A q,r-ANALOGUE FOR THE STIRLING NUMBERS OF THE FIRST KIND OF COXETER GROUPS OF TYPE B

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Summary: The *(unsigned) Stirling numbers of the first kind* $s(n,k)$ are defined by the following identity: $t(t+1)(t+2)\cdots(t+n-1)=\sum_{k=1}^{n} s(n,k)\cdot t^k$. A well-known combinatorial interpretation for these numbers is given by considering them as the number of permutations of the set $[n]=\{1,2,\dots,n\}$ having k cycles. Bala presented a generalization of the Stirling numbers of the first kind to the framework of Coxeter groups of type B, also known as the group of signed permutations. We denote these numbers by $s^B(n,k)$. The definition of these numbers is by the following equation: $(t+1)(t+3)\cdots(t+(2n-1))=\sum_{k=0}^{n} s^B(n,k)\cdot t^k$. Broder gave a variant of Stirling numbers, which is called the *r-Stirling number*. Also, some q-analogues were given by several authors. In this work, we suggest a q,r-analogue for the Stirling numbers of the first kind for the Coxeter groups of type B, together with a combinatorial interpretation and some identities.

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STIRLING NUMBERS OF THE FIRST KIND

The most common combinatorial interpretation of **Stirling number of the first kind**, denoted by $s(n,k)$, is as the number of permutations in S_n that are decomposed into k cycles.

Theorem 1 (classical) *The Stirling numbers of the first kind satisfy the following recursion:* $s(n,k)=s(n-1,k-1)+(n-1)s(n-1,k)$, *with the boundary conditions:* $s(n,n)=1$ *and* $s(n,0)=0$.

BRODER'S r-VARIANT OF STIRLING NUMBERS

Broder defined a generalization of the Stirling number of the first kind by adding the requirement that no two of the first r elements of $\{1,\dots,n\}$ share a cycle. Broder's r-variant, denoted by $s_r(n,k)$, satisfies the same recursion, with the following different boundary conditions: $s_r(n,k)=0$ for $n<r$ and $s_r(n,k)=\delta_{kr}$ for $n=r$.

THE GROUP OF SIGNED PERMUTATIONS

Definition 2 *Denote* $[\pm n]:=\{\pm 1,\dots,\pm n\}$. *A* **signed permutation** *is a bijective function:* $\pi:[\pm n]\to[\pm n]$, *satisfying:* $\pi(-i)=-\pi(i)$, $\forall i$. **The group of signed permutations** *of the set* $[\pm n]$ *(with respect to composition of functions), denoted by* B_n, *is also known as the* **hyperoctahedral group** *or the* **Coxeter group of type** B.

Every signed permutation can be decomposed into a multiplication of disjoint cycles, which may be either split or non-split: A cycle C is called **non-split** if "$i\in C$ if and only if $-i\in C$", and **split** otherwise. A signed permutation, written as a sequence of disjoint cycles, is presented in **standard form** if its cycles are ordered in such a way that the sequence composed by the smallest absolute values of the elements of each cycle increases.

Example 3 *Let* $\pi=(\mathbf{1},-3)(3,-1)(\mathbf{2})(-2)(\mathbf{4},5,-4,-5)$.
Split cycles: $(1,-3)(3,-1)$ *and* $(2)(-2)$; *Non-split cycle:* $(4,5,-4,-5)$.

STIRLING NUMBER OF TYPE B

The **Stirling number of type B of the first kind** $s^B(n,k)$ counts the signed permutations on n elements having k non-split cycles.
$s^B(n,k)$ satisfies the following recursion:

$$s^B(n,k)=s^B(n-1,k-1)+(2n-1)s^B(n-1,k)$$

with the boundary conditions: $s^B(n,n)=s^B(n,0)=1$.

RESTRICTED GROWTH WORDS

Definition 4 *Let* $\Sigma_B=\{(i,j)\in\mathbb{Z}\times\mathbb{Z}\mid 1\le|i|,|j|\le n\}$.
A **restricted growth (RG-)word of type B of the first kind** *is a word* $\omega=\omega_1\cdots\omega_n=(i_1,j_1)\cdots(i_n,j_n)$ *in the alphabet* Σ_B, *which satisfies the following conditions:*

(1) *We have either* $(i_1,j_1)=(1,1)$ *or* $(i_1,j_1)=(-1,1)$.

(2) *For each* $2\le t\le n$, *the following inequality holds:*
$$|i_t|\le\max\{|i_1|,\dots,|i_{t-1}|\}+1.$$

(3a) *If* $|i_t|=\max\{|i_1|,\dots,|i_{t-1}|\}+1$, *we have:* $j_t=1$.

(3b) *If* $|i_t|\le\max\{|i_1|,\dots,|i_{t-1}|\}$, *one of the following pairs exists in* ω: *either* (i_t,j_t-1) *or* $(i_t,-(j_t-1))$.

We denote by $R_B(n,k)$ *the set of all RG-words of type B of the first kind satisfying:* $\#\{i_t\mid 1\le t\le n,\ i_t<0\}=k$, *and by* $R^r_B(n,k)$ *its subset containing the words that begin with the prefix* $\omega_1\cdots\omega_r=1\cdots r$.

FROM SIGNED PERMUTATIONS TO RG-WORDS

Let $\pi=C_1\cdots C_k$ where for each $1\le t\le k$, C_t is a signed cycle in standard form. Define $\Phi_B(\pi)=\omega_1\cdots\omega_n$ according to the rule $\omega_i=(t,s)$, where:

- i (or $-i$) appears in the cycle C_t,
- $t>0$ if and only if C_t is split.
- $|s|$ is the location of i in the cycle C_t.
- The sign of s is the sign of the first appearance of i or $-i$ in the cycle C_t.

