutations.

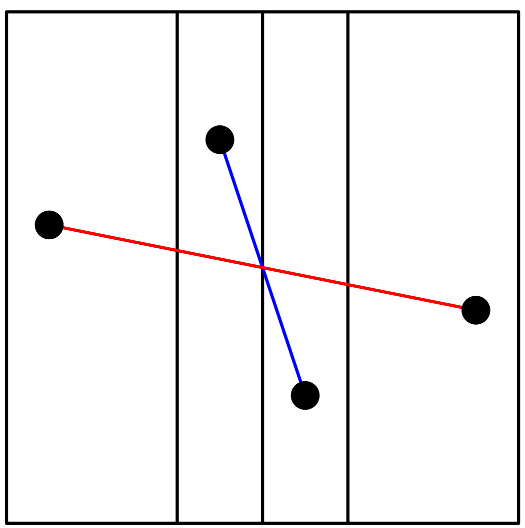
Co-two-clumped permutations.



2[41]3



3[14]2



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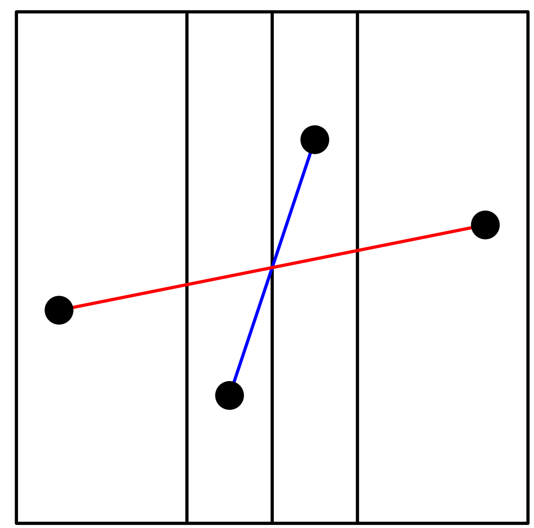

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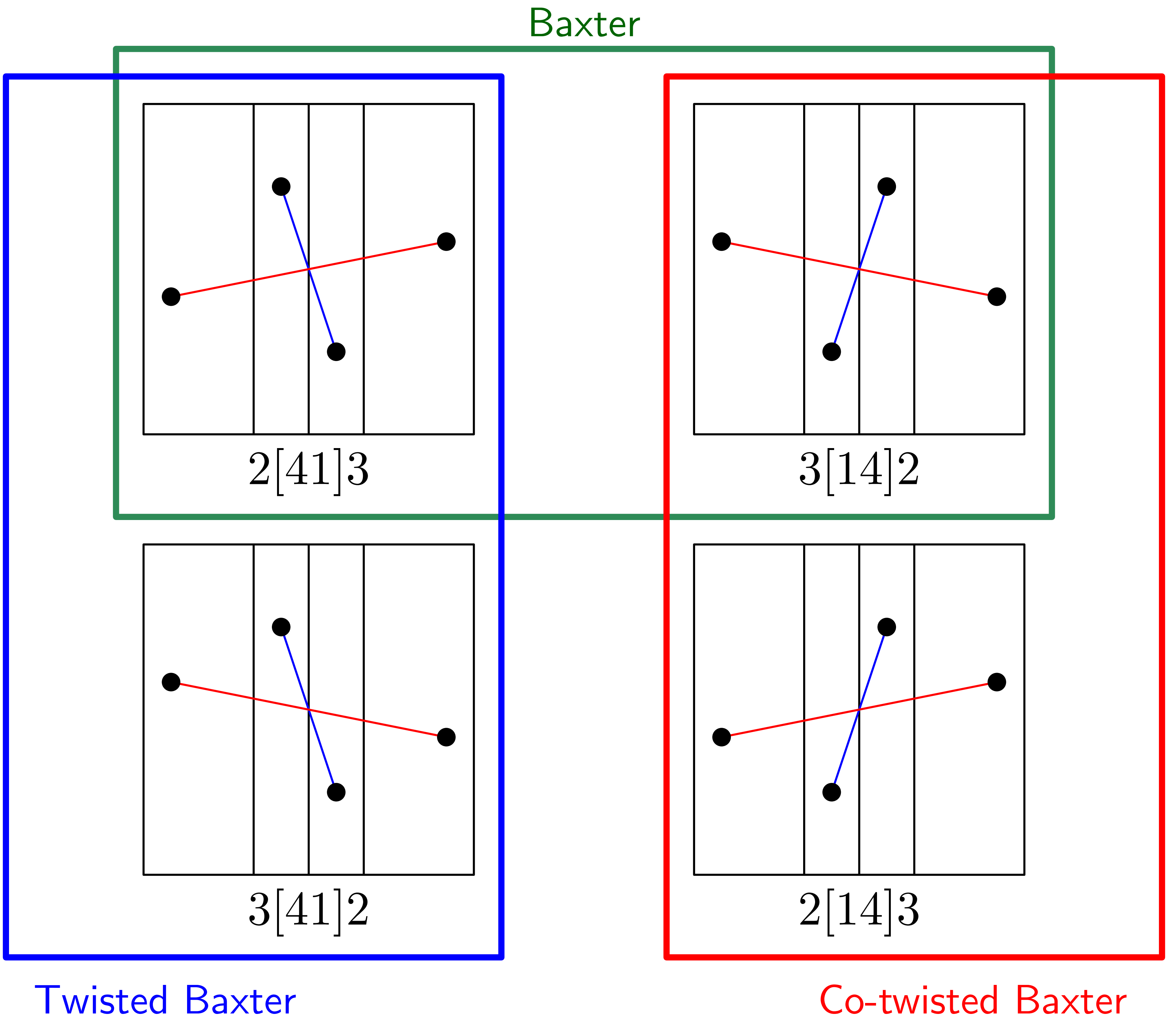
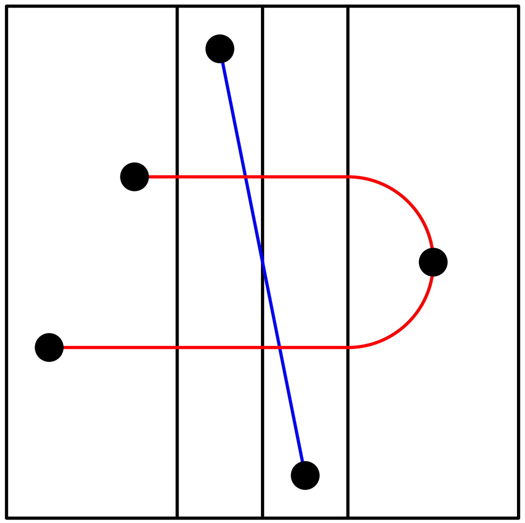
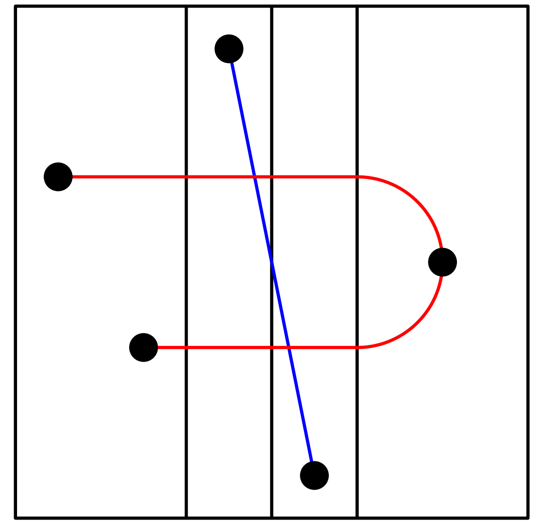

Baxter
2[41]3
3[14]2
3[41]2
2[14]3
Twisted Baxter
Co-twisted Baxter

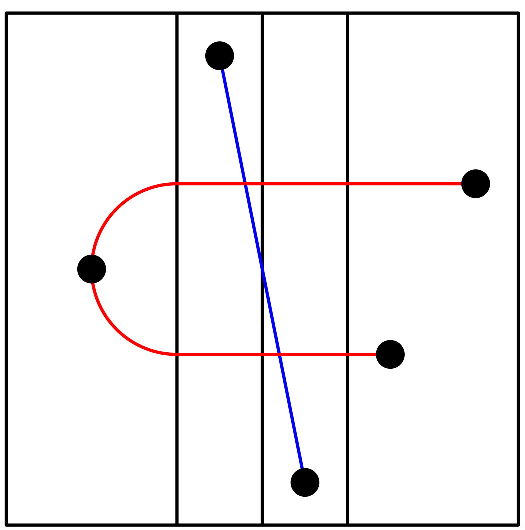
Two-clumped:



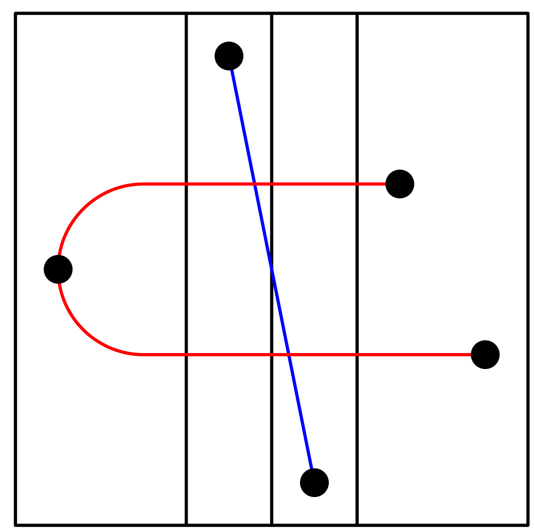
$24[51]3$



$42[51]3$

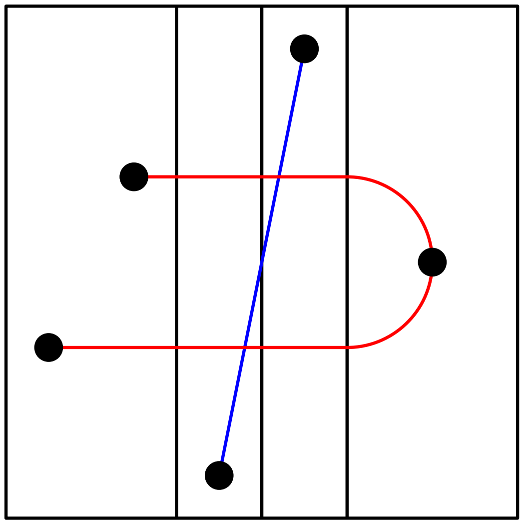


$3[51]24$

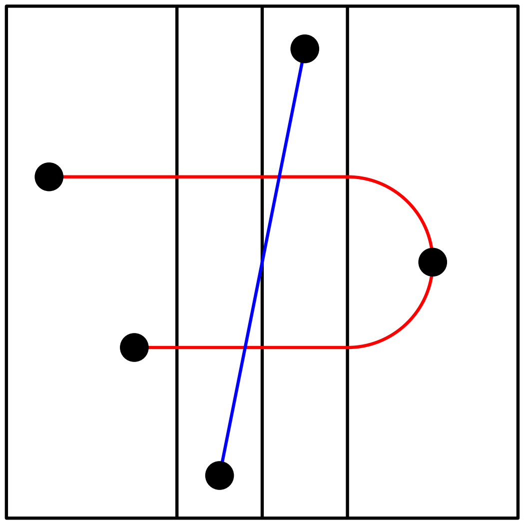


$3[51]42$

Co-two-clumped:



24[15]3



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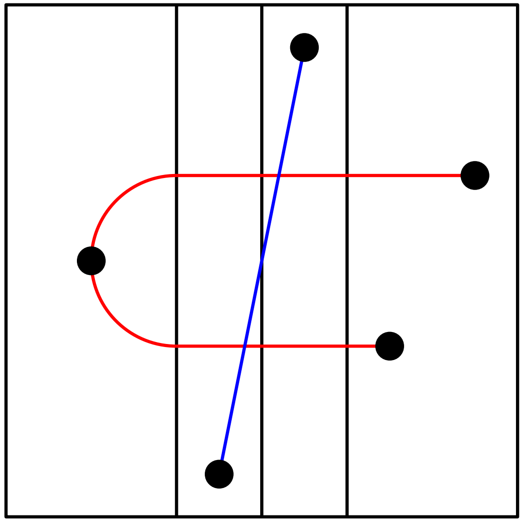

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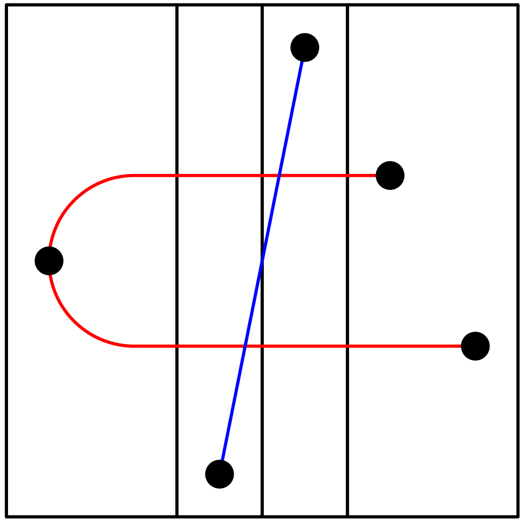

3[15]42

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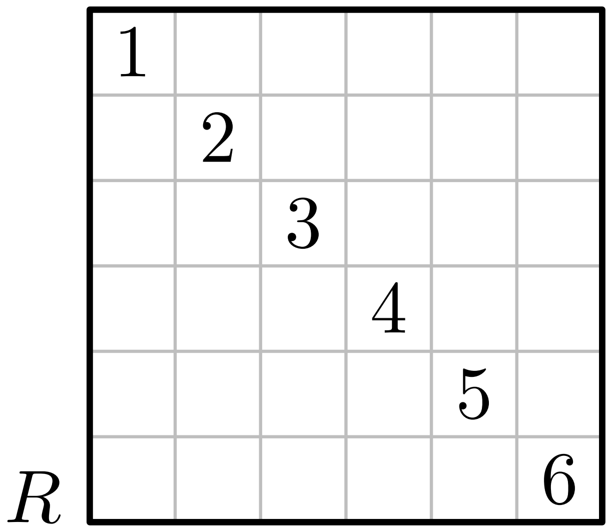

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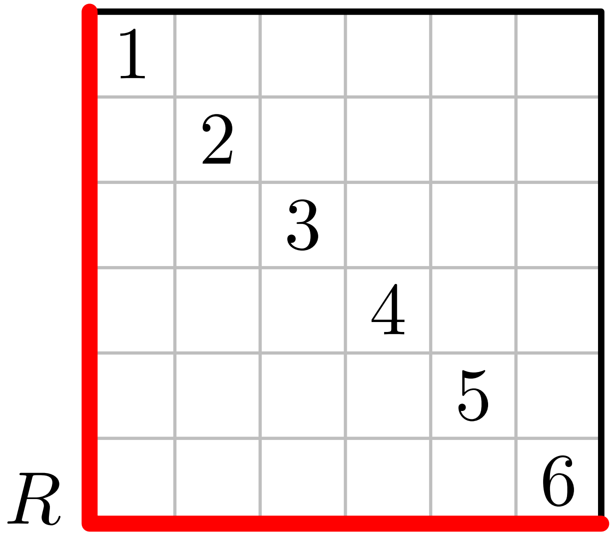

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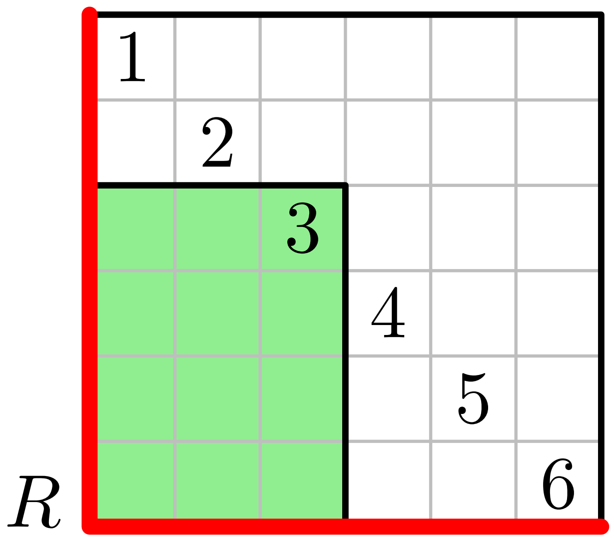

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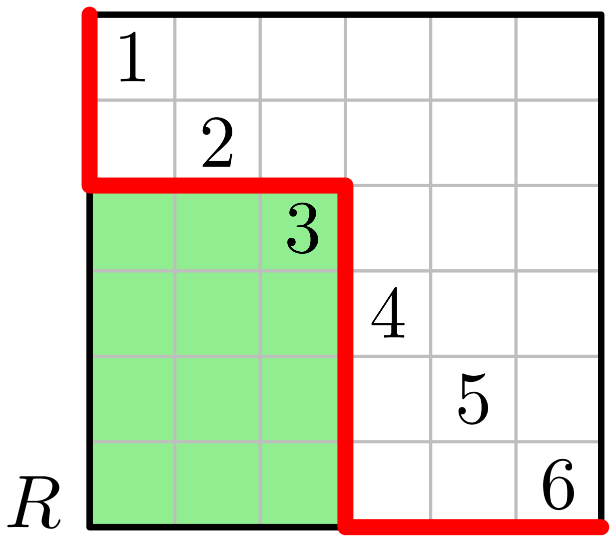

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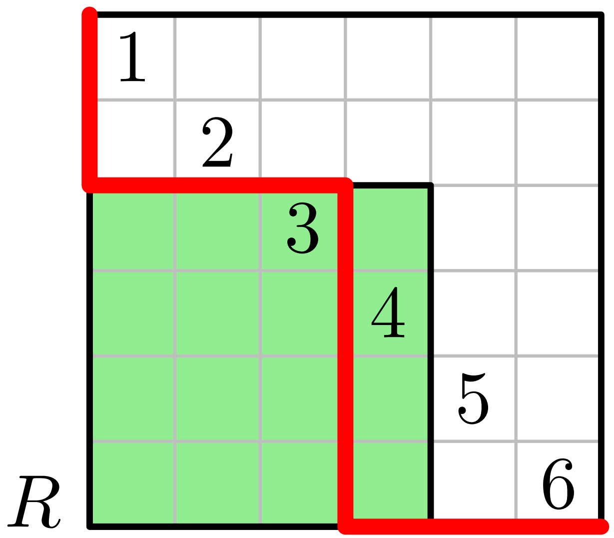

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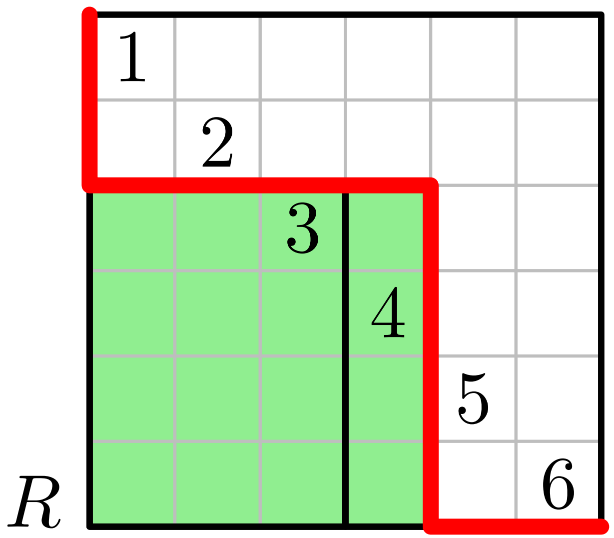

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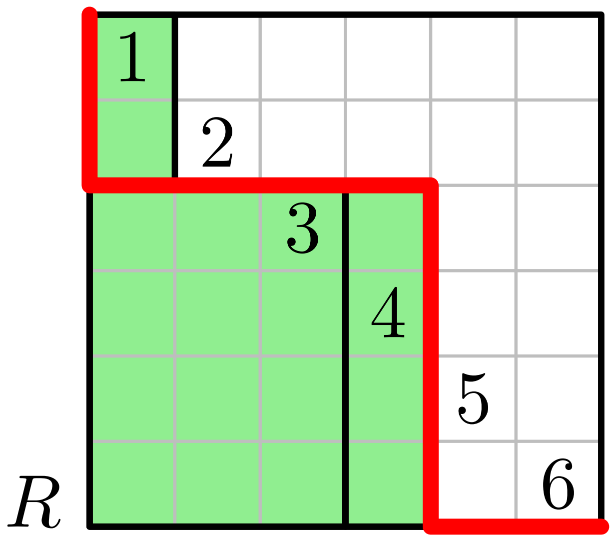

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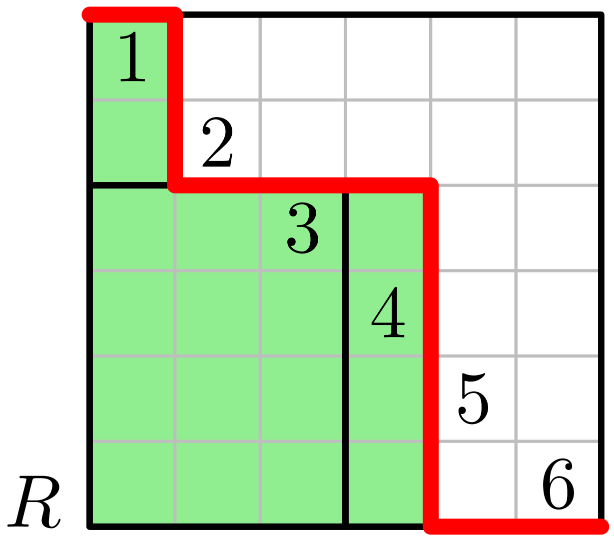

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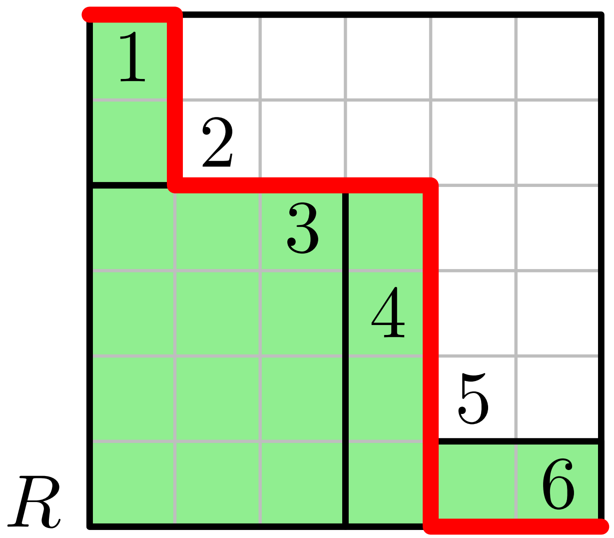

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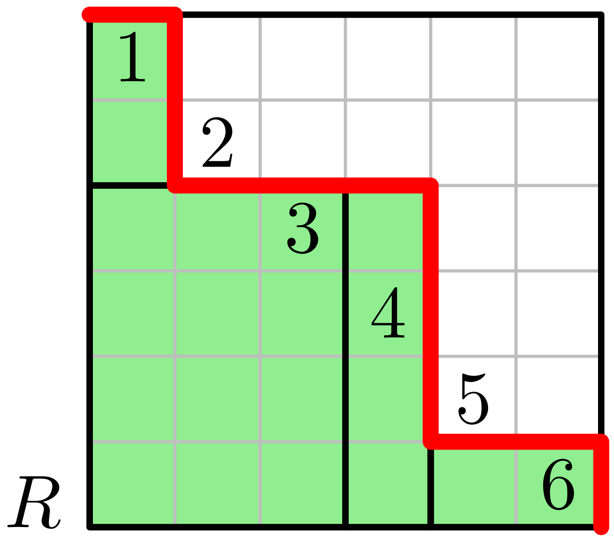

