# Restricted permutations and bounded Dyck paths

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## Notation

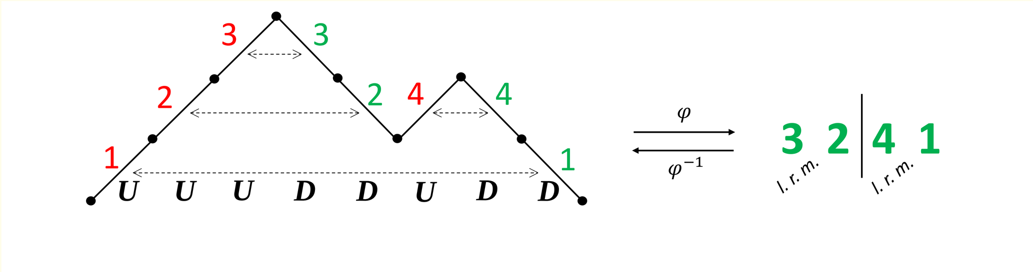
| | |
|---|---|
| $\mathcal{D}_n$ | Dyck paths of semilength $n$ |
| $\mathcal{D}_n^{(h)}$ | Dyck paths of semilength $n$ with height at most $h$ |
| $\mathcal{D}_n^{(h,t)}$ | Dyck paths of semilength $n$ with height at most $h$ avoiding $t-1$ consecutive valleys at height $h-1$ |
| $\mathcal{S}_n(312)$ | 312-avoiding permutations of length $n$ |

## Abstract


- Combinatorial interpretation of $\mathcal{D}_n^{(h,2)}$ (i.e. Dyck paths in $\mathcal{D}_n^{(h)}$ without valleys at height $h-1$) by means of 312-avoding permutations with some restrictions on their *left-to-right maxima* (*l.r.m.*).
- Enumeration of these two classes using the ECO method and the theory of production matrices.
- We obtain a family of combinatorial identities involving Catalan numbers.


## Bijection $\varphi$

Fix a Dyck path $P$ and label its up steps by enumerating them from left to right (so that the $\ell$-th up step is labelled $\ell$). Next assign to each down step the same label of the up step it corresponds to. Now consider the permutation whose entries are constituted by the labels of the down steps read from left to right. Such a permutation $\pi = \varphi(P)$ is easily seen to be 312-avoiding.

