Block-wise Simple Permutations

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Summary: A permutation is called *block-wise simple* if it contains no interval of the form $p_1 \oplus p_2$ or $p_1 \ominus p_2$. We present this new set of permutations and explore some of its combinatorial properties. We present a generating function for this set, as well as a recursive formula for counting block-wise simple permutations. Following Tenner, who founded the notion of interval posets, we characterize and count the interval posets corresponding to block-wise simple permutations. We also present a bijection between these interval posets and certain tiling's of the n-gon.

Block-wise simple permutation - Definition

Let $\mathcal{S}_n$ be the group of permutations of the set $\{1, \dots, n\}$.

1. The identity permutation $\pi = 1$ is *block-wise simple*.
2. A permutation $\pi \in \mathcal{S}_n$ is *block-wise simple* if there is $\sigma \in Simp_k$ ($k \geq 4$) where $Simp_k$ is the set of simple permutations of order k, and there are $\alpha_1, \dots, \alpha_k$ which are block-wise simple permutations, such that $\pi = \sigma[\alpha_1, \dots, \alpha_k]$, the inflation of σ by $\alpha_1, \dots, \alpha_k$.

   Denote by W_n the set of block-wise simple permutations.

Block-wise simple - Example

$$2413[3142, 1, 1, 1] = 4253716 \in W_7.$$

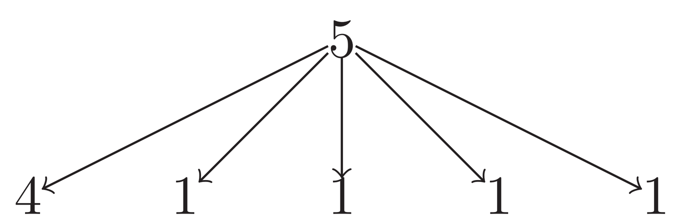
There are no block-wise simple permutations of orders 2 and 3.
For $n \in \{4, 5, 6\}$, a permutation is block-wise simple, if and only if it is simple.

An alternate definition of block-wise simple permutations

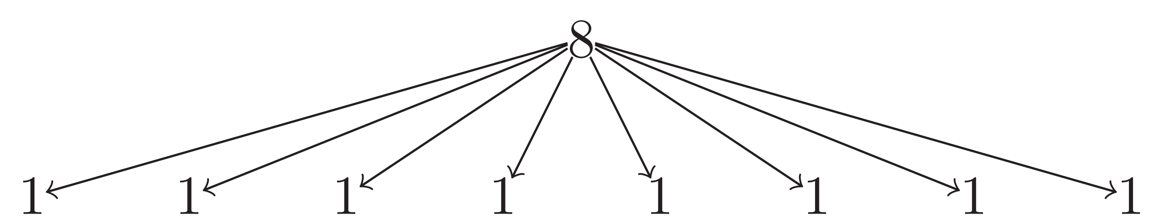
A permutation $\pi \in \mathcal{S}_n$ is block-wise simple if and only if it has no interval of the form $p_1 \oplus p_2$ or $p_1 \ominus p_2$.

Counting block-wise simple permutations for $n = 8$

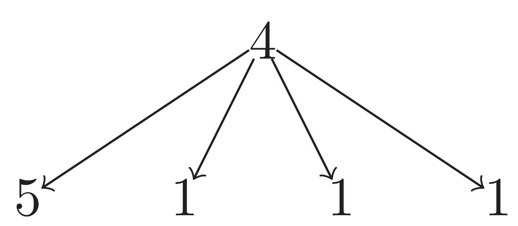
1. $\pi = \sigma[1, 1, 1, 1, 1, 1, 1, 1]$ with $\sigma \in Simp_8$.
2. $\pi \in \{\sigma[\tau, 1, 1, 1], \sigma[1, \tau, 1, 1], \sigma[1, 1, \tau, 1], \sigma[1, 1, 1, \tau]\}$, $\sigma \in Simp_4, \tau \in Simp_5$.
3. $\pi \in \{\sigma[\tau, 1, 1, 1, 1], \sigma[1, \tau, 1, 1, 1], \sigma[1, 1, \tau, 1, 1], \sigma[1, 1, 1, \tau, 1], \sigma[1, 1, 1, 1, \tau]\}$, $\sigma \in Simp_5, \tau \in Simp_4$.