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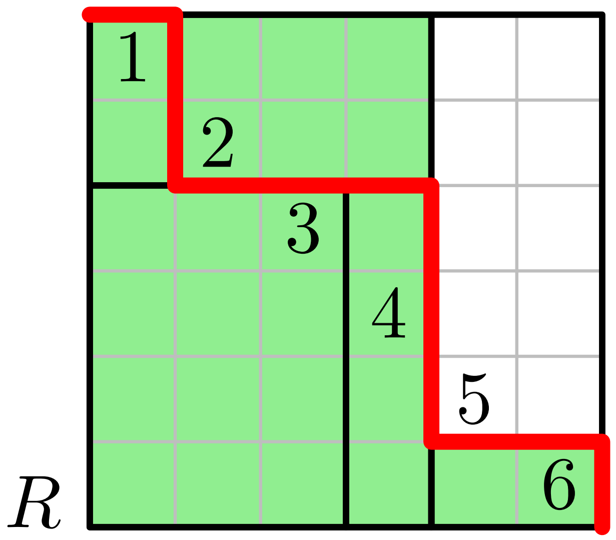

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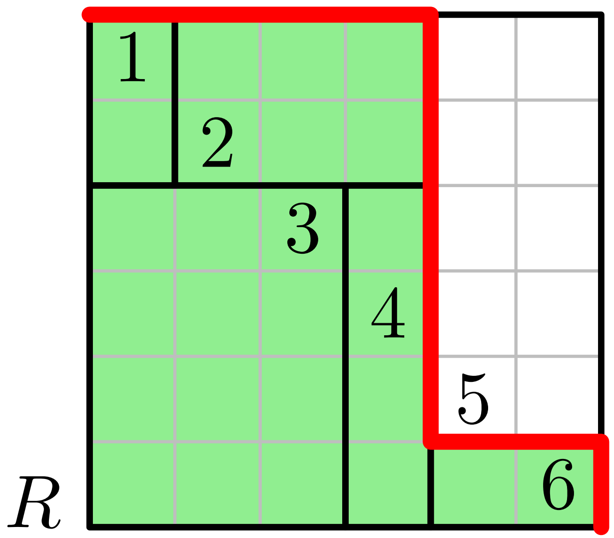

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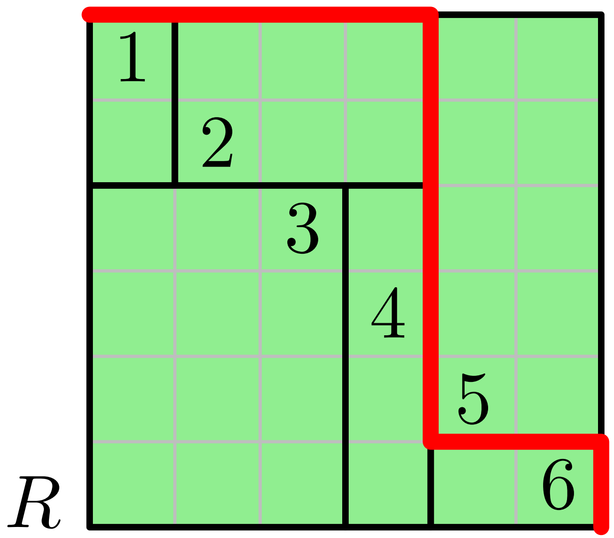

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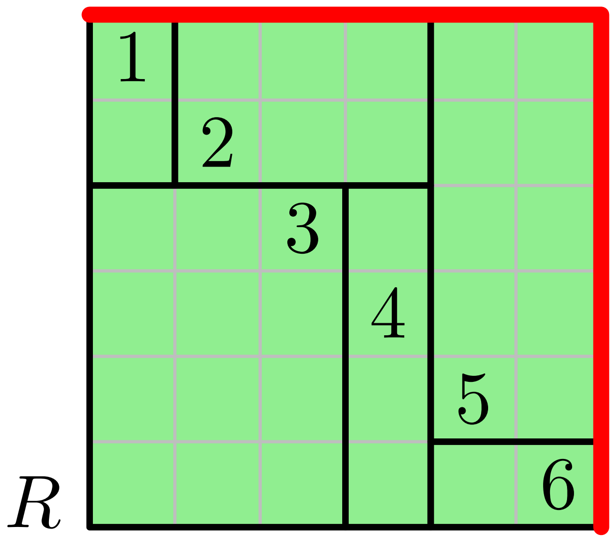

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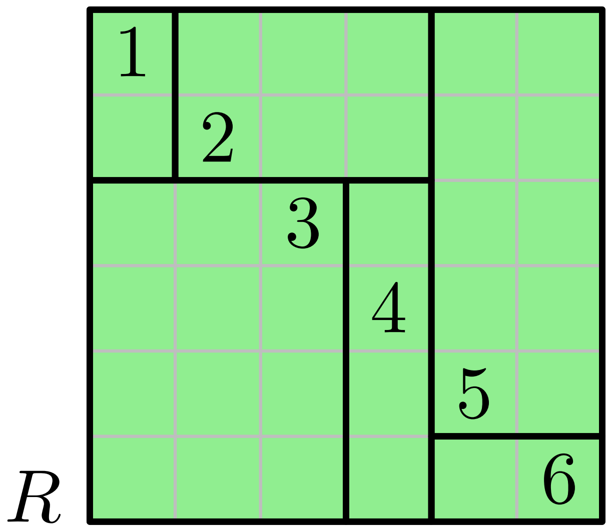

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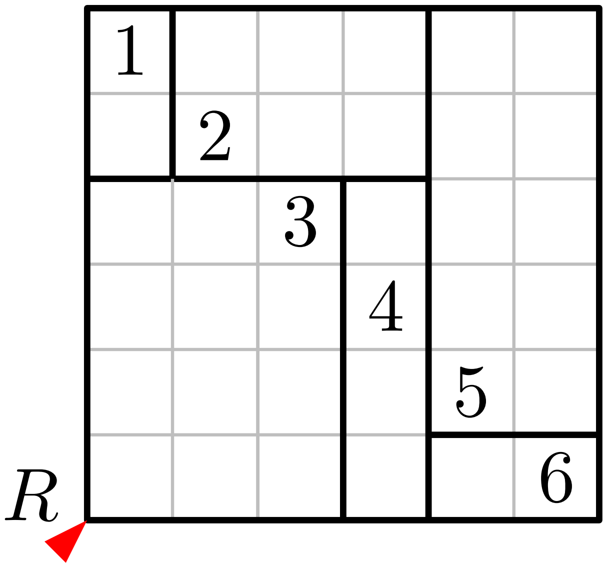

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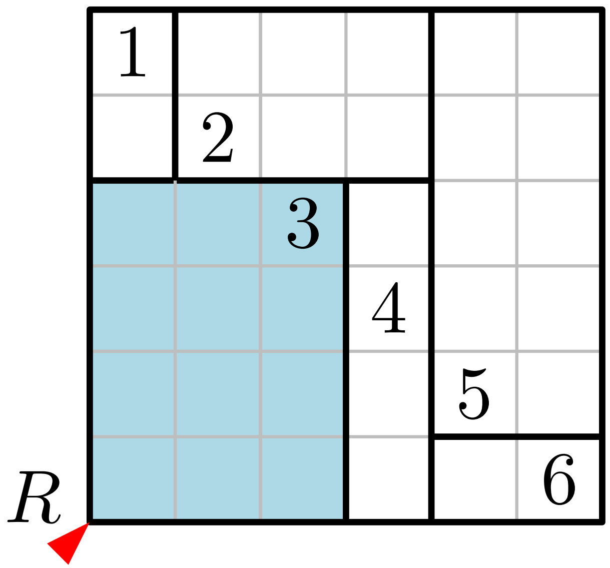

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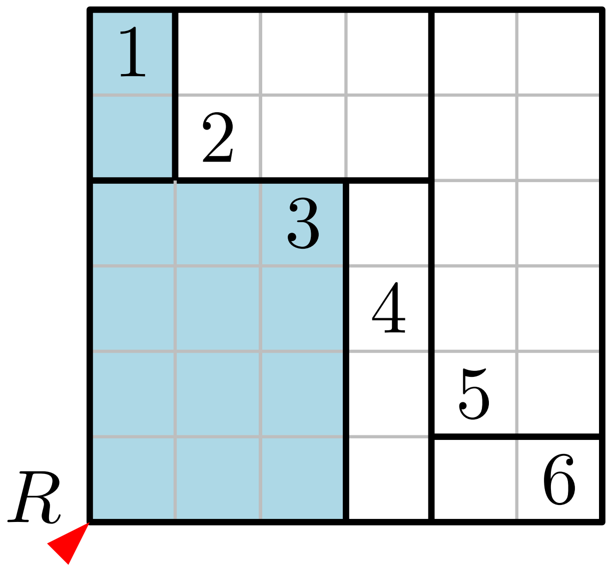
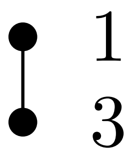


the left rule



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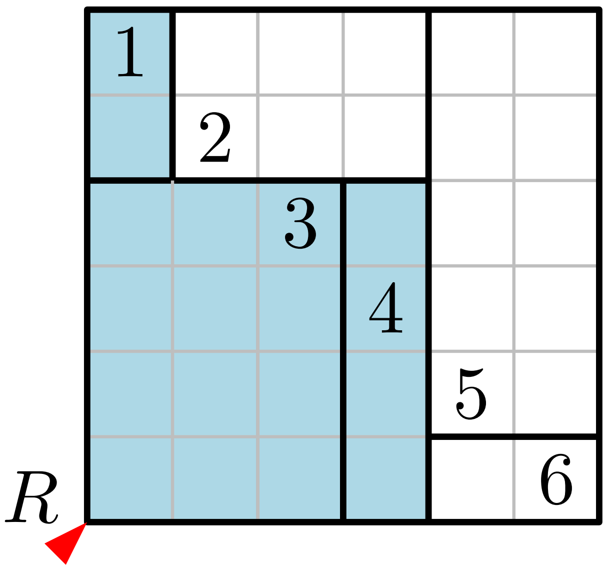

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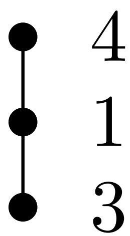

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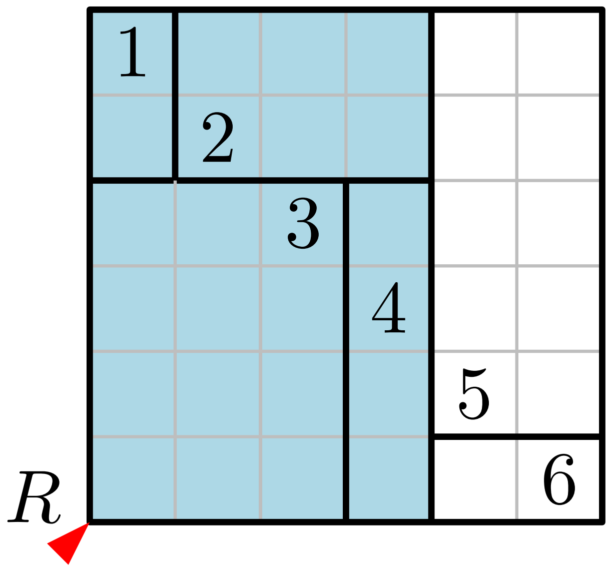

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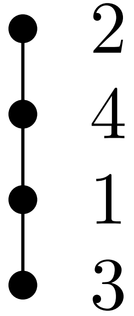

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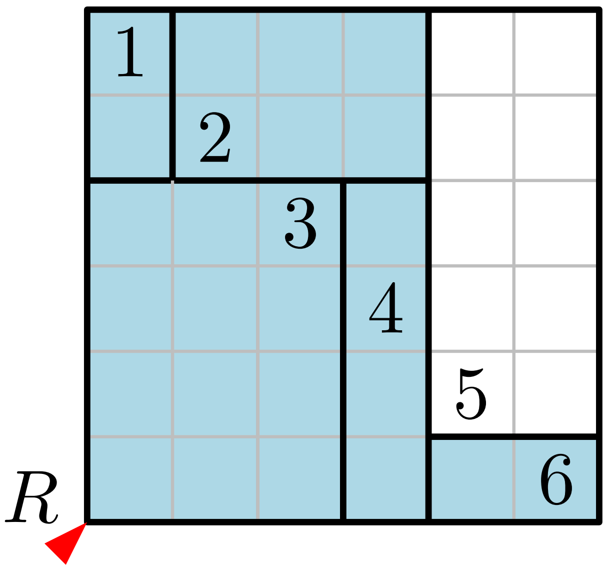

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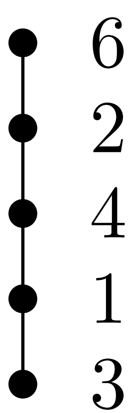

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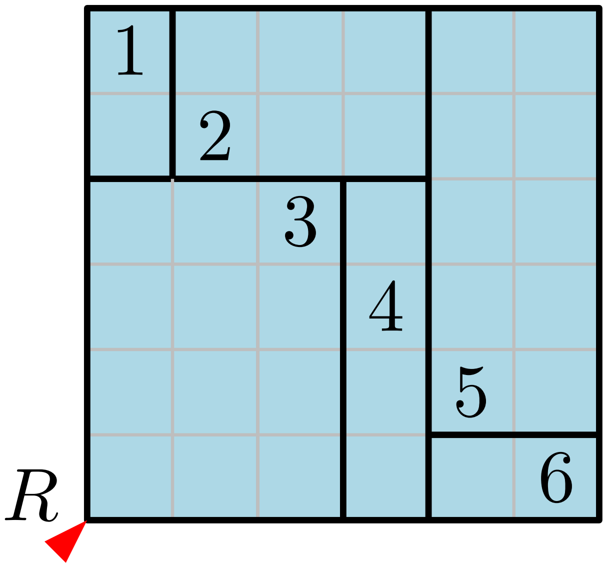

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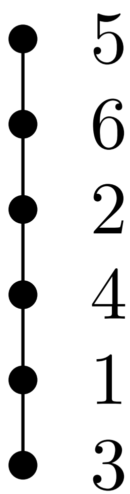

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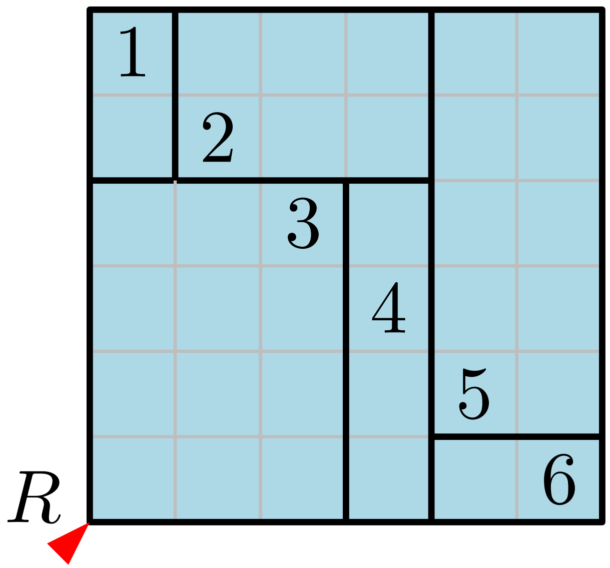

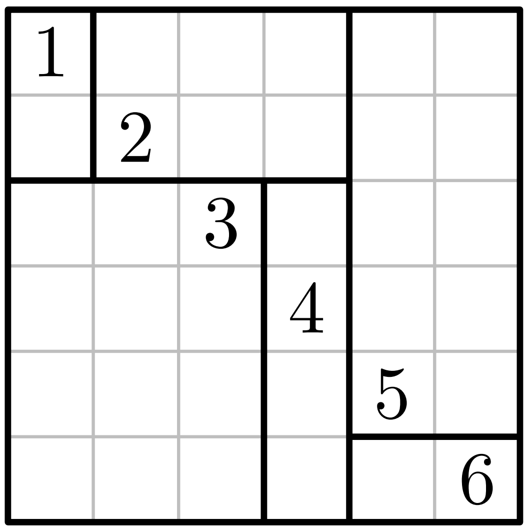

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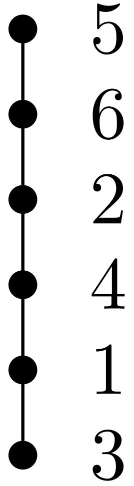

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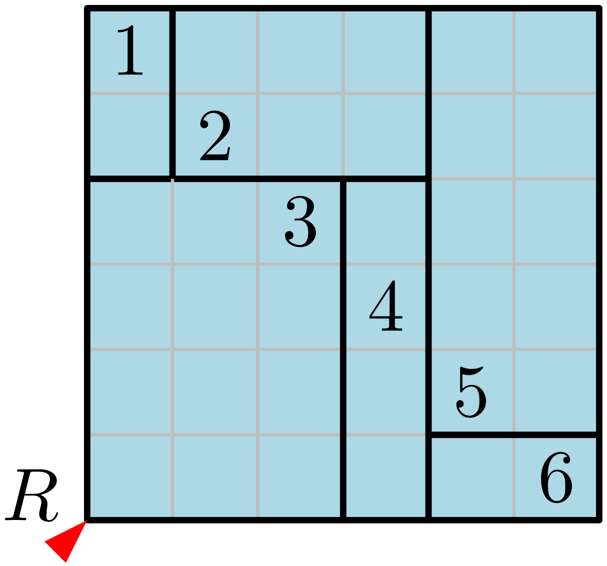

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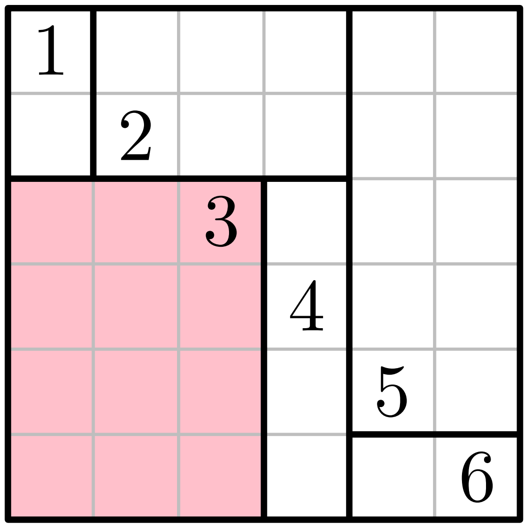

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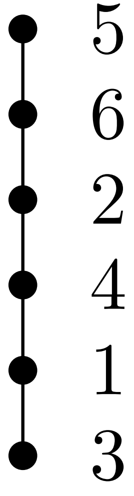

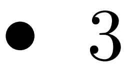

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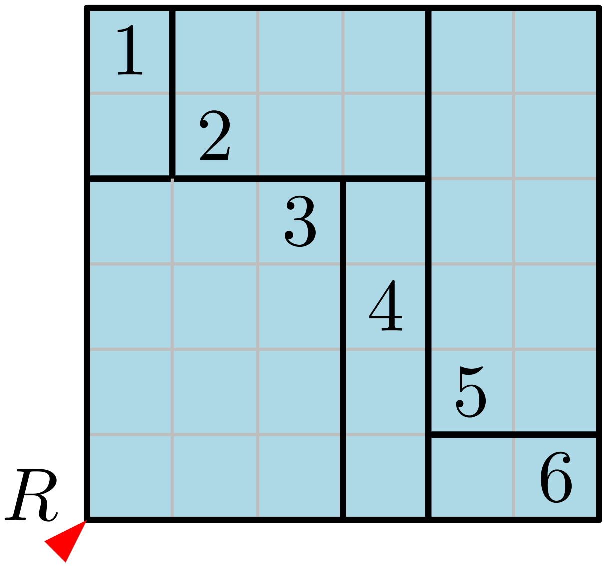

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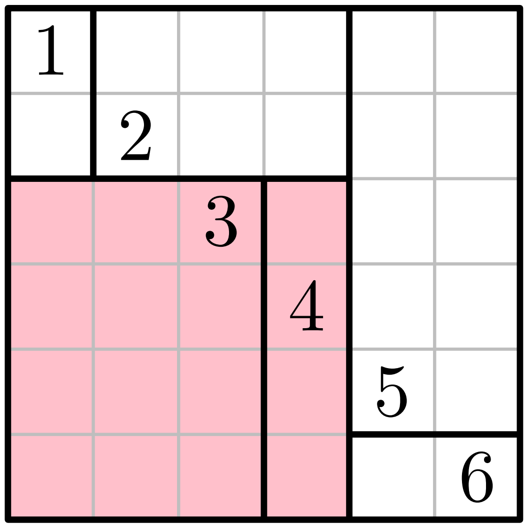
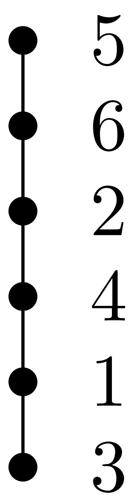
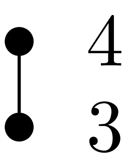


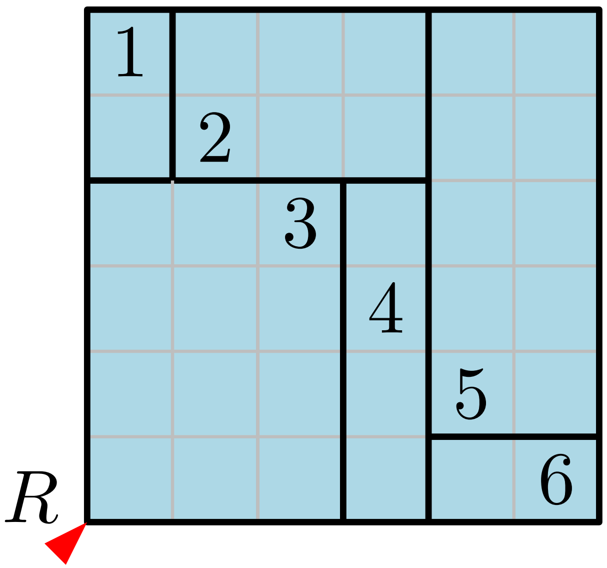
the right rule



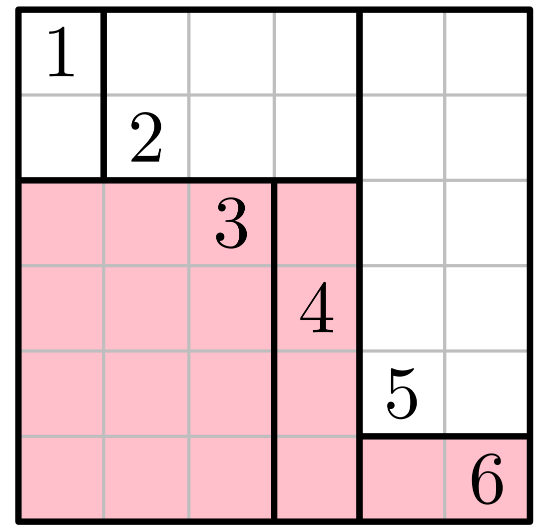


The mapping $\gamma_w \colon \; S_n \to$ **weak** rectangulations of size n

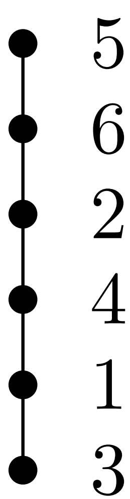
$\pi = 341625$ Now we construct the preimage $\gamma_w^{-1}(R)$