Example 5 *The signed permutation*

$$\underbrace{(1,-7)}_{C_1}\underbrace{(-1,7)}_{-C_1}\underbrace{(2,-5,4,-9)}_{C_2}\underbrace{(-2,5,-4,9)}_{-C_2}\underbrace{(\mathbf{3},\mathbf{8},-\mathbf{3},-\mathbf{8})}_{C_3=-C_3}\underbrace{(\mathbf{6},-\mathbf{6})}_{C_4=-C_4},$$

corresponds to the following RG-word of type B of the first kind:

$$\omega=\underbrace{(1,1)}_{\omega_1}\underbrace{(2,1)}_{\omega_2}\underbrace{(-3,1)}_{\omega_3}\underbrace{(2,3)}_{\omega_4}\underbrace{(2,-2)}_{\omega_5}\underbrace{(-4,1)}_{\omega_6}\underbrace{(1,-2)}_{\omega_7}\underbrace{(-3,2)}_{\omega_8}\underbrace{(2,-4)}_{\omega_9}.$$

FLAG-INVERSION STATISTIC ON RG-WORDS

Definition 6 *Define* $\preceq_{\text{abslex}}$ *on* Σ_B *as follows:*
$(i,j)\preceq_{\text{abslex}}(i',j') \iff (|i|<|i'|)$ or $(|i|=|i'|$ and $|j|\le|j'|)$.

Definition 7 *Let* $\omega=\omega_1\cdots\omega_n\in R_B(n,k)$. *Define:*
$\mathrm{inv}_B(\omega)=\#\{(\omega_i,\omega_j)\mid i<j,\ \omega_j\prec_{\text{abslex}}\omega_i\}$,
$\mathrm{neg}(\omega)=\#\{\omega_t=(i_t,j_t)\mid j_t<0\}$ *and* $\mathrm{finv}(\omega)=2\mathrm{inv}_B(\omega)+\mathrm{neg}(\omega)$.

EXAMPLE FOR THE FLAG-INVERSION STATISTIC

Given the RG-word of type B of the first kind:

$$\omega=\underbrace{(1,1)}_{\omega_1}\underbrace{(-2,1)}_{\omega_2}\underbrace{(-3,1)}_{\omega_3}\underbrace{(-2,3)}_{\omega_4}\underbrace{(-2,-2)}_{\omega_5}\underbrace{(-4,1)}_{\omega_6}\underbrace{(1,-2)}_{\omega_7}\underbrace{(-3,2)}_{\omega_8}\underbrace{(-2,-4)}_{\omega_9},$$

we have:

$$\mathrm{inv}_B(\omega)=\#\left\{\begin{matrix}(\omega_2,\omega_7),(\omega_3,\omega_4),(\omega_3,\omega_5),(\omega_3,\omega_7),(\omega_3,\omega_9),(\omega_4,\omega_5),\\(\omega_4,\omega_7),(\omega_5,\omega_7),(\omega_6,\omega_7),(\omega_6,\omega_8),(\omega_6,\omega_9),(\omega_8,\omega_9)\end{matrix}\right\}=12,$$

and $\mathrm{neg}(\omega)=\#\{\omega_5,\omega_7,\omega_9\}=3$.
Therefore:

$$\mathrm{finv}(\omega)=2\mathrm{inv}_B(\omega)+\mathrm{neg}(\omega)=27.$$

A q-ANALOGUE OF TYPE B

Definition 8 *Define a q-analogue of type B,* s^B_q, *using the recursion:*

$$s^B_q(n,k)=s^B_q(n-1,k-1)+(1+[2n-2]_q)\cdot s^B_q(n-1,k),$$

where $[k]_q=1+q+q^2+\cdots+q^{k-1}$, *and the boundary conditions:*

$$s^B_q(n,0)=\sum_{k=0}^{n} s^A_{q^2}(n,k)\cdot(1+q)^{n-k}\ \text{for}\ n\ge 1$$

and $s^B_q(0,k)=\delta_{0k}$.

A q,r-ANALOGUE OF TYPE B

Definition 9 *Define a q,r-analogue of type B,* $s^B_{q,r}$, *using the recursion:*

$$s^B_{q,r}(n,k)=s^B_{q,r}(n-1,k-1)+(1+[2n-2]_q)\cdot s^B_{q,r}(n-1,k),$$

and the boundary conditions:

$$s^B_{q,r}(n,r)=\sum_{\ell=0}^{n-2r} s^A_{q^2}(n-r,\ell+r)\cdot(1+q)^{n-r-\ell},$$

$s^B_{q,r}(n,k)=0$ *for* $0\le k<r$, *and* $s^B_{q,r}(0,k)=\delta_{0k}$.

COMBINATORIAL REALIZATION

Theorem 10 (Bagno-Garber, 2023) *The generating functions of the statistic* $\mathrm{finv}=2\mathrm{inv}_B+\mathrm{neg}$ *over* $R_B(n,k)$ *and* $R^r_B(n,k)$ *satisfy:*

$$\sum_{\omega\in R_B(n,k)} q^{\mathrm{finv}(\omega)}=s^B_q(n,k);\qquad \sum_{\omega\in R^r_B(n,k)} q^{\mathrm{finv}(\omega)}=s^B_{q,r}(n,k).$$

Hence, they are combinatorial realizations of the q-Stirling and the q, r-Stirling numbers of the first kind of type B, respectively.