$|Simp_5| \cdot |Simp_4| \cdot 5 = 60$



$|Simp_8| = 2926$



$|Simp_4| \cdot |Simp_5| \cdot 4 = 48$

A formula to count simple block permutations

Let $w_n = |W_n|$ and $s_n = |Simp_n|$. Then, for $n \geq 4$ we have

$$w_n = \sum_{l=4}^{n} s_l \sum_{\lambda(\lambda_1, \dots, \lambda_l) \in Comp(n,l)} w_{\lambda_1} \cdots w_{\lambda_l} \qquad (1)$$

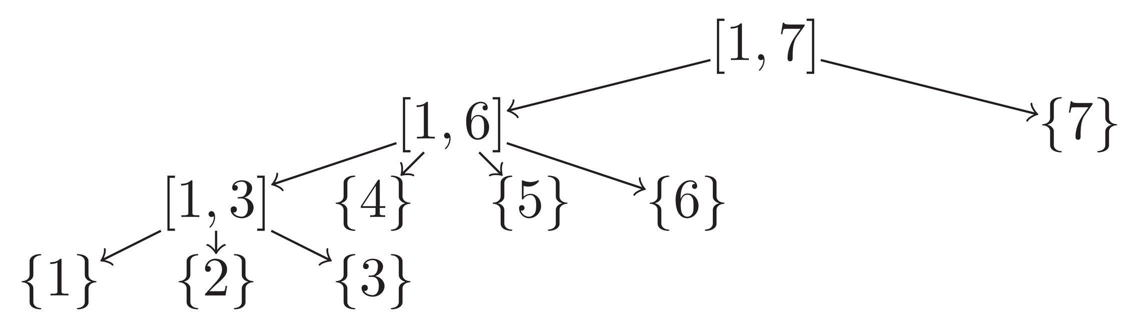
where $Comp(n, l)$ is the set of compositions of n in l parts.

B. E. Tenner's interval poset - Definition

The interval poset of a permutation $\pi \in \mathcal{S}_n$ is the poset $P(\pi)$ whose elements are the non-empty intervals of π; the order is defined by set inclusion. The minimal elements are the intervals of size 1.

Interval poset - Example

Interval poset of the permutations:
5123647, 5321647, 4612357, 4632157, 7463215, 7461235, 7532164, 7512364



Theorem (Bagno-Eisenberg-Reches-Sigron, 2022)

The number of interval posets that represent a block-wise simple permutation of order n is equal to

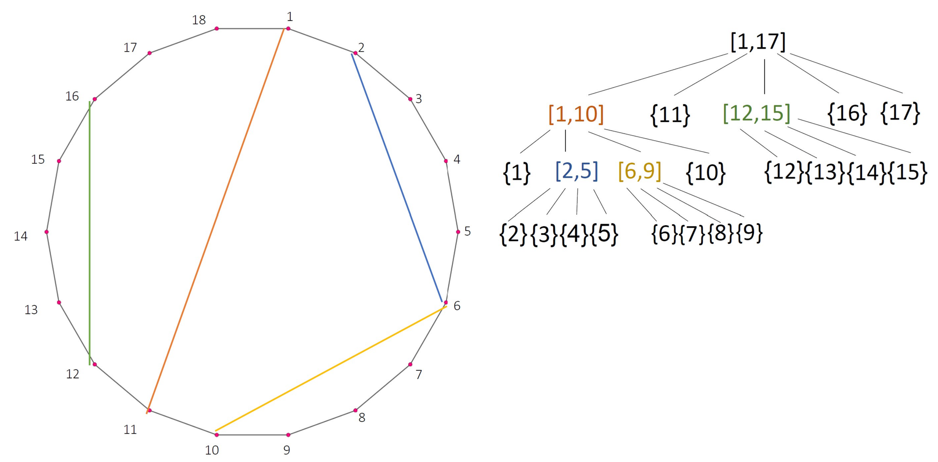
$$\frac{1}{n} \sum_{i=1}^{\lfloor \frac{n-1}{3} \rfloor} \binom{n+i-1}{i} \binom{n-2i-2}{i-1}$$

which is also the number of ways to place non-crossing diagonals in a convex $(n+1)$-gon such that no triangles or quadrilaterals are present.

Bijective proof

Associate with the interval poset corresponding to $\pi \in W_n$, the convex $(n+1)$-gon whose set of diagonals is

$$\{(a, b+1) | \, [a, b] \text{ is an internal node of the interval poset}\}.$$



Asymptotics - an open question

Use Equation (1) to obtain an upper bound for the proportion of the number of block-wise simple permutations to the magnitude of the entire set of permutations. Write $A_n = W_n - Simp_n$ and $R_n = \frac{|A_n|}{|S_n|}$. Experimental checks show that $\lim_{n \to \infty} R_n = 0$. Actually, we have corroborating data up to $n = 23$.

n	1	2	3	4	5	6	7	8	9	10	11	12
R_n	0	0	0	0	0	0	0.0031746	0.00267857	0.00303131	0.0029343	0.00273389	0.00247482

Conjecture: The proportion R_n tends to 0 when n tends to infinity.