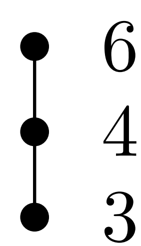


the left rule



the right rule





The mapping $\gamma_w\colon\ S_n \to$ **weak** rectangulations of size n

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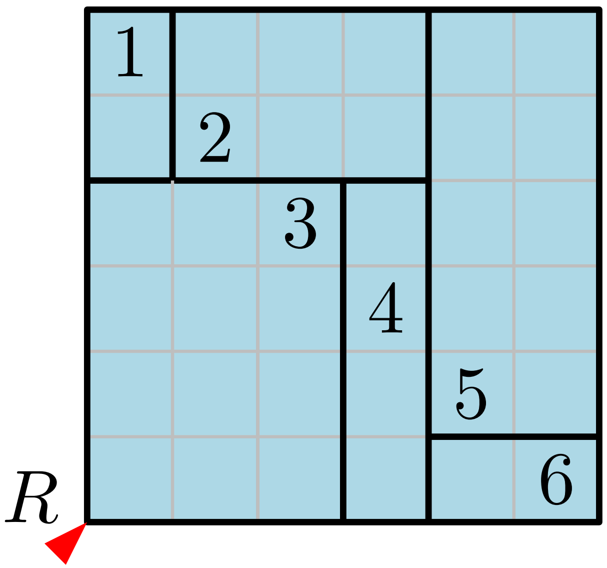
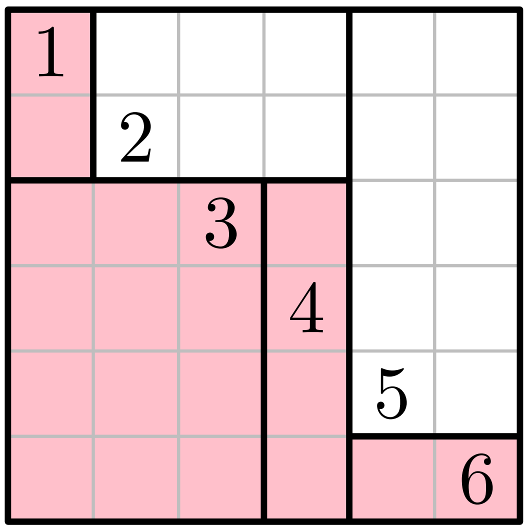
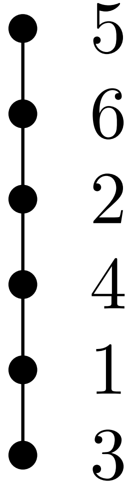
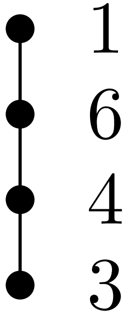


the left rule



the right rule





The mapping $\gamma_w \colon\ S_n \to$ **weak** rectangulations of size n

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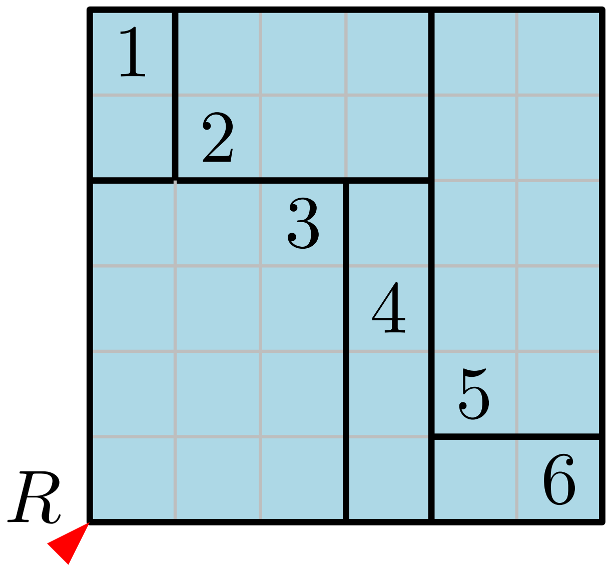
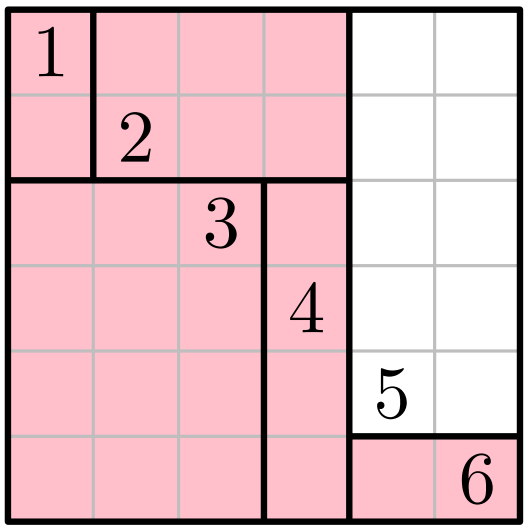
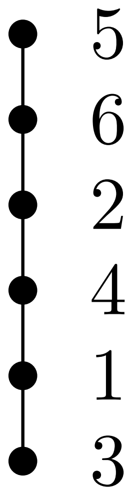
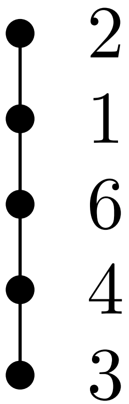


the left rule



the right rule





The mapping $\gamma_w\colon\ S_n \to$ **weak** rectangulations of size n

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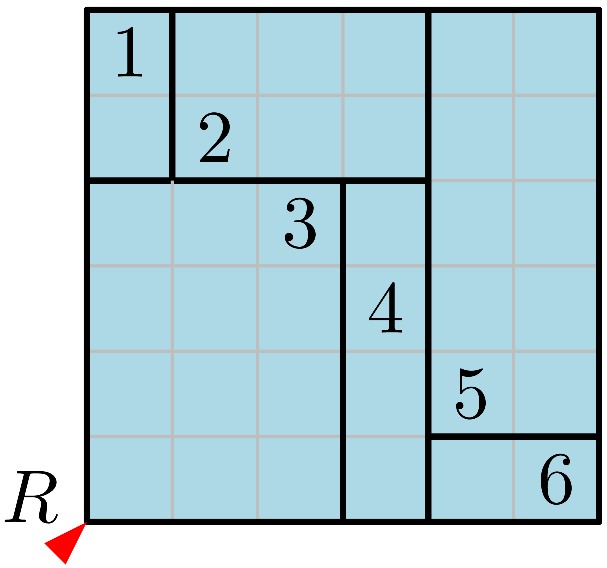
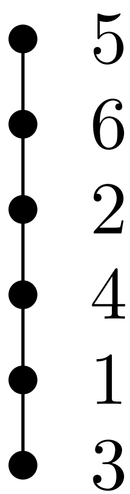
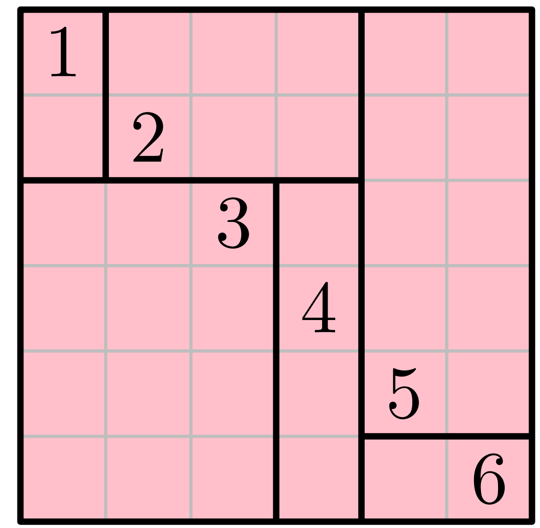
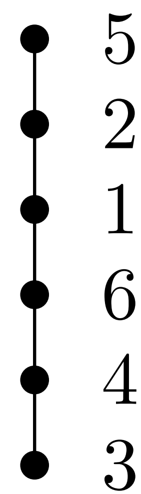


the left rule



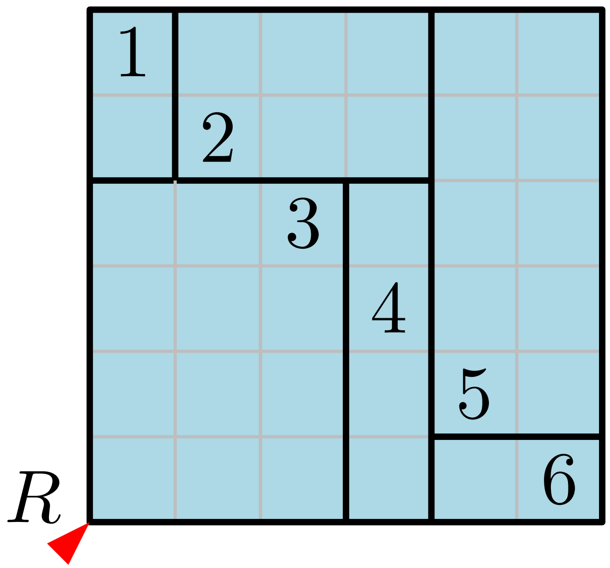


the right rule

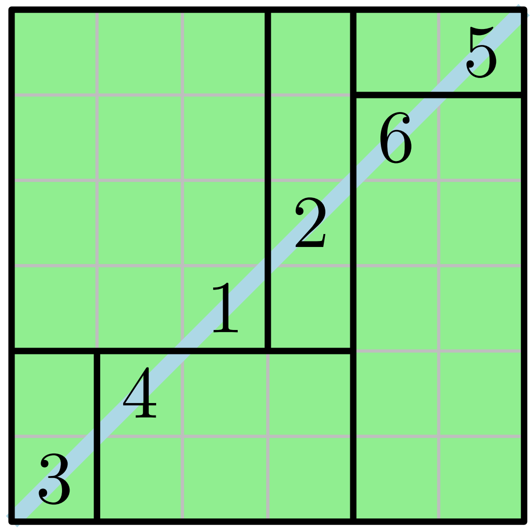


The mapping $\gamma_w\colon\ S_n \to$ **weak** rectangulations of size n

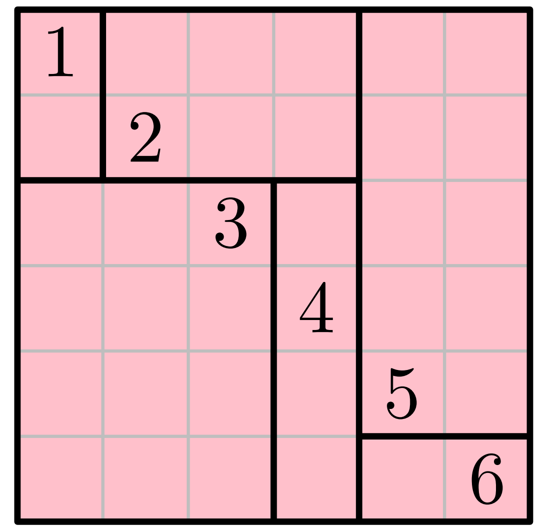
$\pi = 341625$ Now we construct the preimage $\gamma_w^{-1}(R)$



the left rule



the SW–NE rule

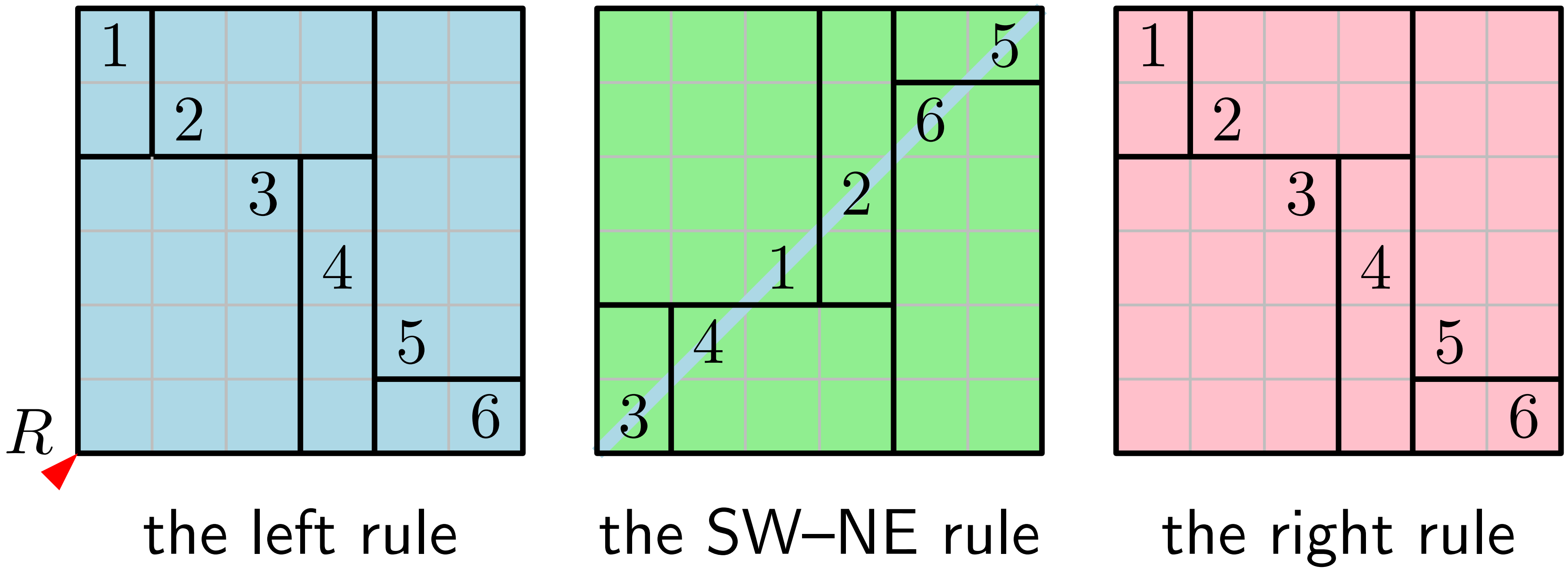


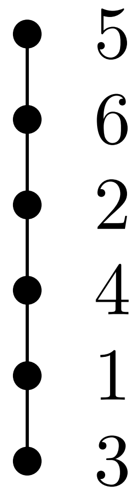
the right rule

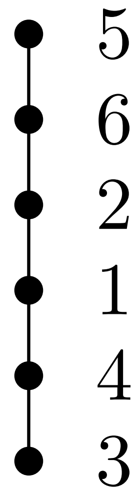
the left rule	the SW–NE rule	the right rule
5	5	5
6	6	2
2	2	1
4	1	6
1	4	4
3	3	3

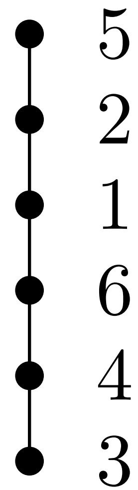
The mapping $\gamma_w\colon\ S_n \to$ **weak** rectangulations of size n

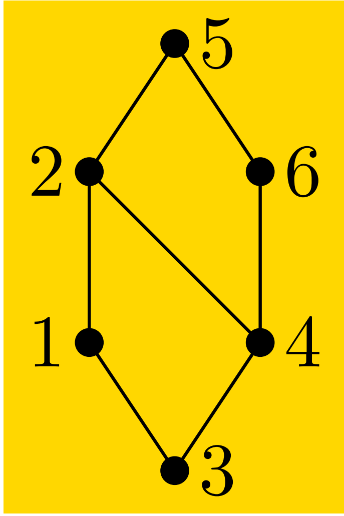
$\pi = 341625$ Now we construct the preimage $\gamma_w^{-1}(R)$







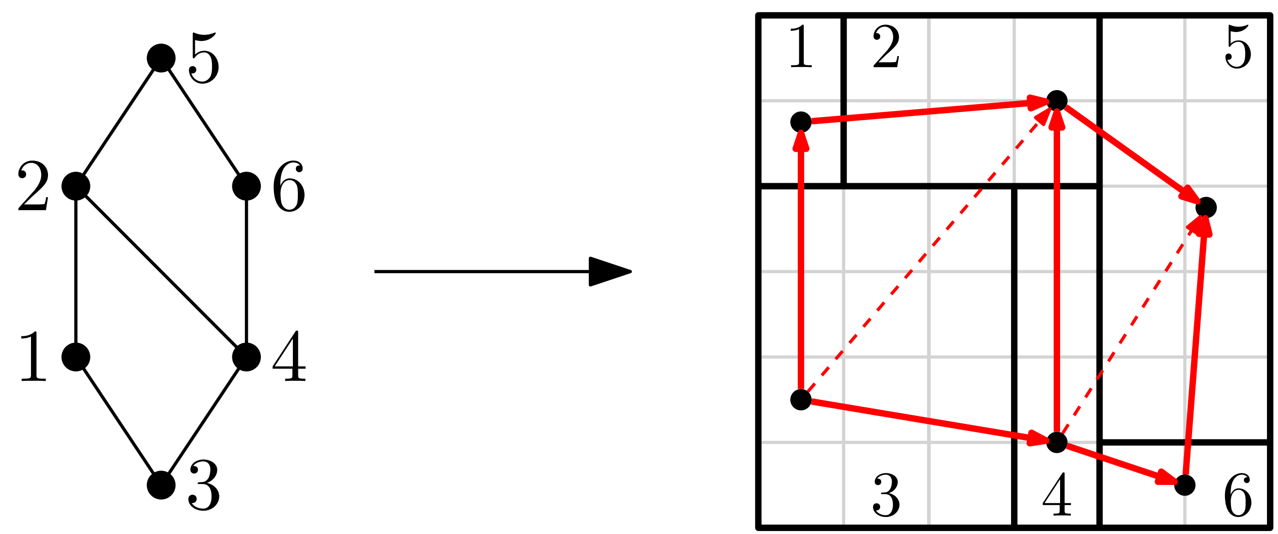




Summary. (Reading$^+$, Meehan, Ackerman$^+$, Cardinal$^+$, ...)

- The mapping γ_w is surjective.
- The preimage of a rectangulation R is the set of linear extensions of a poset $P_w(R)$.
- $P_w(R)$ is the transitive closure of the contact relations in the diagonal representative of R.

- The set $L(P_w(R))$ contains precisely one Baxter / twisted Baxter / co-twisted Baxter permutation.
- The twisted Baxter / the co-twisted Baxter permutation is the minimal / the maximal element of $L(P_w(R))$, with respect to the weak Bruhat order.



The mapping $\gamma_s\colon\ S_n \to$ **strong** rectangulations of size n

- Reading (2011):
  - A "2-step" description of γ_s:
    First, apply γ_w, then shuffle the walls.
  - Restriction of γ_s to a bijection β between two-clumped permutations and strong rectangulations.
- Our contribution:
  - A "1-step" description of γ_s.
  - An explicit desription of the preimage of a rectangulation under γ_s (a poset) and β (a 2-clumped permutation).

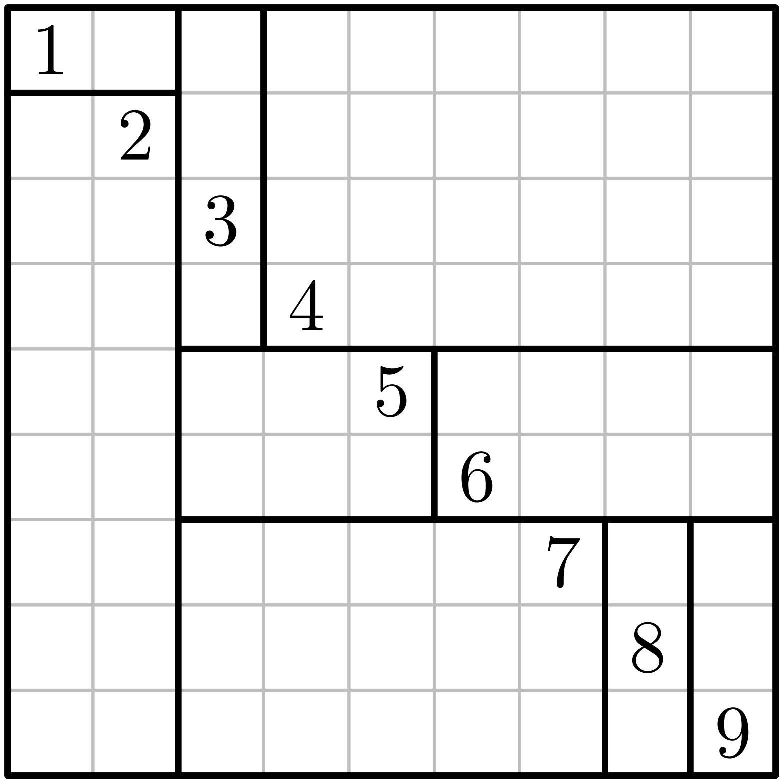
Reading's mapping

$$\pi = 278153964$$

1								
	2							
		3						
			4					
				5				
					6			
						7		
							8	
								9

Reading's mapping

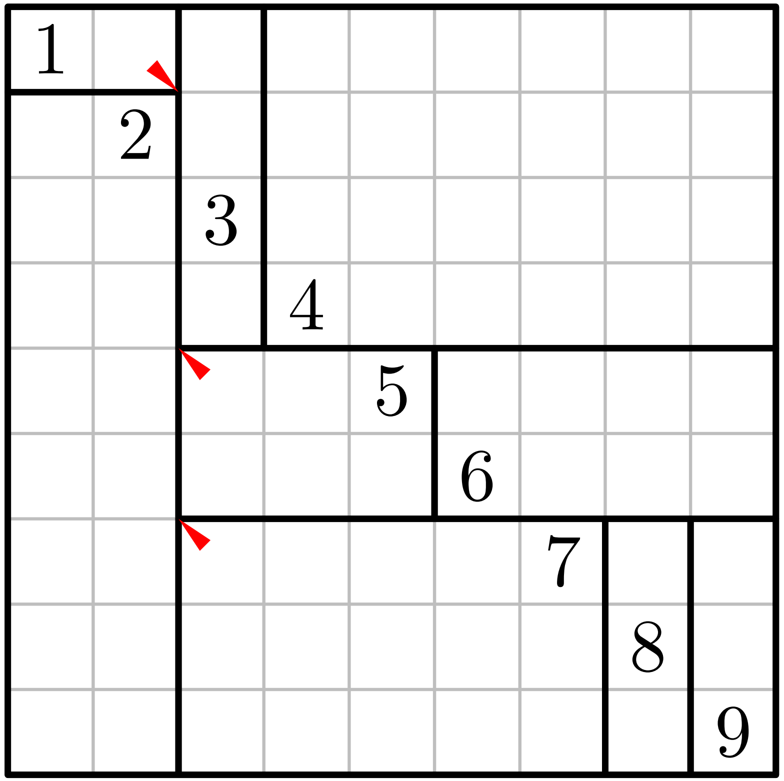
$$\pi = 278153964$$



Step 1:
diagonal rectangulation
Result: $\gamma_w(\pi)$.

Reading's mapping

$\pi = 278153964$



Step 1:
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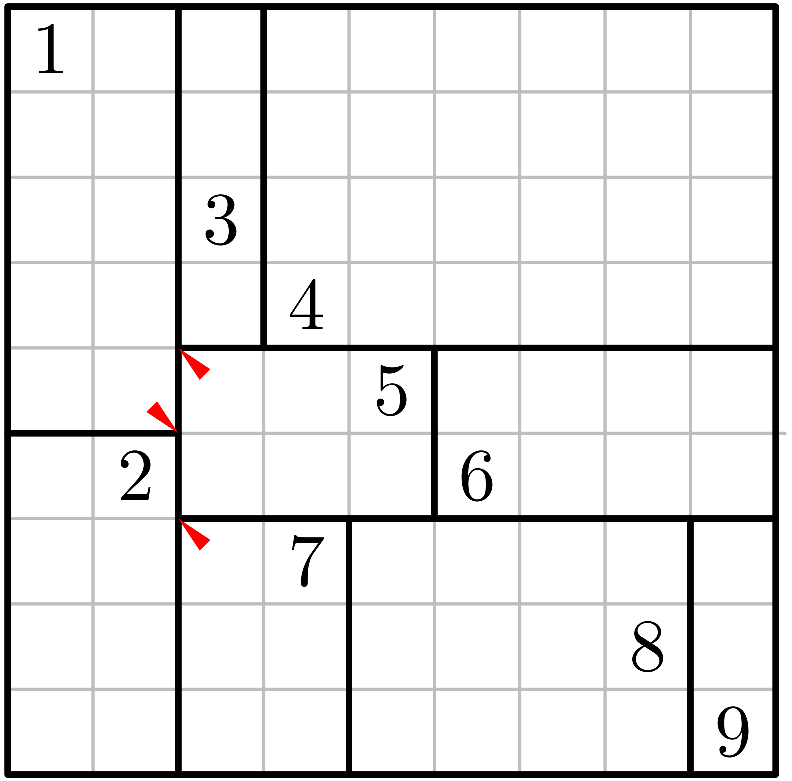
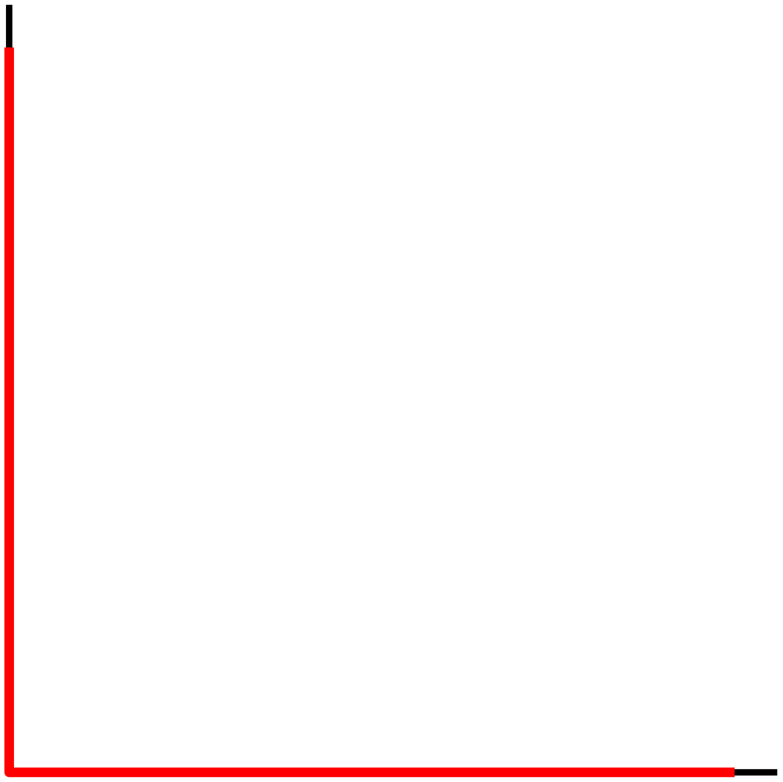
Result: $\gamma_w(\pi)$.



Step 2:
wall shuffles.
Result: $\gamma_s(\pi)$.

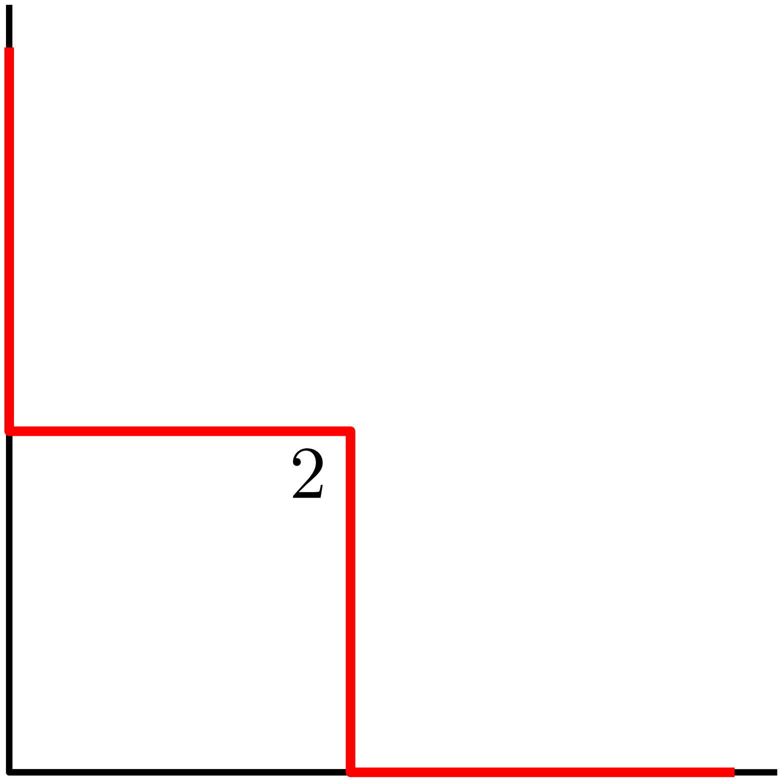
Our mapping

$$\pi = 278153964$$



Our mapping

$$\pi = 278153964$$



Our mapping

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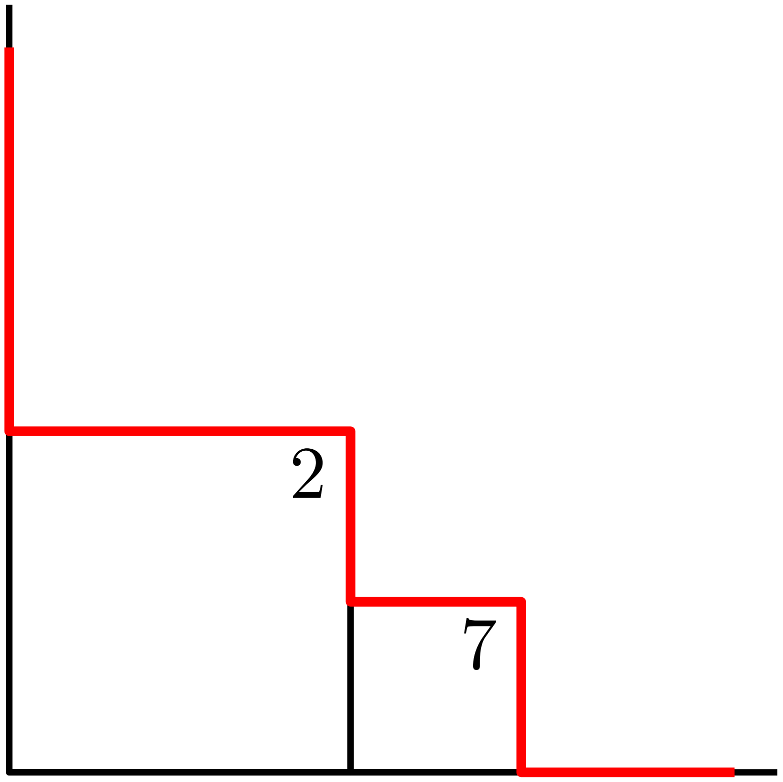

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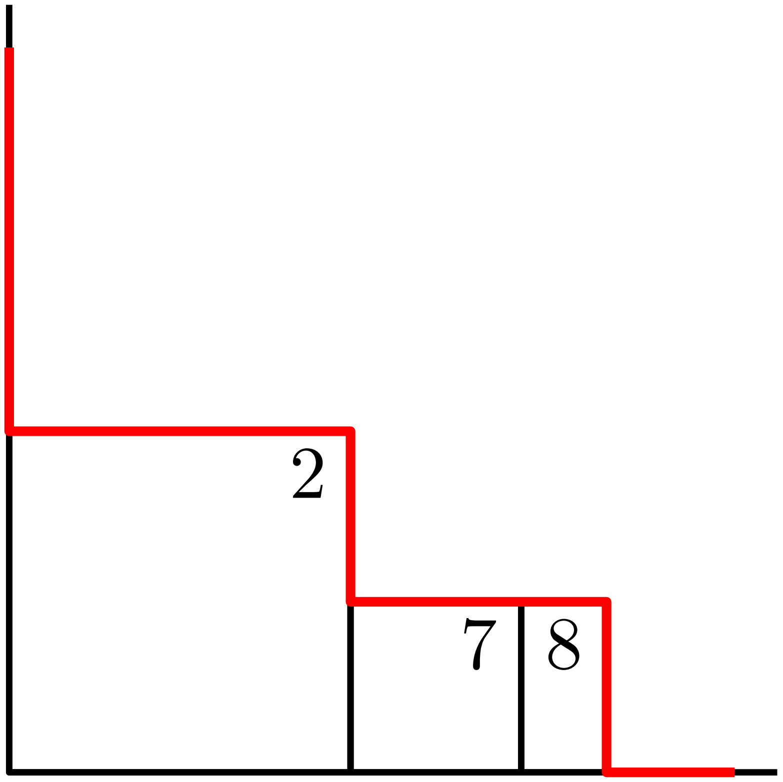

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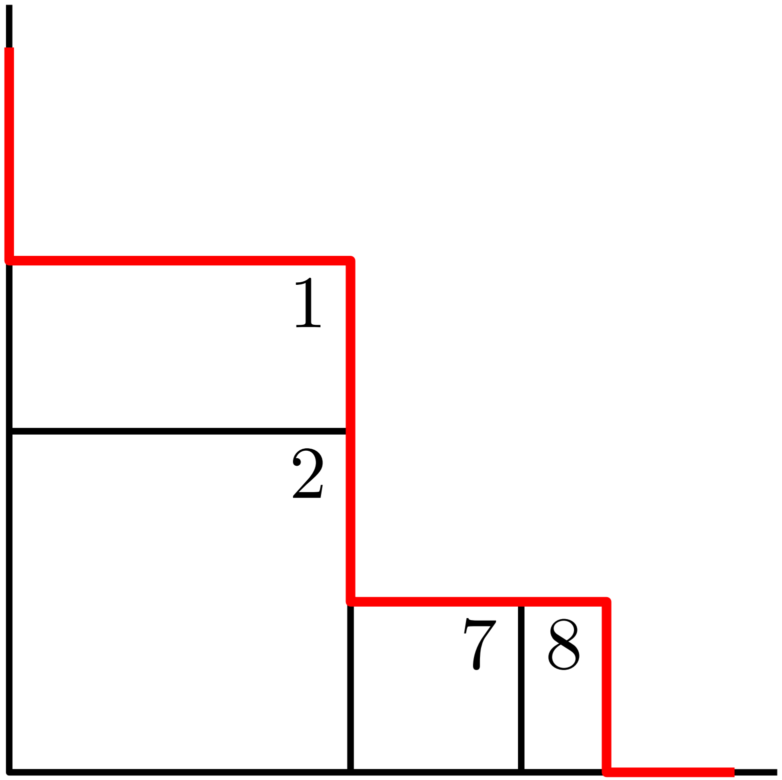

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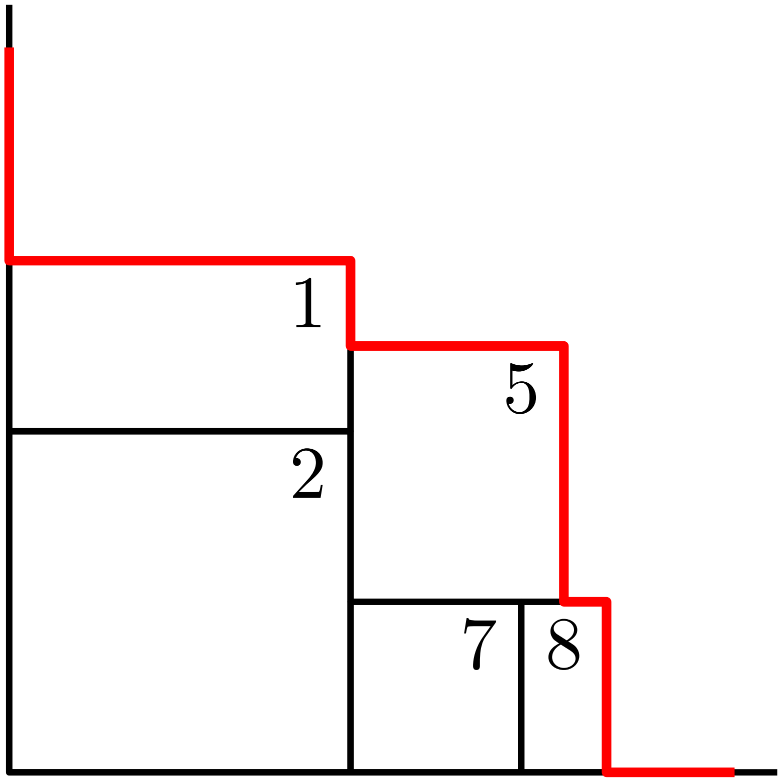

Our mapping

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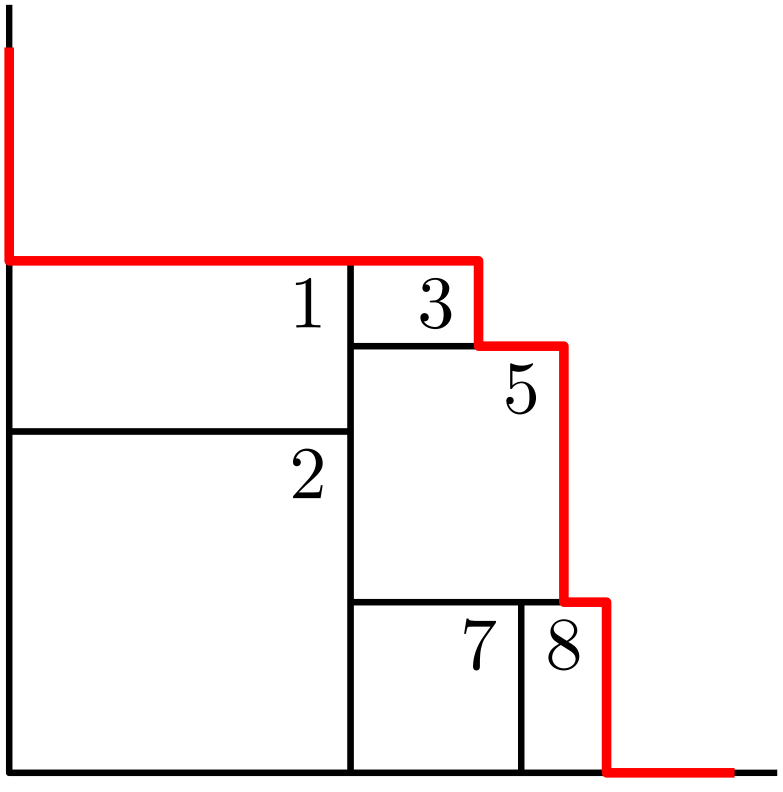

Our mapping

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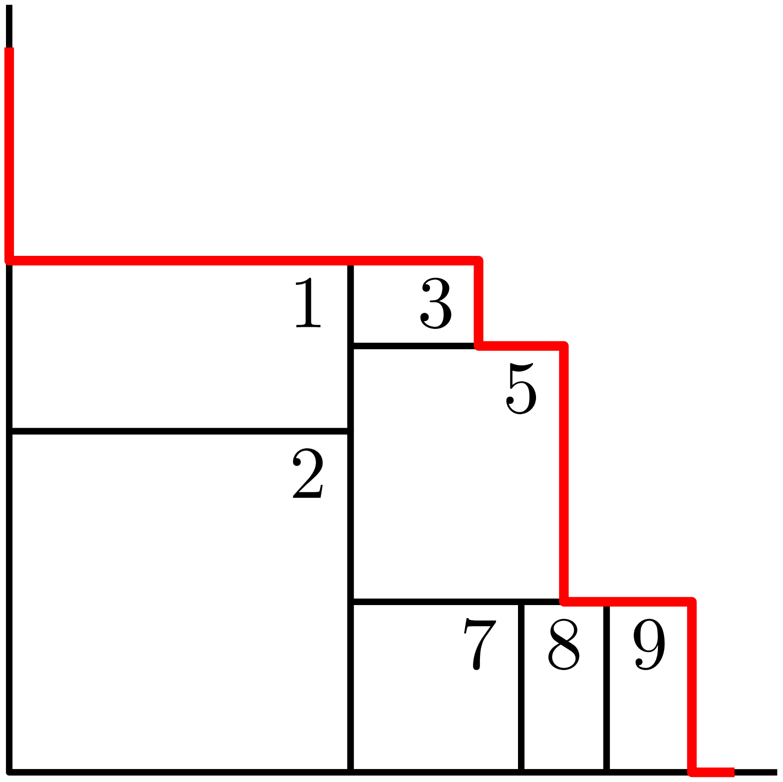

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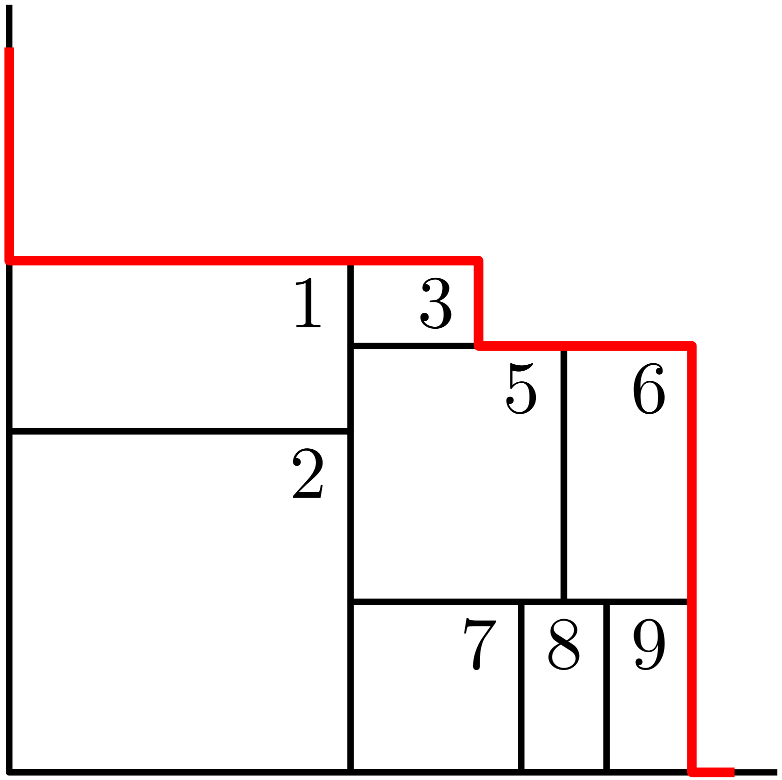

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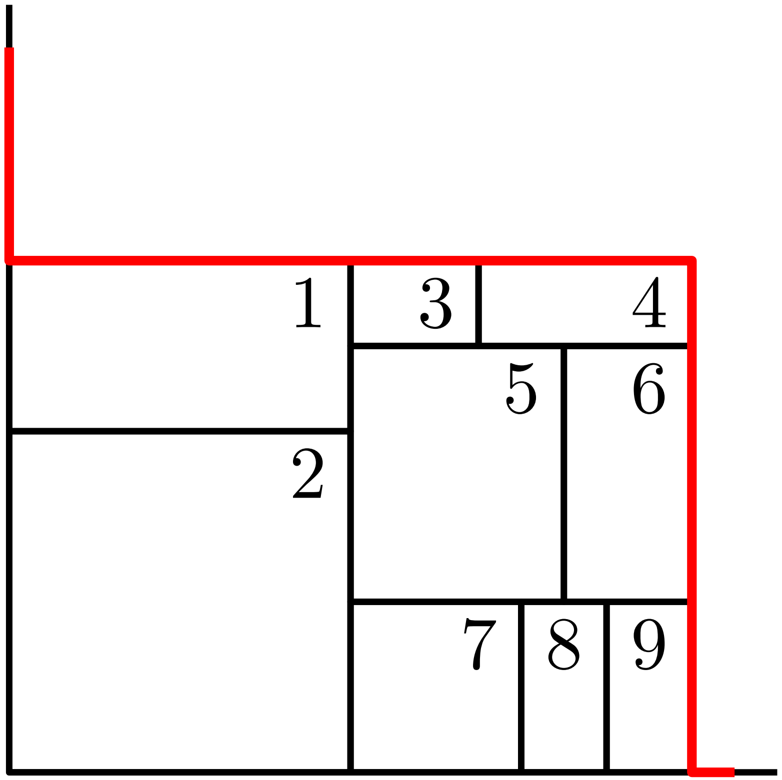

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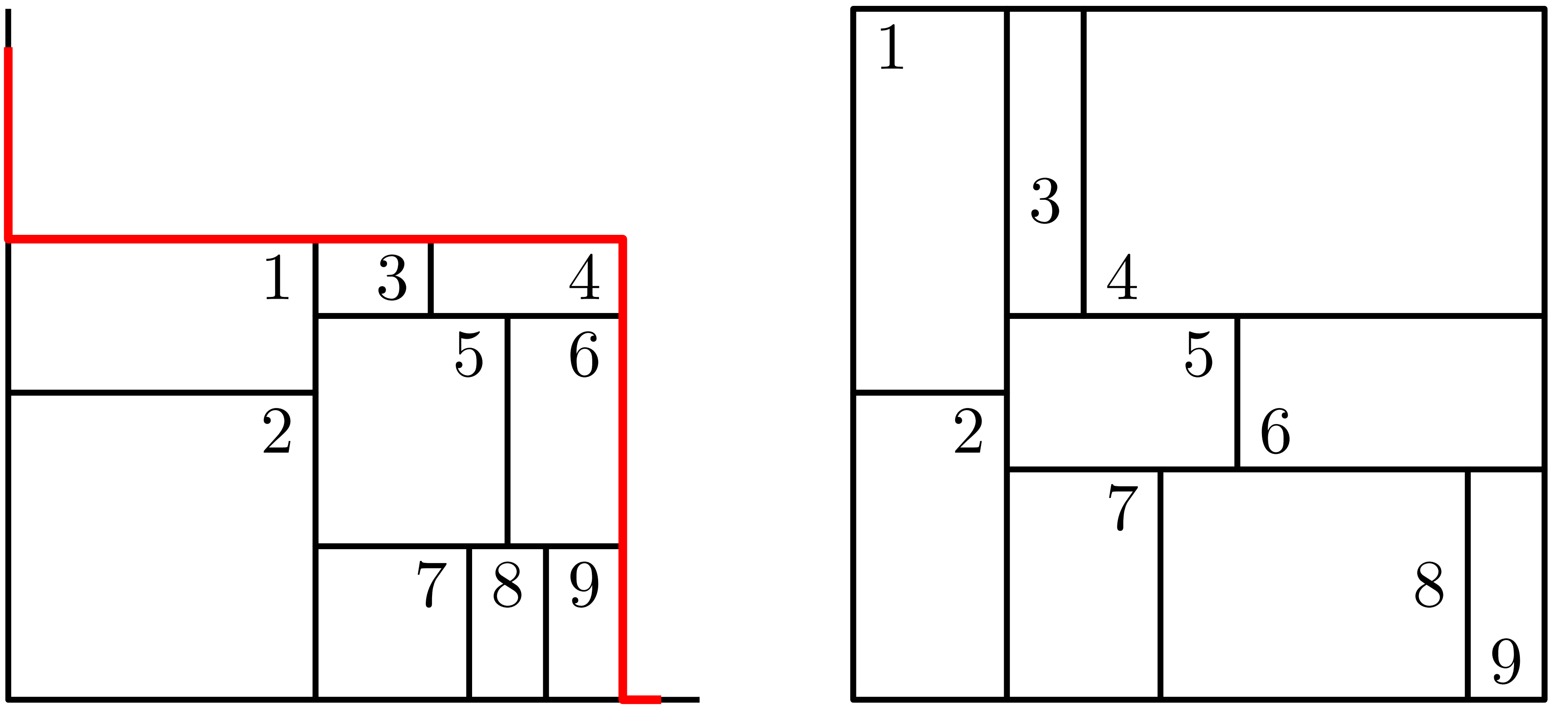
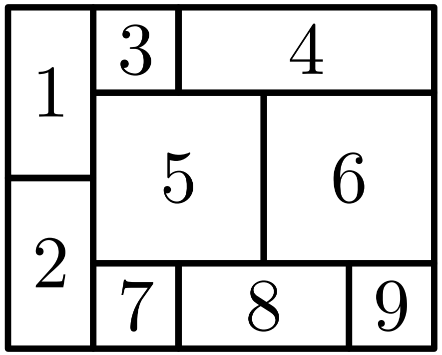


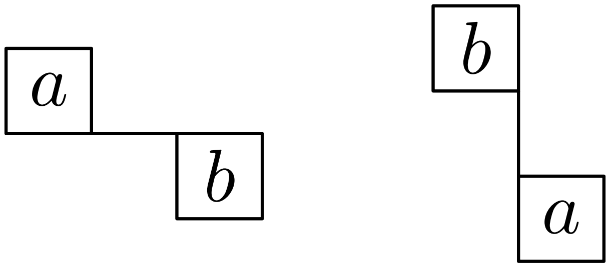
Preimage

To find the preimage, we act similarly to the weak case. We use the blocking relations and two "special relations":

$$a < b$$

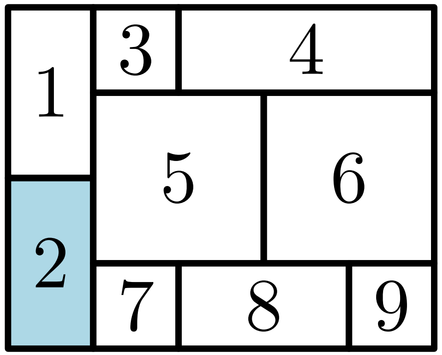
Preimage



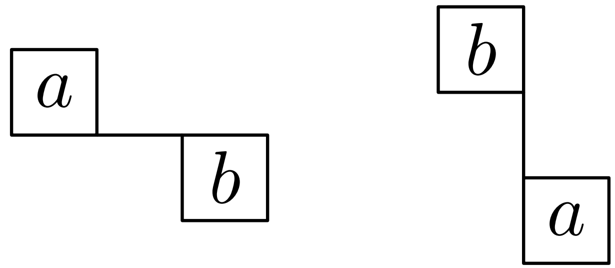


$$a < b$$

Preimage



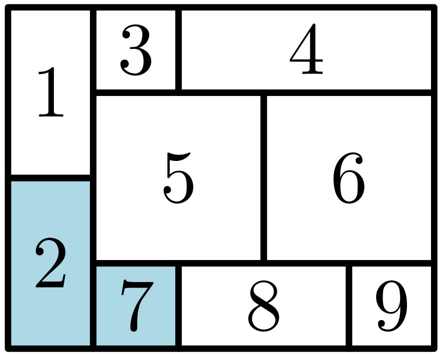
the left rule



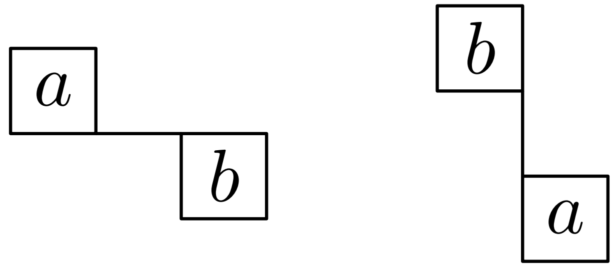
$a < b$

- 2

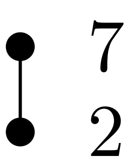
Preimage



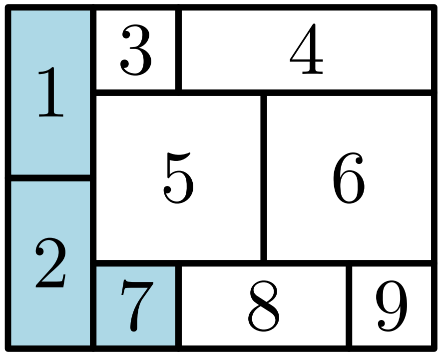
the left rule



$a < b$



Preimage



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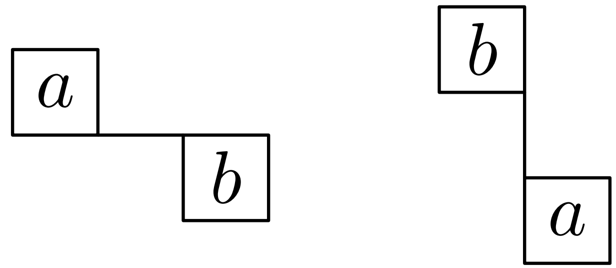
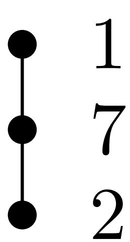


$a < b$



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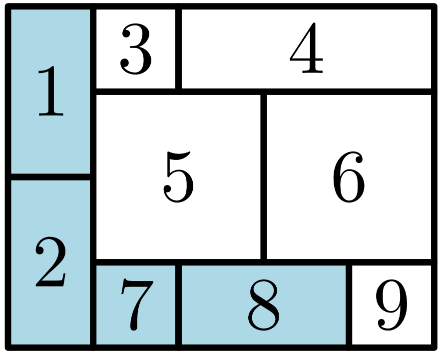
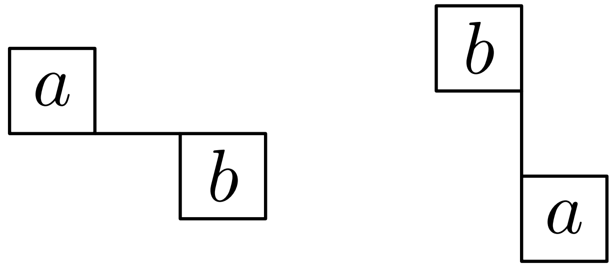
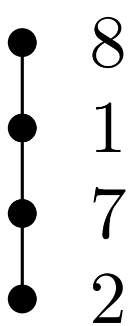


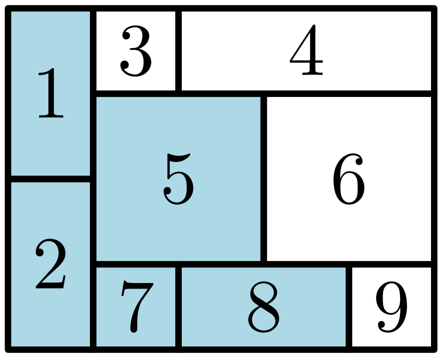
the left rule



$a < b$



Preimage



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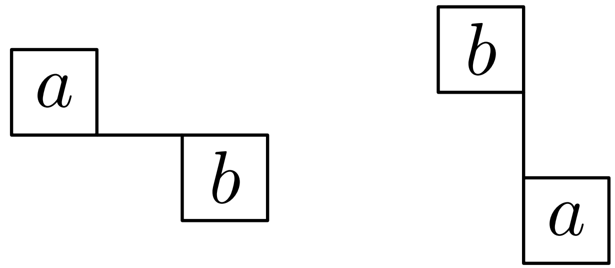
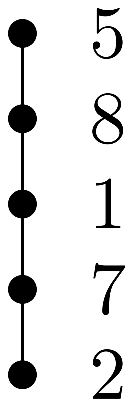
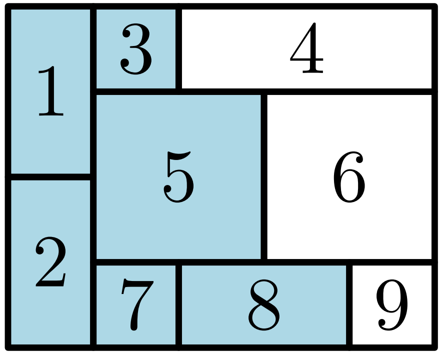


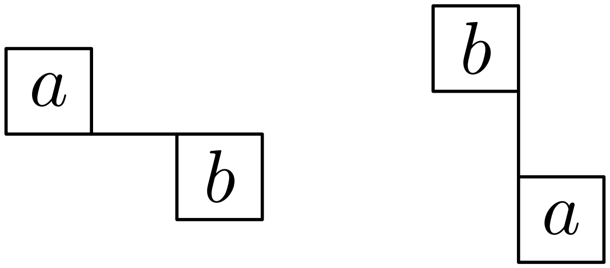
$a < b$



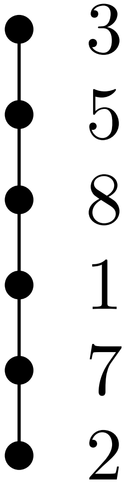
Preimage



the left rule



$a < b$



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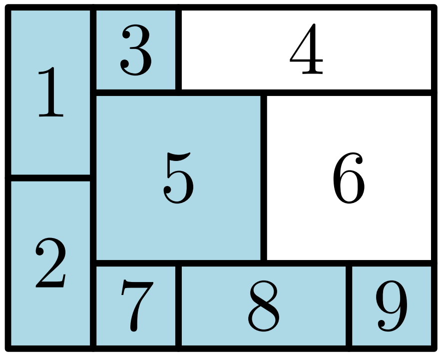
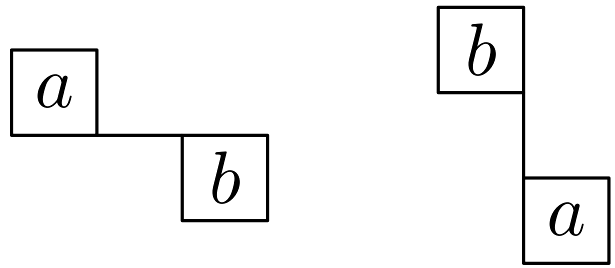
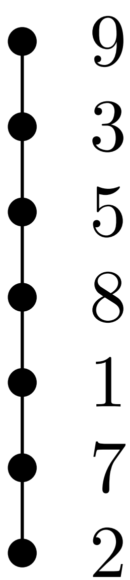


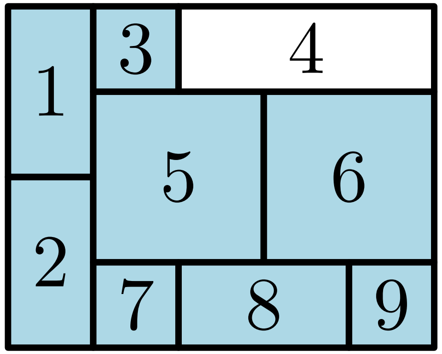
the left rule



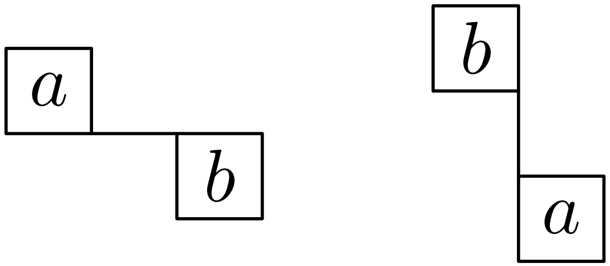
$a < b$



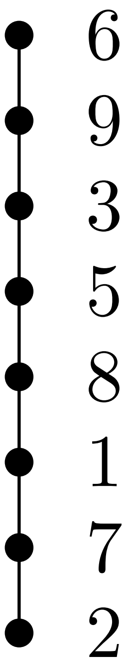
Preimage



the left rule



$a < b$



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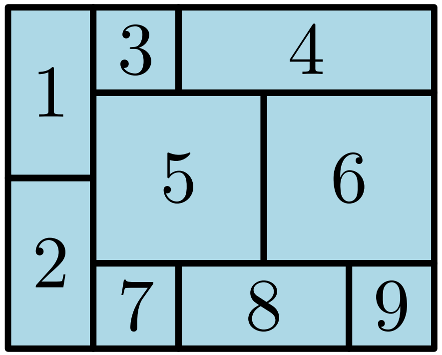
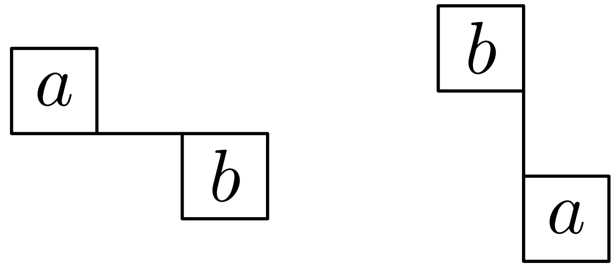
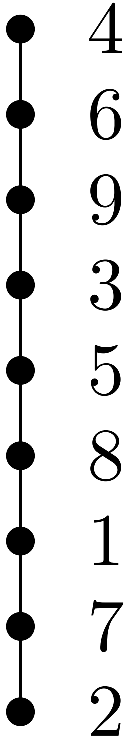




$$a < b$$

the left rule



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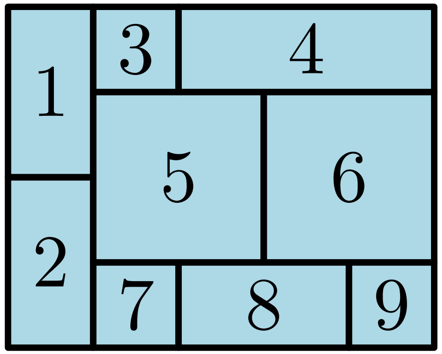
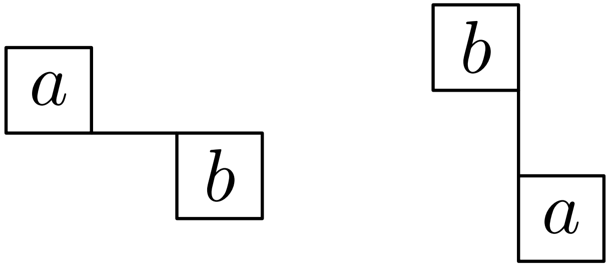
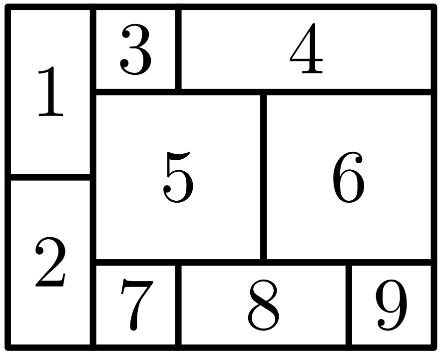


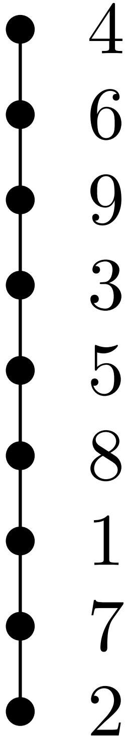
the left rule



$a < b$



the right rule



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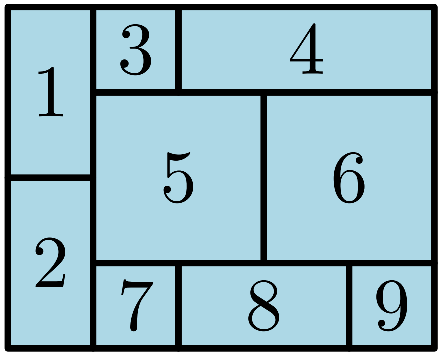

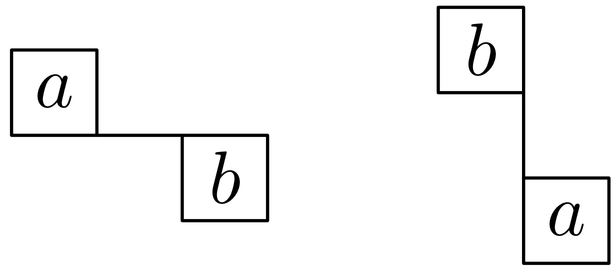

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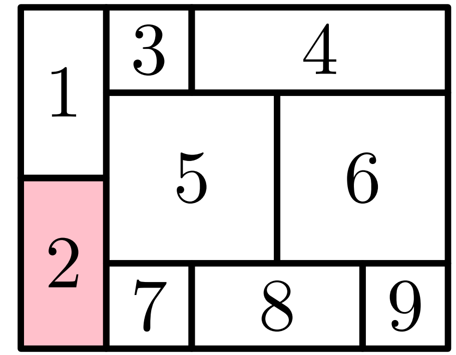
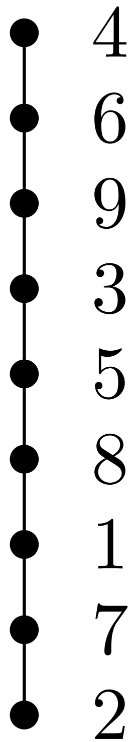
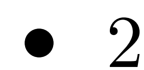


the left rule

the right rule





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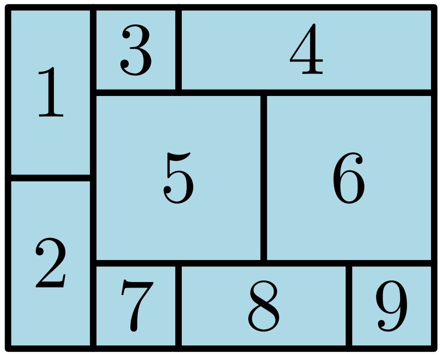

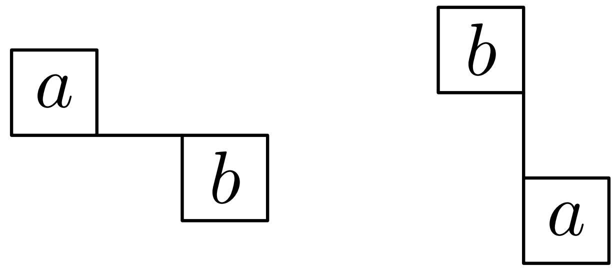

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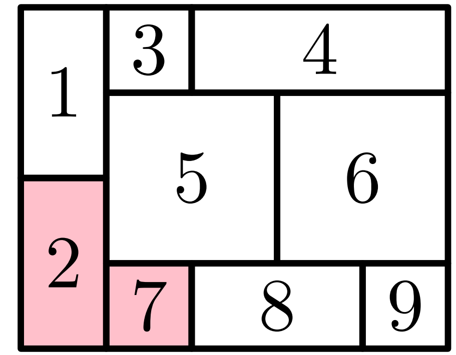
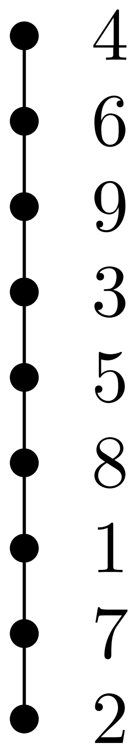
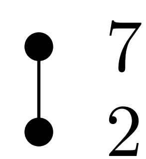


the left rule



the right rule



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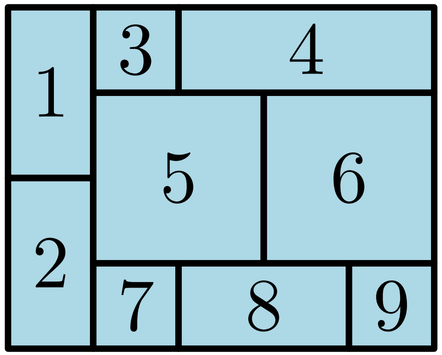

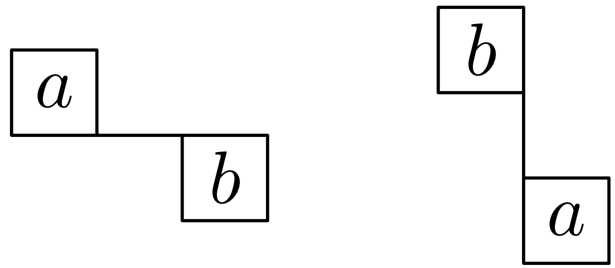

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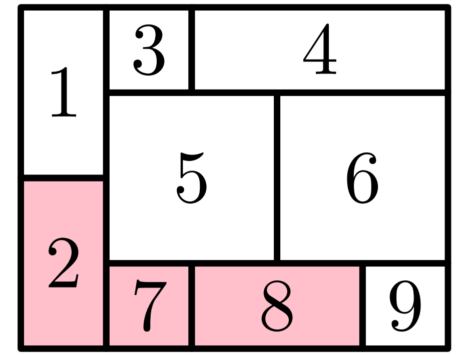
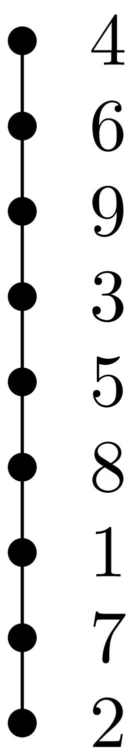
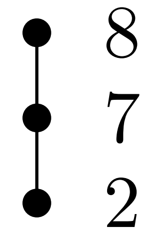


the left rule

the right rule





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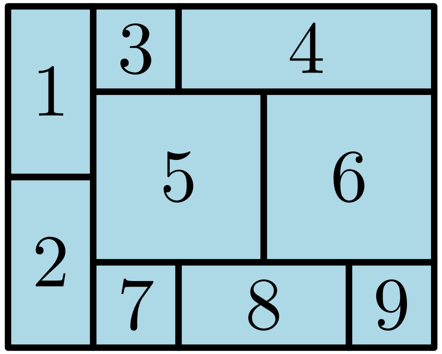

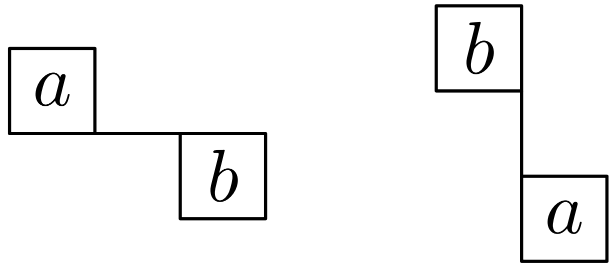

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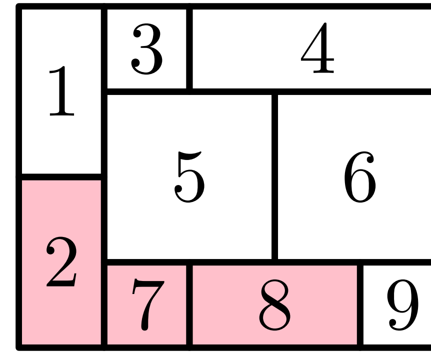
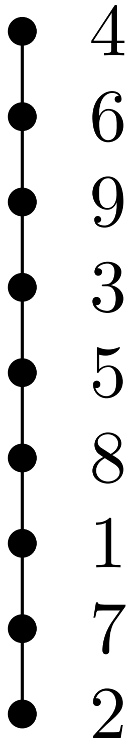
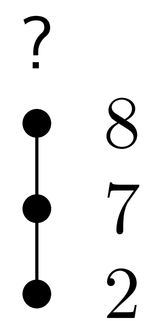


the left rule

the right rule





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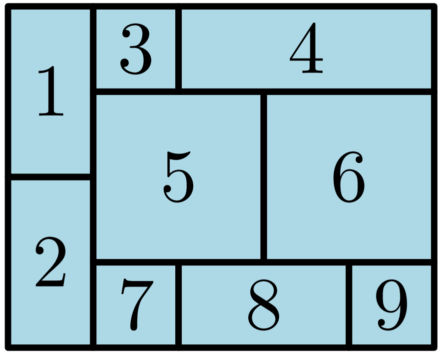

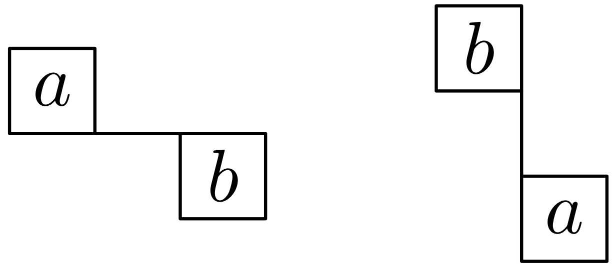

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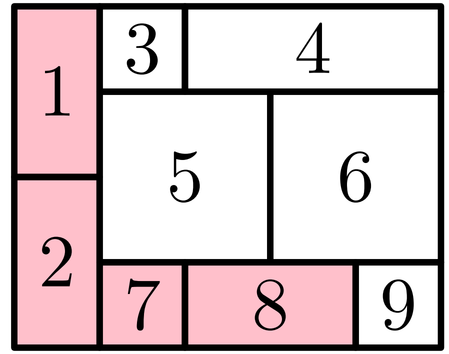
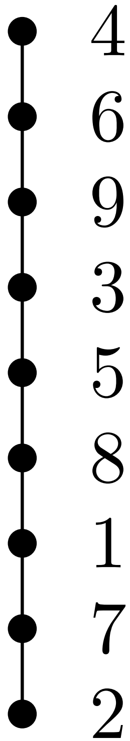
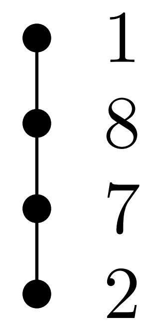


the left rule



the right rule



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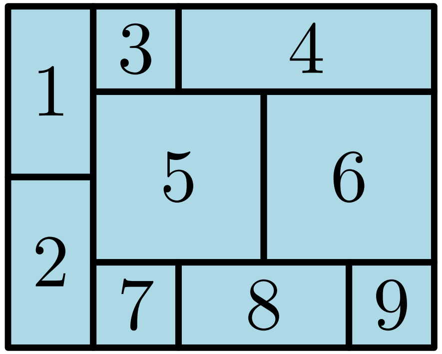

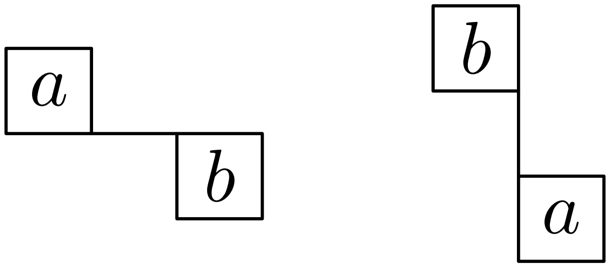

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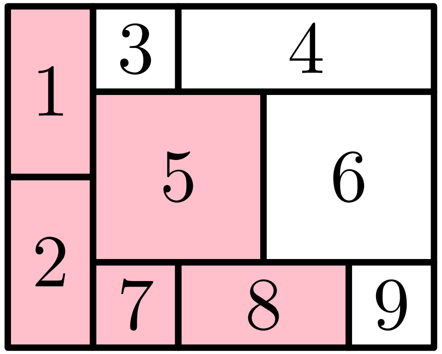
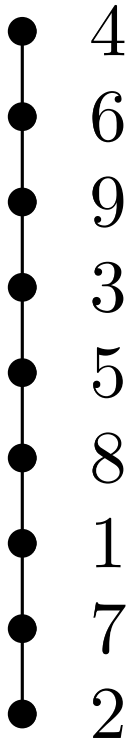
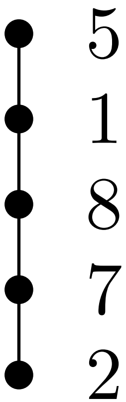


the left rule

the right rule





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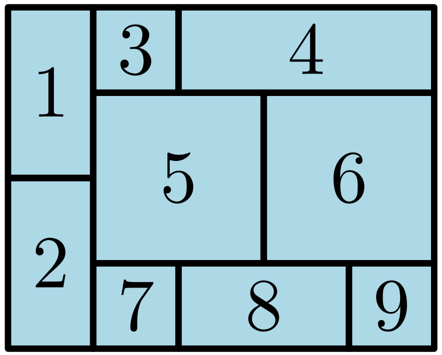

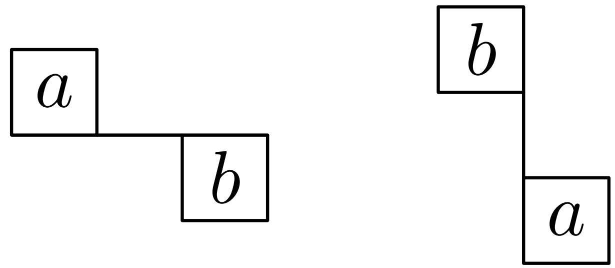

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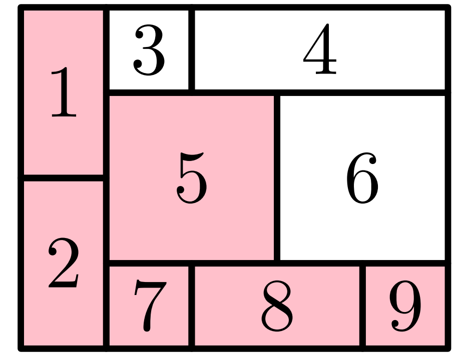
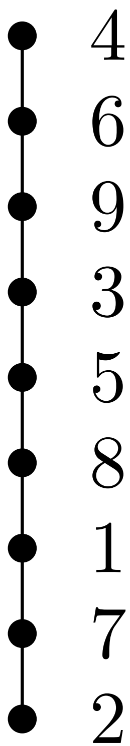
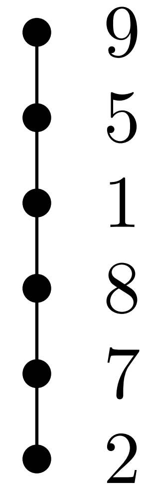


the left rule



the right rule



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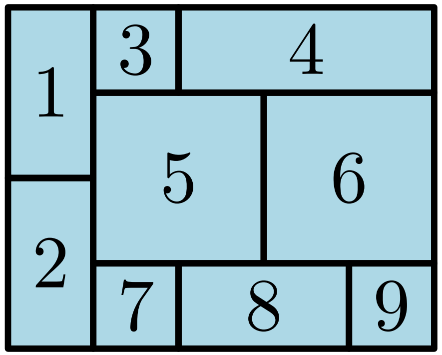

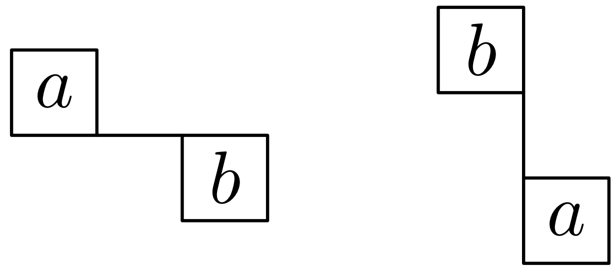

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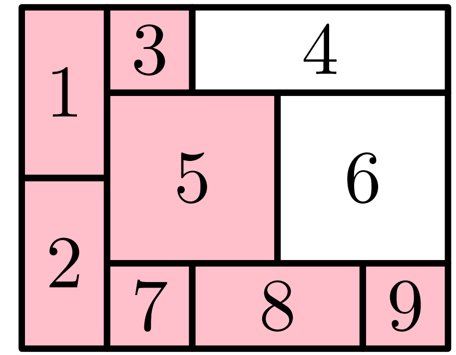
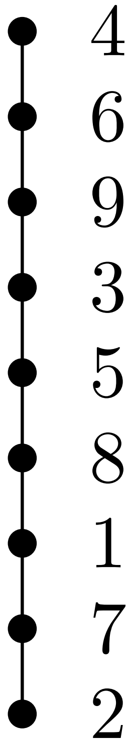
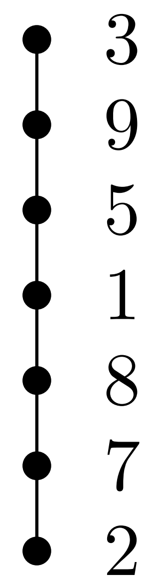


the left rule

the right rule





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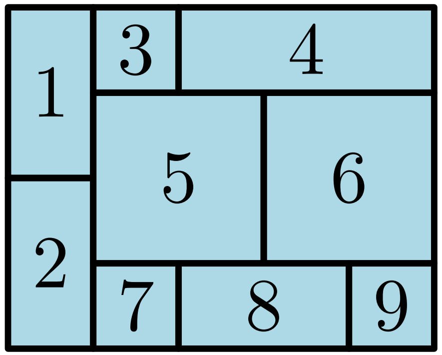

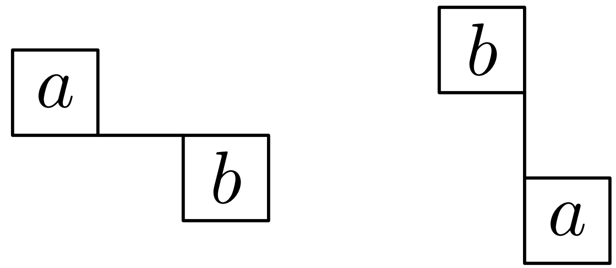

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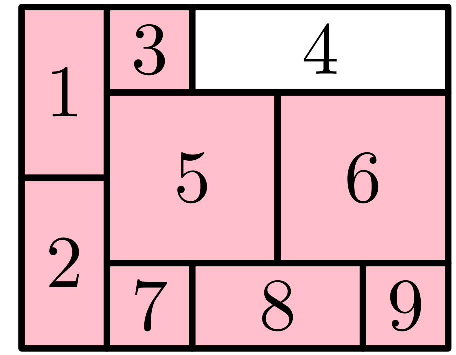
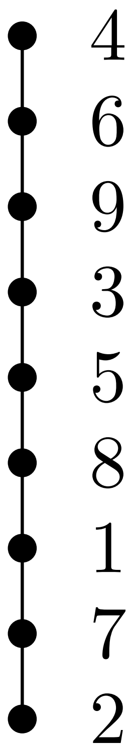
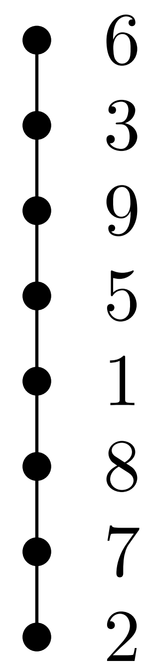


the left rule



the right rule



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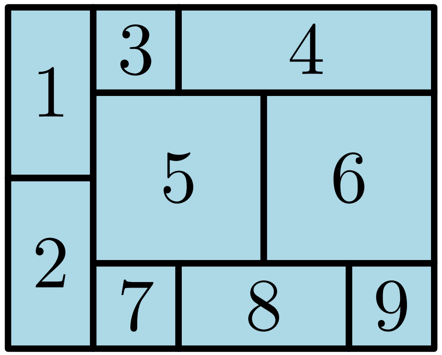

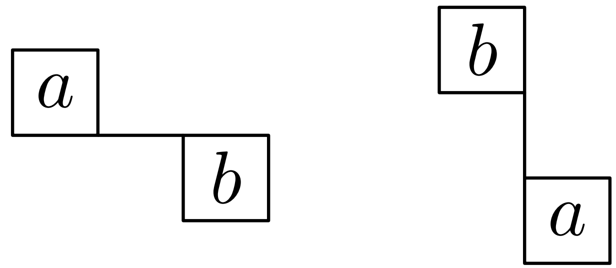

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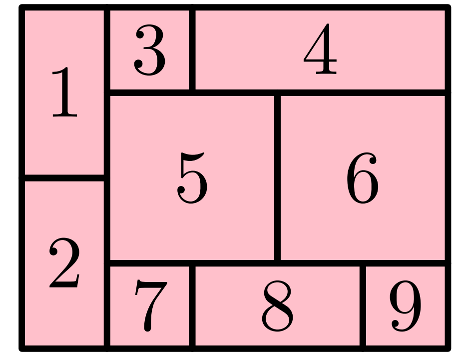
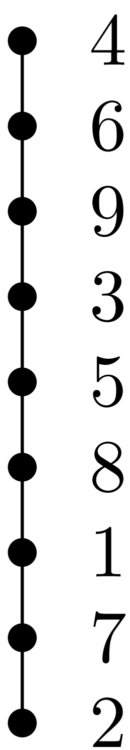
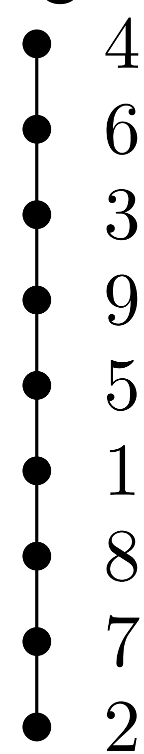


the left rule



the right rule



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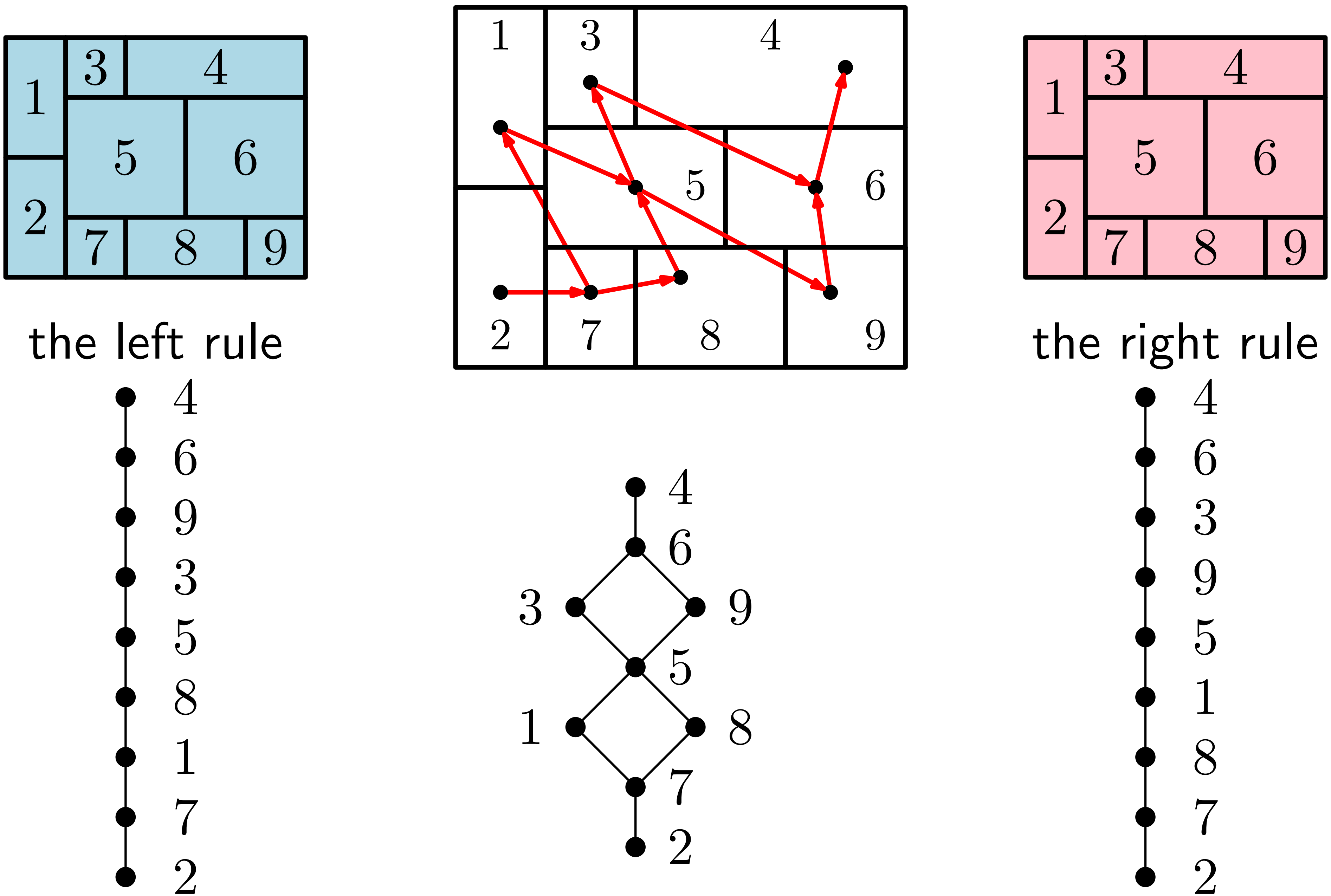

1
3
4
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2
7
8
9
the left rule
4
6
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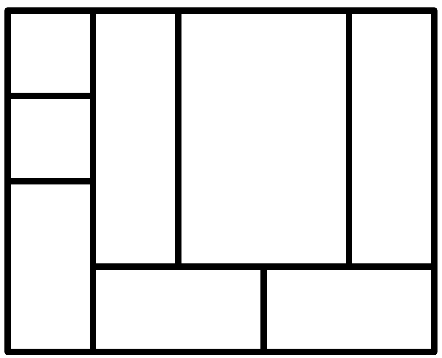
1
7
2
the right rule
4
6
3
9
5
1
8
7
2

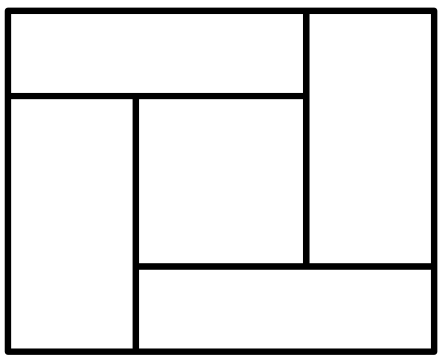
Summary. (Reading, Takahashi$^+$, ...)

- The mapping γ_s is surjective.
- The preimage of a rectangulation R is the set of linear extensions of a poset $P_s(R)$.
- $P_s(R)$ is the transitive closure of the contact $+$ special relations in R.
- The set $L(P_s(R))$ contains precisely one two-clumped / co-two-clumped permutation.
- The two-clumped / the co-two-clumped permutation is the minimal / the maximal element of $L(P_s(R))$.

A **guillotine rectangulation** is either the rectangulation of size 1, or a rectangulation obtained recursively by gluing two guillotine rectangulations along a horizontal or a vertical segment.



guillotine



non-guillotine

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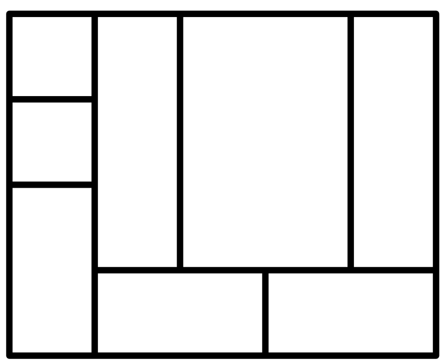
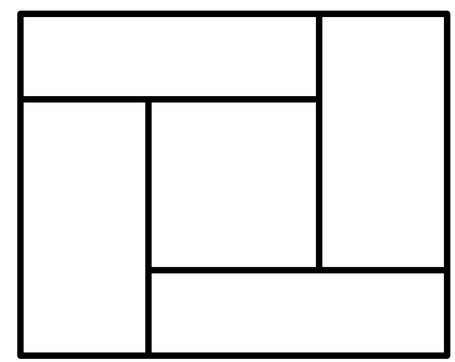
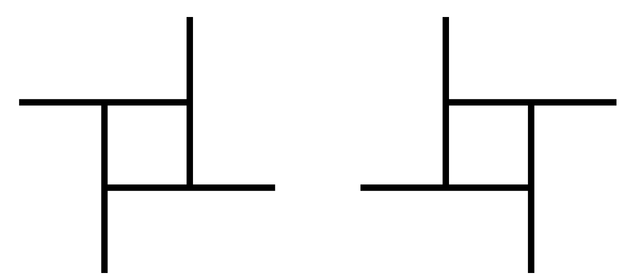


guillotine



non-guillotine

Observation. A rectangulation is guillotine if and only if it does not contain “windmills” – configurations of segments as shown here.



w_1 w_2

windmills

Thm (Ackerman^{+}). The bijection between weak rectangulations and Baxter permutations restricts to a bijection between weak guillotine rectangulations and separable permutations.

$$\text{Baxter} = \mathrm{Av}(2[41]3, 3[14]2)$$
$$\text{Separable} = \mathrm{Av}(2413, 3142)$$

Weak guillotine rectangulations are enumerated by Schröder numbers.

Thm (Ackerman$^+$). The bijection between weak rectangulations and Baxter permutations restricts to a bijection between weak guillotine rectangulations and separable permutations.

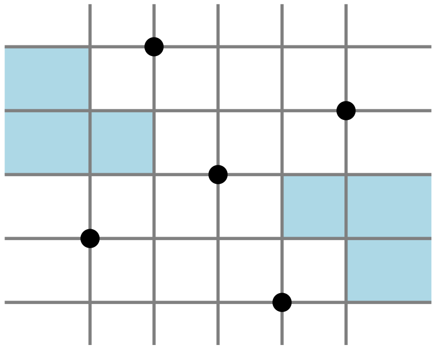
$$\text{Baxter} = \mathrm{Av}(2[41]3, 3[14]2)$$

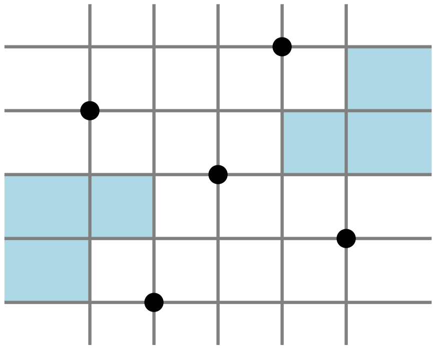
$$\text{Separable} = \mathrm{Av}(2413, 3142)$$

Weak guillotine rectangulations are enumerated by Schröder numbers.

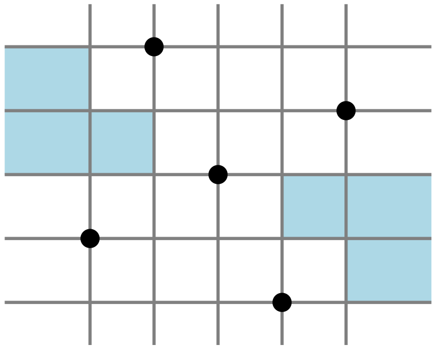
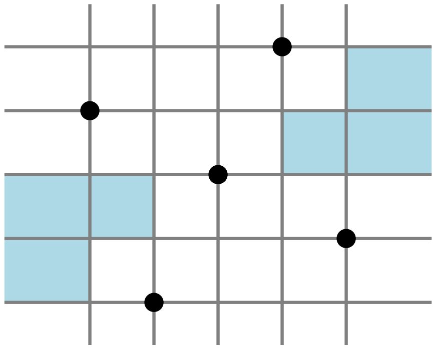
Weak: Baxter permutations Enumeration: 1970s GF D-finite	**Weak guillotine:** Separable permutations Enumeration is elementary GF algebraic
Strong: Two-clumped permutations First results: 2011 GF not D-finite (Fusy$^+$ 2021)	**Strong guillotine:** ???????????????

Conjecture (Merino and Mütze, 2021). Weak guillotine rectangulations are in bijection with two-clumpled permutations that avoid the following mesh patterns:

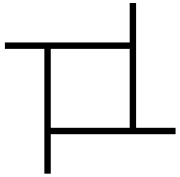
$p_1 =$ 

$p_2 =$ 

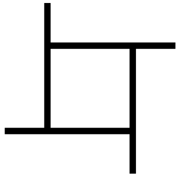
Conjecture (Merino and Mütze, 2021). Weak guillotine rectangulations are in bijection with two-clumpled permutations that avoid the following mesh patterns:

$p_1 =$  $p_2 =$ 

Theorem (ACFF, 2023). A permutation π avoids the mesh pattern p_1 resp. p_2 if and only if the rectangulations $\gamma_w(\pi)$ and $\gamma_s(\pi)$ avoid the windmill w_1 resp. w_2.



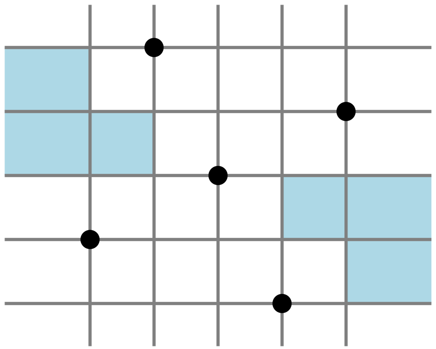
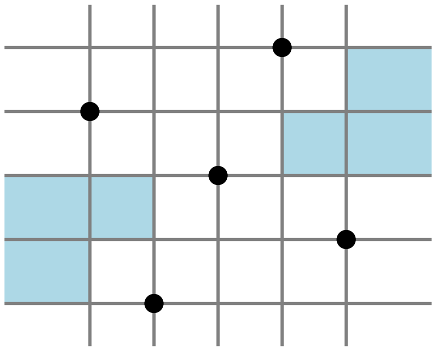
w_1



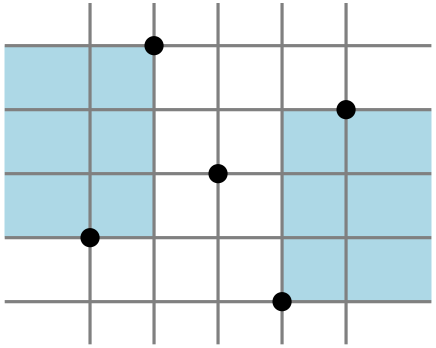
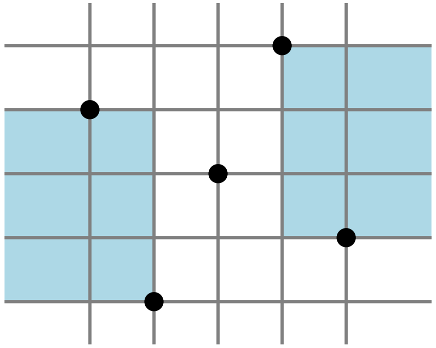
w_2

Proof, step 1:

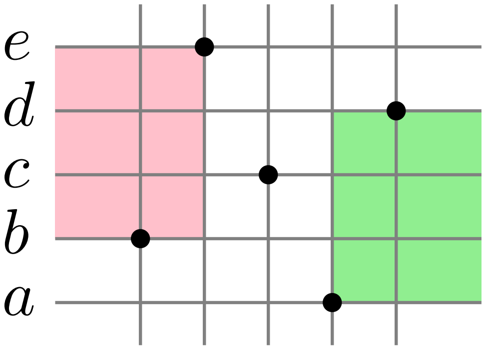
The mesh patterns

$p_1 =$  $p_2 =$ 

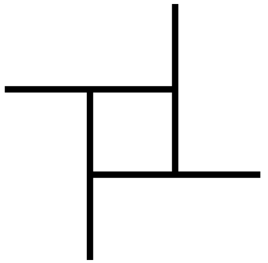
are (respectively) equivalent to the mesh paterns

$q_1 =$  $q_2 =$ 

Proof, step 2:
π contains $q_1 \implies$
$\gamma_w(\pi)$ contains w_1.



$\implies$



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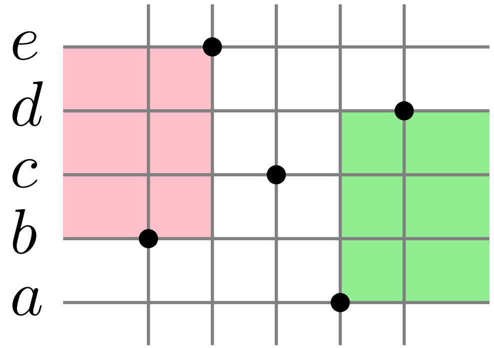
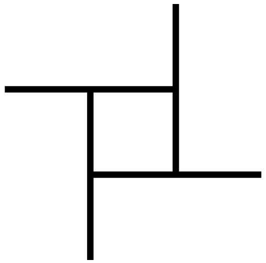
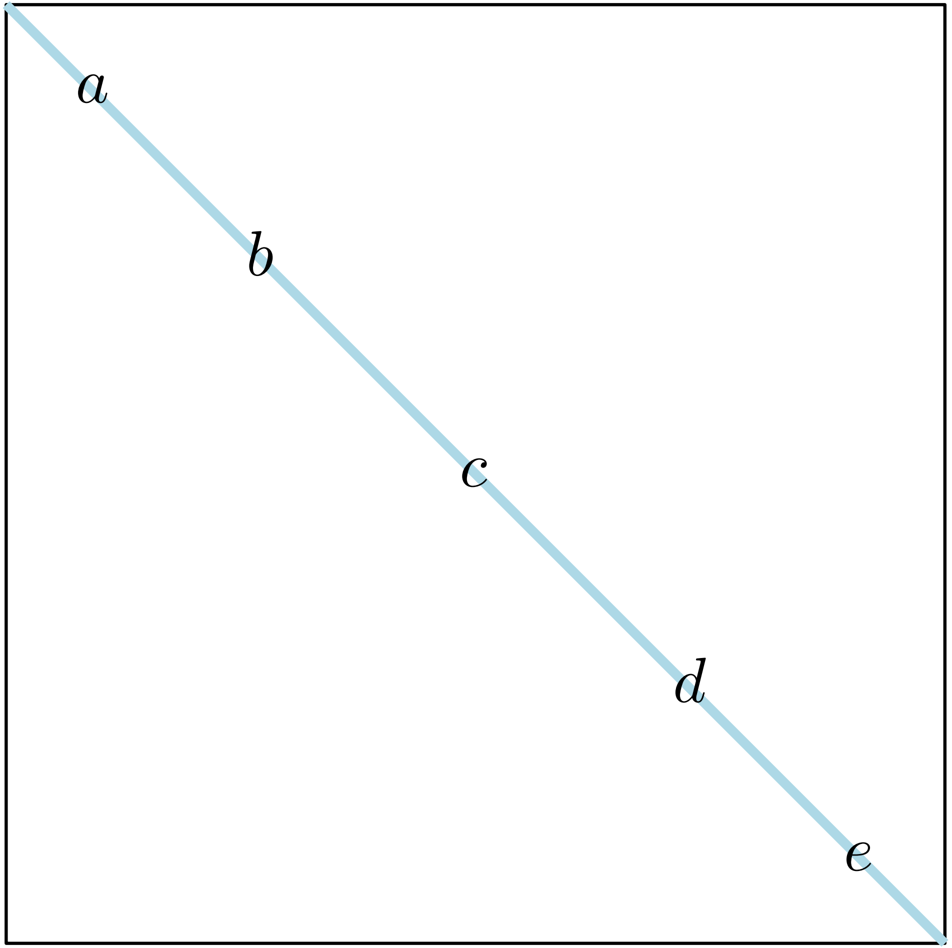


$\implies$

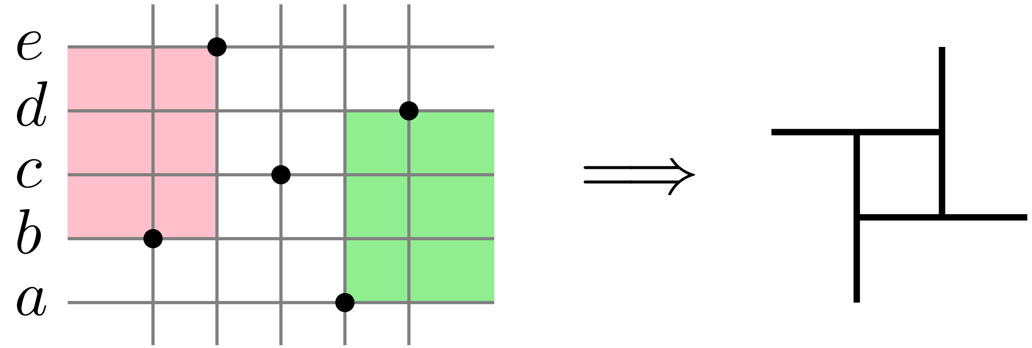


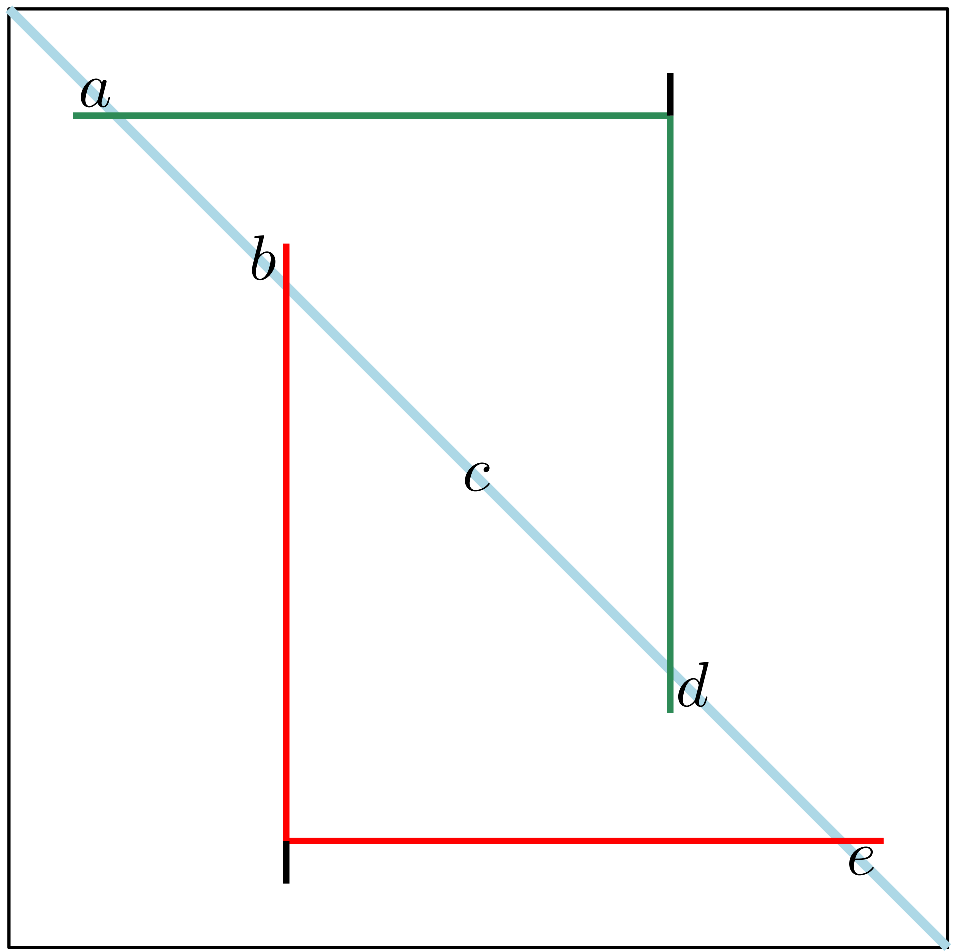


$\gamma_s(\pi)$

Proof, step 2:

π contains q_1 $\Longrightarrow$ $\gamma_w(\pi)$ contains w_1.





$\gamma_s(\pi)$

No x with $b < x < e$ is inserted earlier than e

The staircase just after inserting e

No x with $a < x < d$ is inserted later than a.

The staircase just before inserting a

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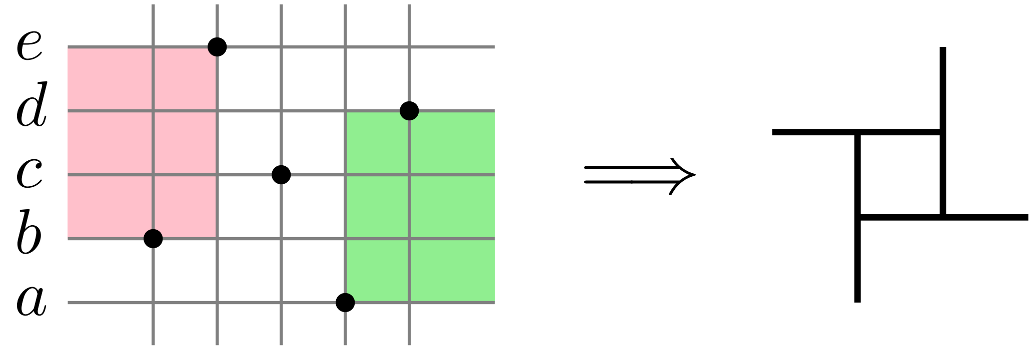

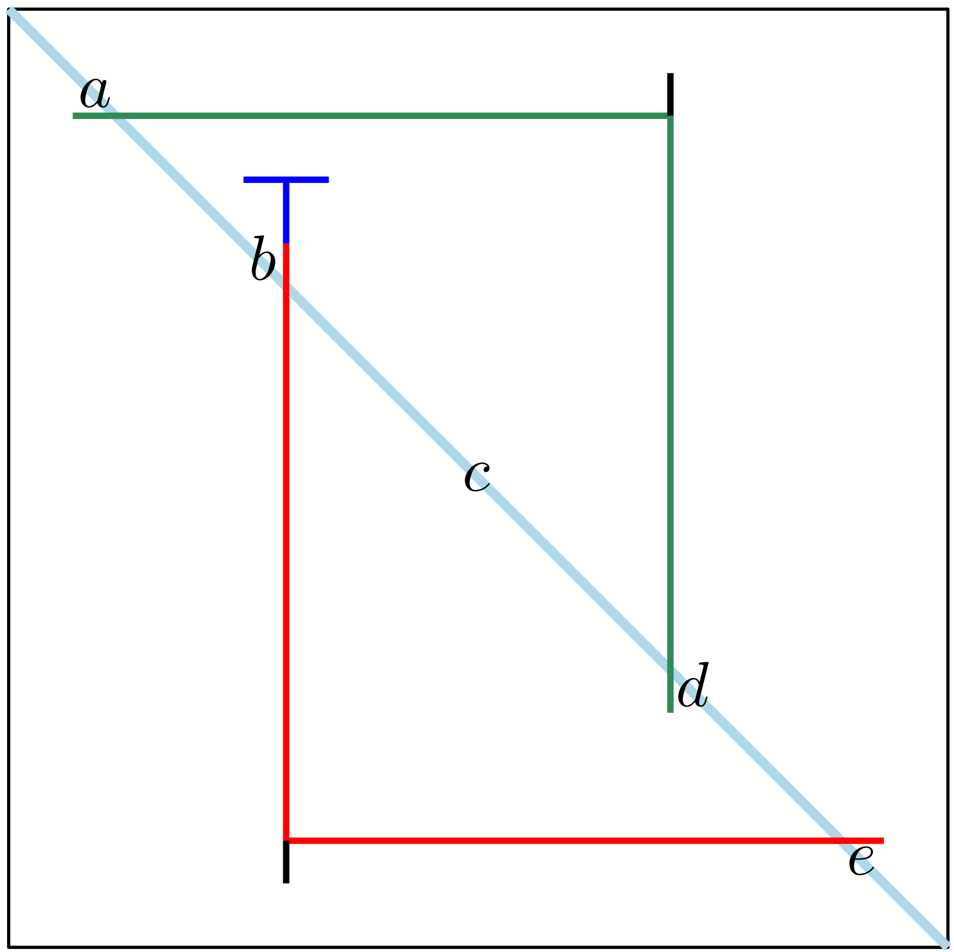

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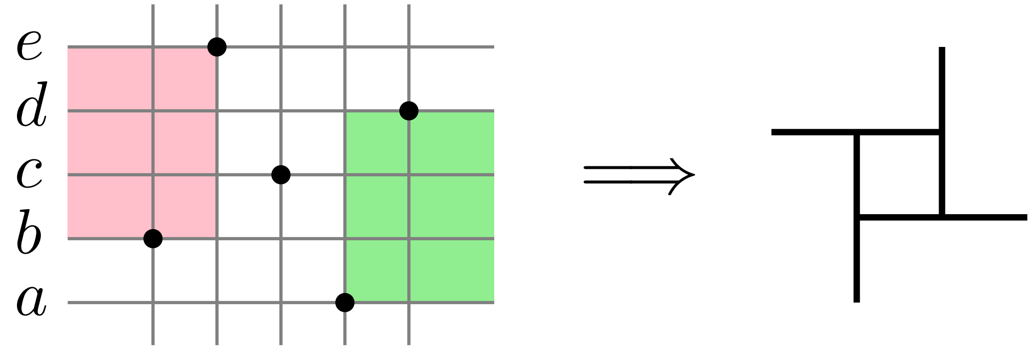

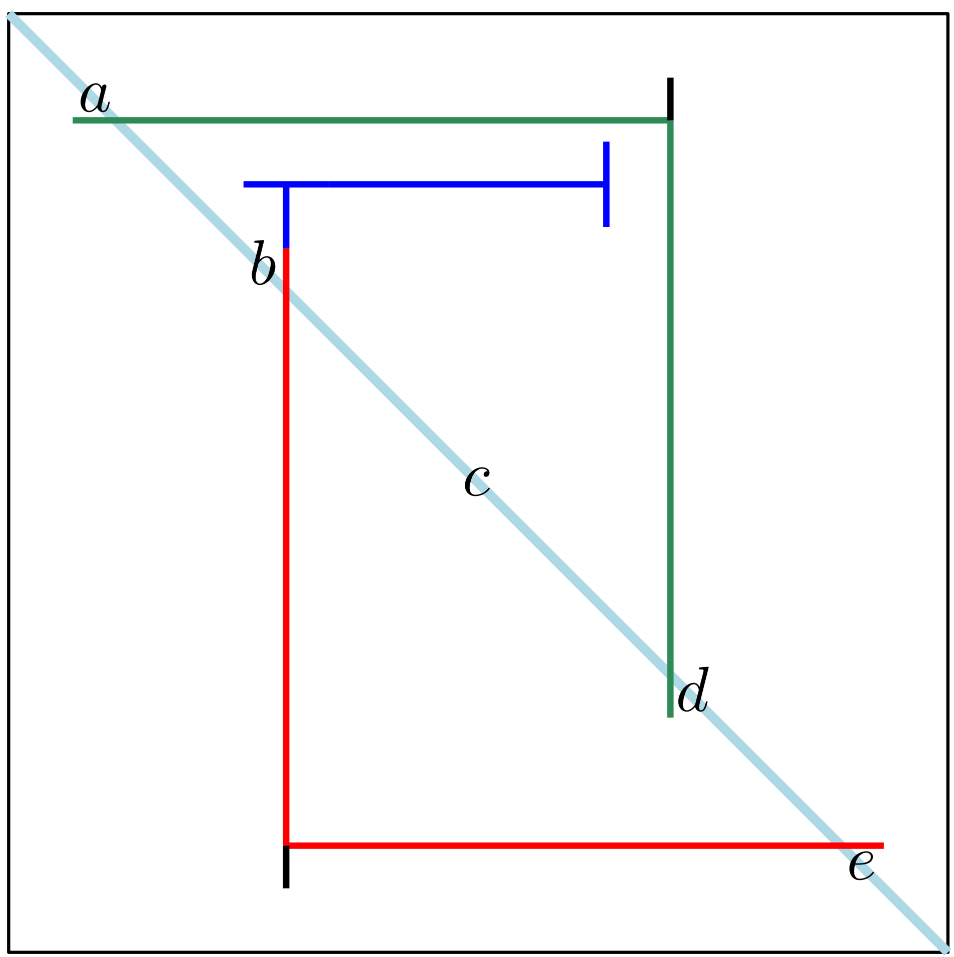

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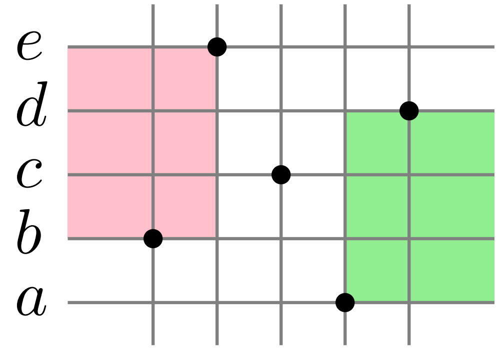
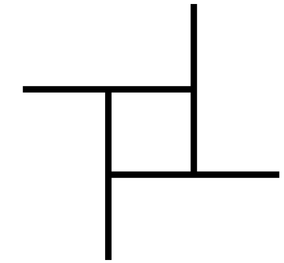
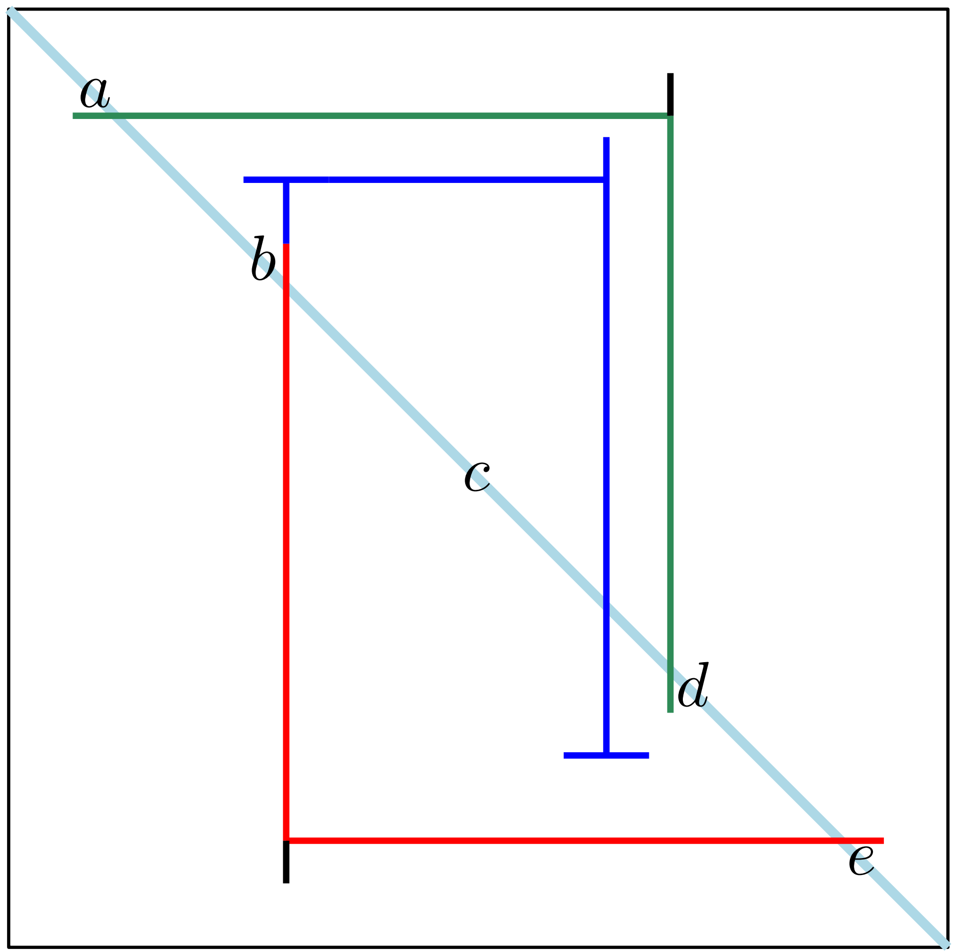


$\implies$





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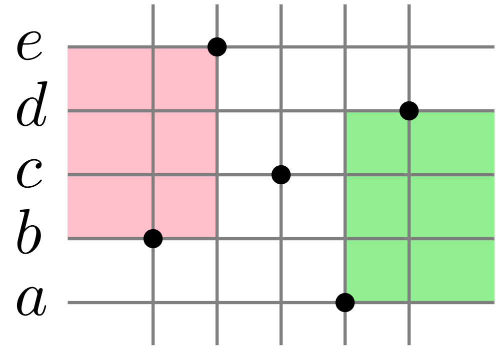
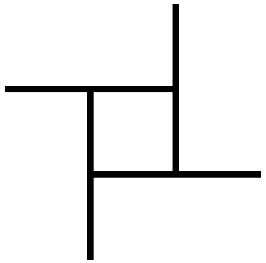
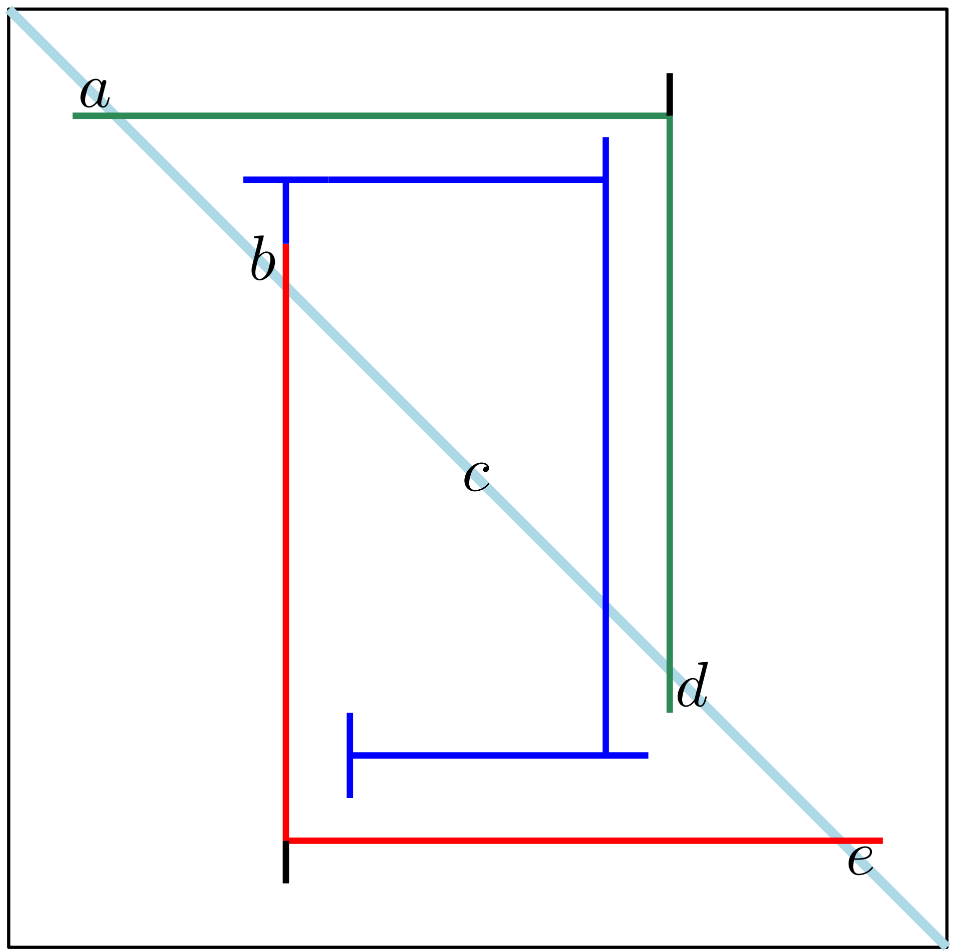


$\implies$





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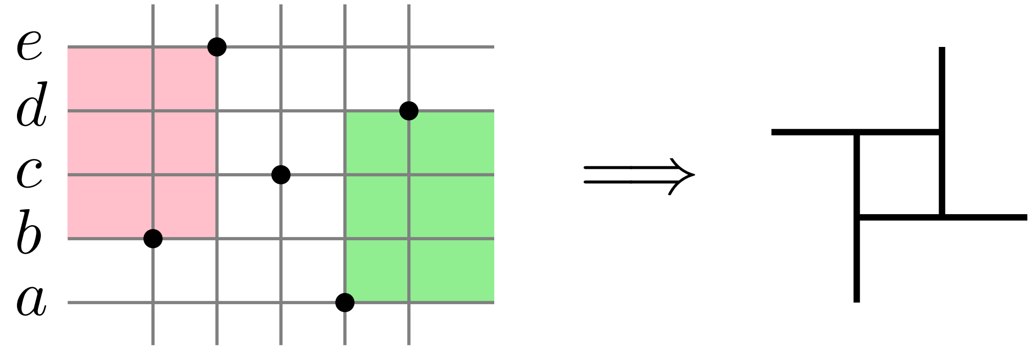
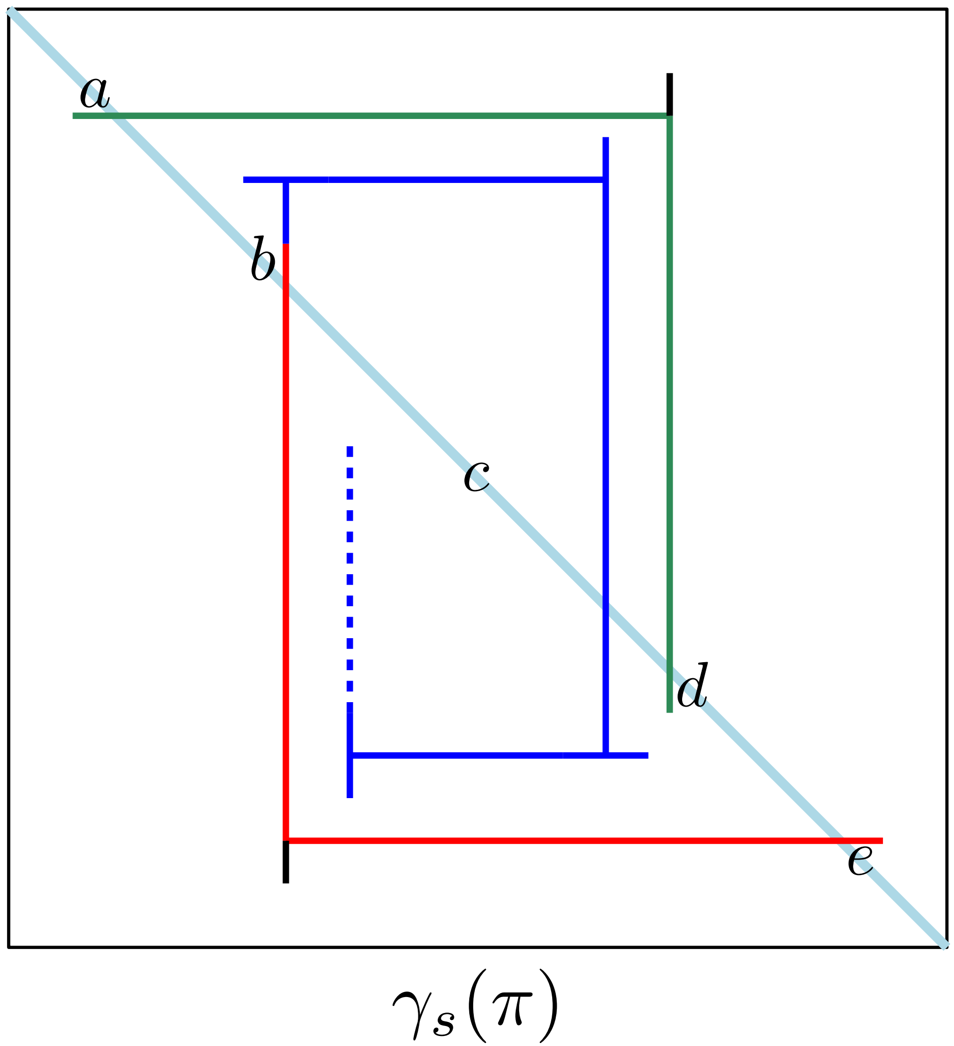
π contains $q_1 \implies \gamma_w(\pi)$ contains w_1.





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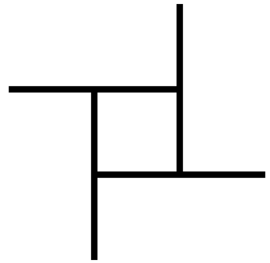
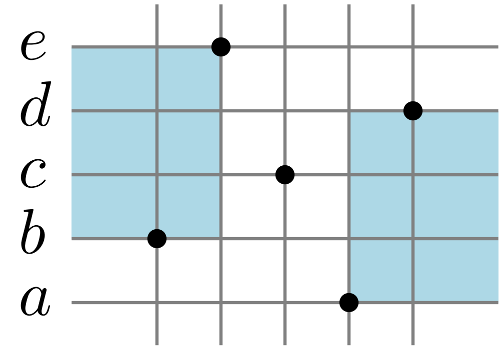
Proof, step 3:

$\gamma_s(\pi)$ contains w_1

$\implies$ π contains q_1.



$\implies$



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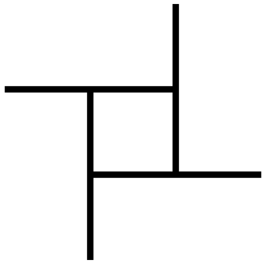
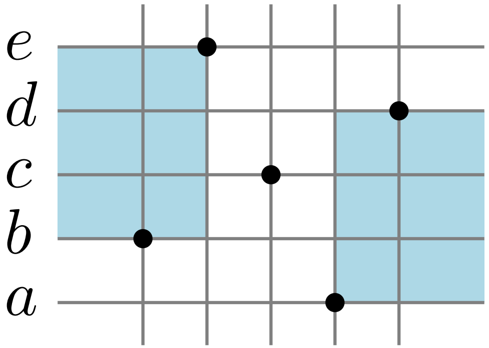
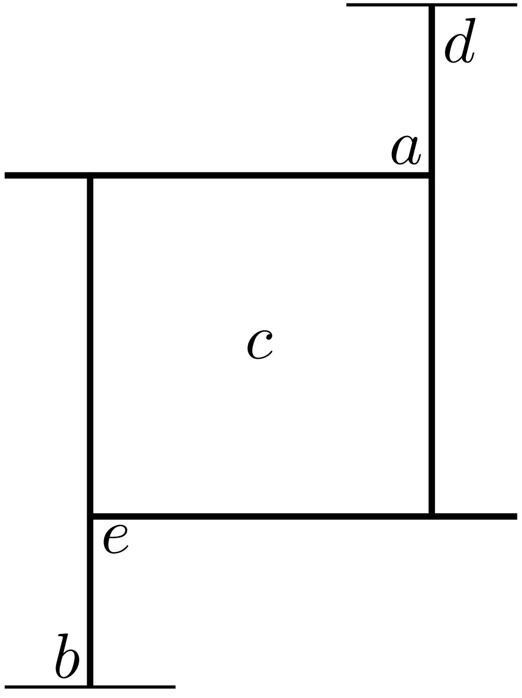


$\implies$





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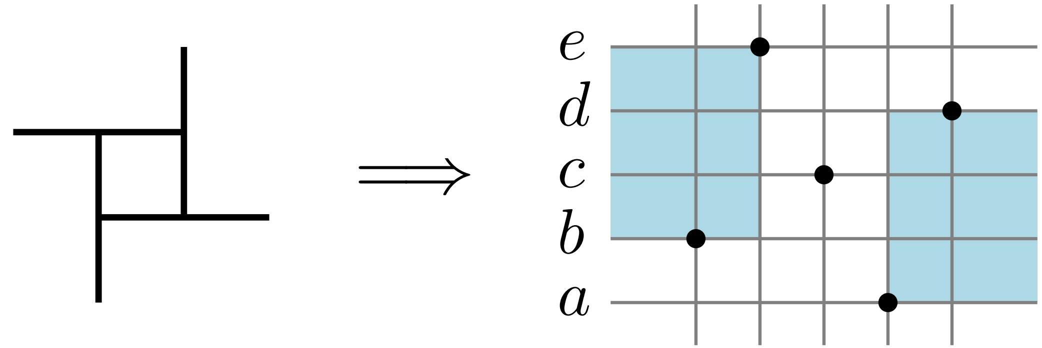

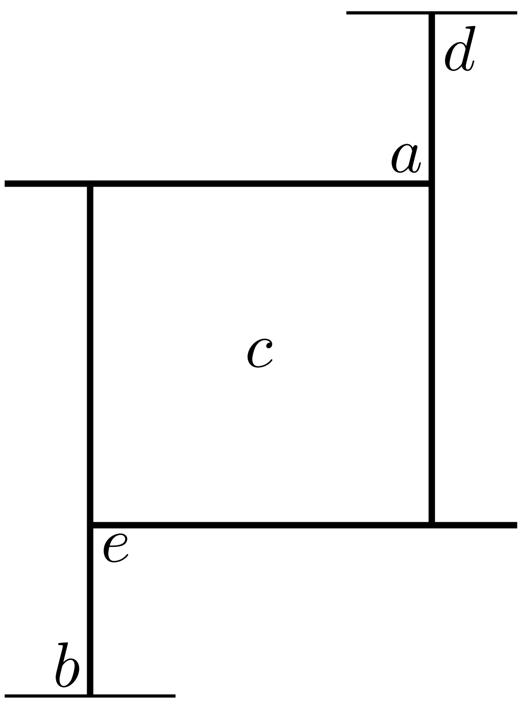

Labeling: $a < b < c < d < e$

Insertion: $b \prec e \prec c \prec a \prec d$

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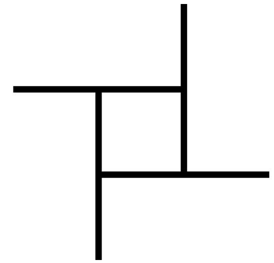
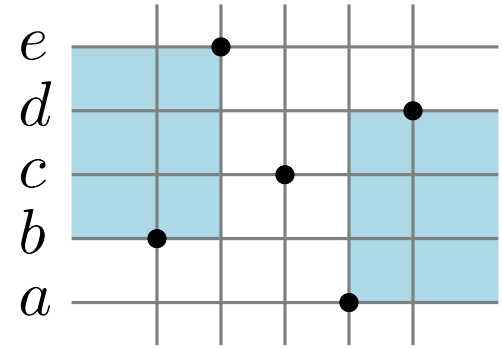
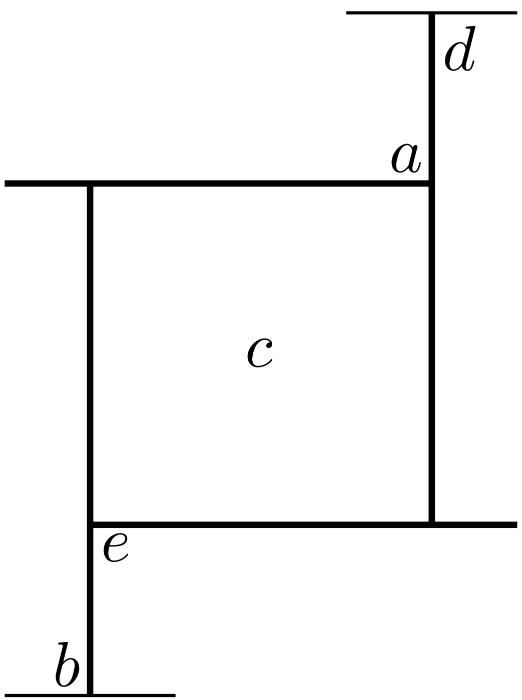
$\Longrightarrow$ π contains q_1.



$\Longrightarrow$





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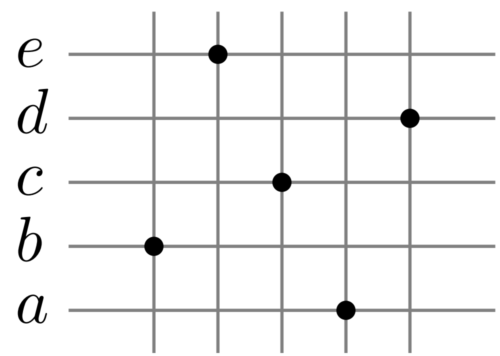

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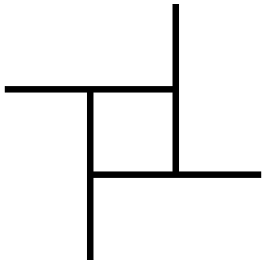

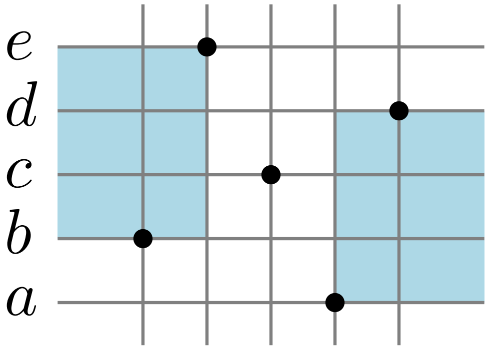

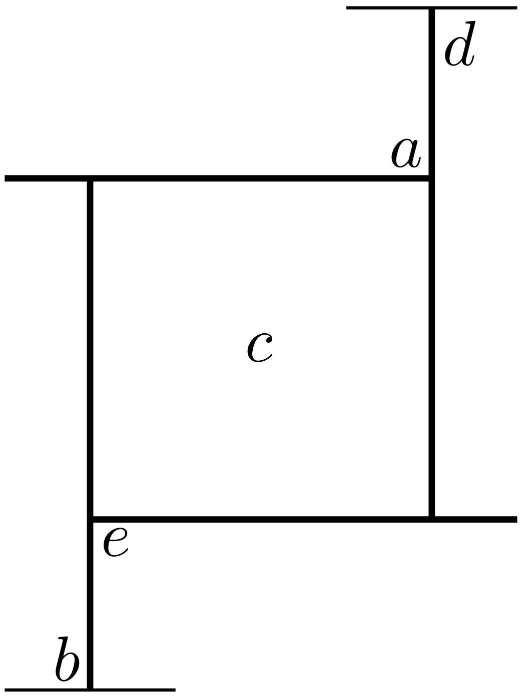

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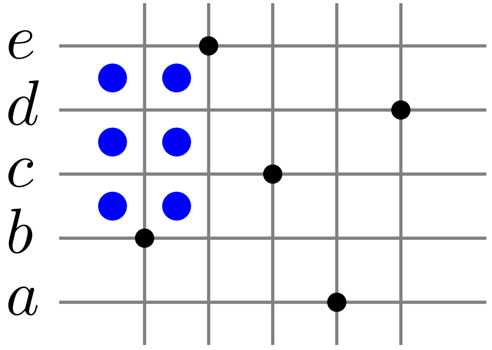

$b < x < e, \quad x \prec e$

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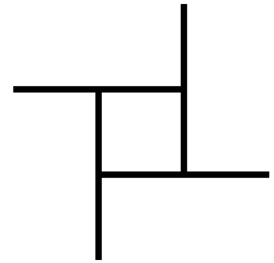
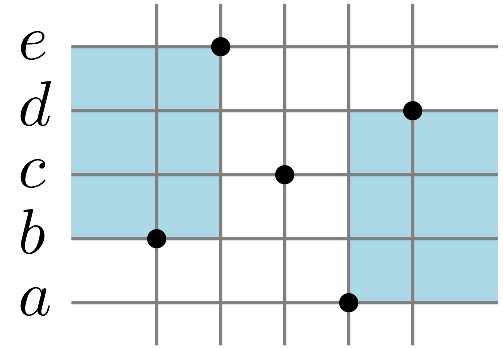
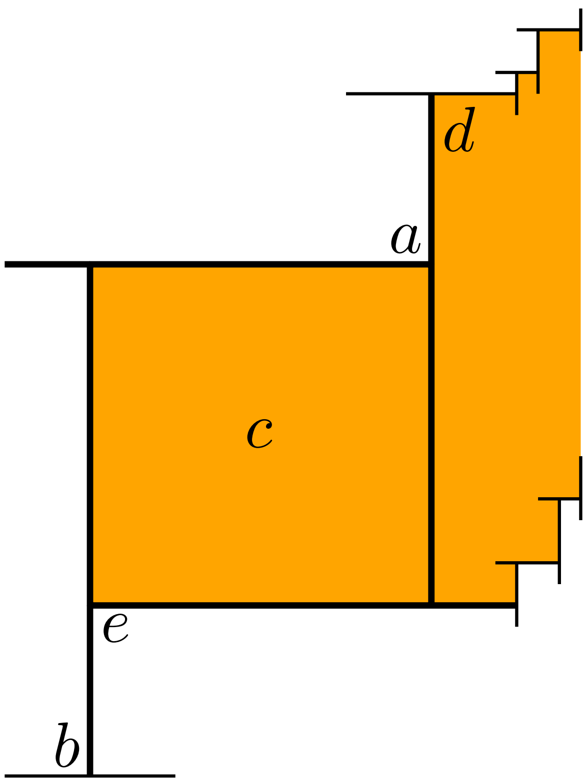


$\Longrightarrow$





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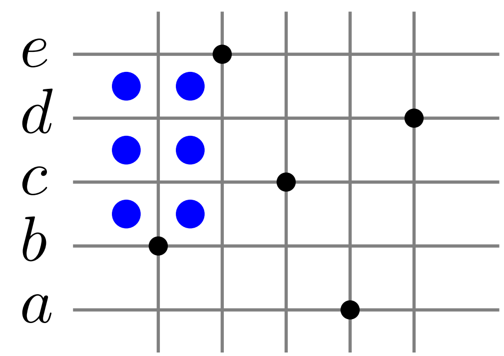

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Theorem (ACFF, 2023). A permutation π avoids p_1 and p_2 if and only if the rectangulations $\gamma_w(\pi)$ and $\gamma_s(\pi)$ are guillotine.

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Corollary. There are bijections between

1. weak guillotine rectangulations and
   (a) (p_1, p_2)-avoiding Baxter permutations,
   (b) (p_1, p_2)-avoiding twisted Baxter permutations,
   (c) (p_1, p_2)-avoiding co-twisted Baxter permutations.
2. strong guillotine rectangulations and
   (a) (p_1, p_2)-avoiding two-clumped permutations.
   (b) (p_1, p_2)-avoiding co-two-clumped permutations.

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Implications.

- 1(a) implies $\mathrm{Av}(2413,\, 3142) = \mathrm{Av}(2[41]3,\, 3[14]2,\, p_1,\, p_2)$.
- 2(a) is the first representation of strong guillotine rectangulations by a permutation class (conjectured by Merino and Mütze).

END