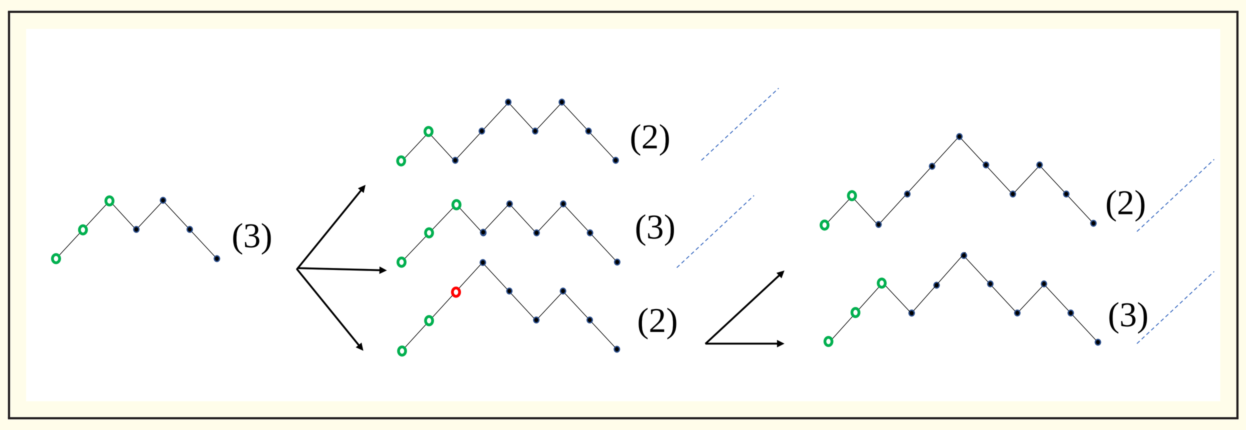


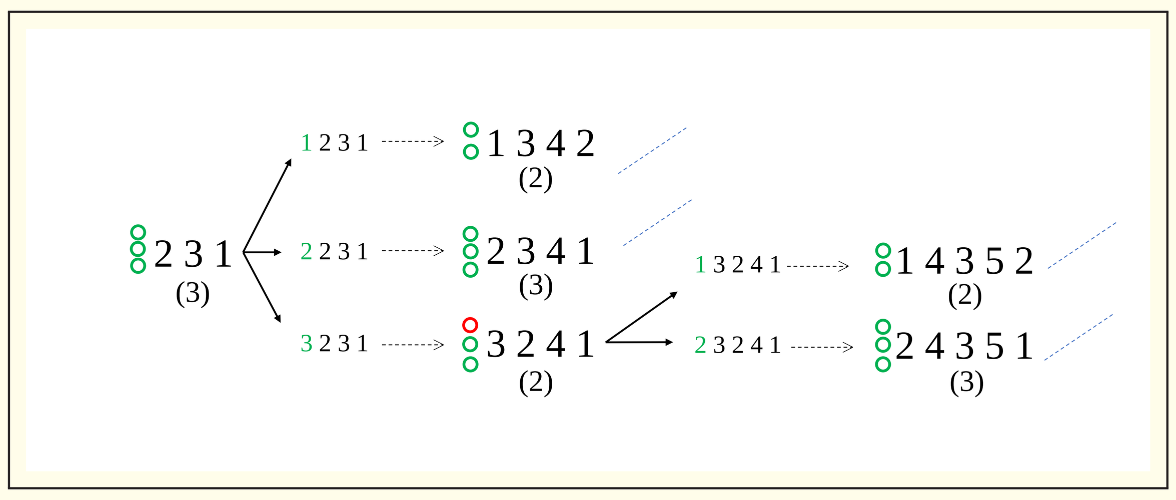

Fix $\pi \in \mathcal{S}_n(312)$ and consider its factors starting with its *l.r.m.* $\pi_{i_1}, \pi_{i_2}, \dots, \pi_{i_\ell}$.

- write as many $U$'s as $\pi_{i_1} (= \pi_1)$ followed by as many $D$'s as the cardinality of the first descending subsequence headed by $\pi_{i_1}$;
- for each $j = 2, \dots, \ell$, add as many $U$'s as $\pi_{i_j} - \pi_{i_{j-1}}$ followed by as many $D$'s as the cardinality of the sub-sequence headed by $\pi_{i_j}$.

## Generating algorithm

Example with $h = 3$ and $t = 2$.




### ECO operator $\vartheta : \mathcal{D}_n^{(h,2)} \to \mathcal{D}_{n+1}^{(h,2)}$

(Insert a peak in the $k$ active sites $\circ$)

If $k < h$ then the $k$ paths in $\vartheta(P)$ start, respectively, with $1, 2, \dots$ or $k$ up steps, so that, in turns, they are labelled $(2), (3), \dots (k+1)$. Then we can write the production

$$(k) \rightsquigarrow (2)(3)\cdots(k)(k+1),\ \ 2 \le k < h.$$

If $k = h$ then the $k$ paths in $\vartheta(P)$ start, respectively, with $1, 2, \dots$ or $h$ up steps. Since the path having $h$ starting up steps is labelled $(h-1)$, then we can write the production

$$(h) \rightsquigarrow (2)(3)\cdots(h-1)^2(h).$$

The two paths having label $(h-1)$ are precisely the one starting with $h$ up steps and the one starting with $h-2$ up steps.

$$\Omega_h : \begin{cases} (1) & \\ (1) & \rightsquigarrow\ (2) \\ (k) & \rightsquigarrow\ (2)(3)\cdots(k)(k+1) \\ (h) & \rightsquigarrow\ (2)(3)\cdots(h-1)^2(h) \end{cases}$$

Succession rule

### $\mathcal{S}_n^{(h,2)}(312)$ : 312-avoiding permutations with no l.r.m $\pi_{i_j}$ such that

1. $\pi_{i_j} - i_j = h - 1$ and
2. $\pi_{i_{j+1}} = \pi_{i_j} + 1$.

The parameter $(k)$ in $\Omega_h$ has the following interpretation according to the value of $\pi_1$ in $\pi \in S_n^{(h,2)}(312)$:

$$(k) = \begin{cases} \pi_1 + 1 & \text{if } \pi_1 \neq h\,; \\ \pi_1 - 1 & \text{if } \pi_1 = h. \end{cases}$$

If $\pi_1 < h$, a permutation $\pi = \pi_1 \dots \pi_n \in \mathcal{S}_n^{(h,2)}(312)$ at level $n$, produces $k = \pi_1 + 1$ sons at level $n+1$ by inserting the element $\ell$, with $\ell = 1, 2, \dots, \pi_1 + 1$, before $\pi_1$ and rescaling the sequence. If $\pi_1 = h$, a permutation $\pi$ at level $n$, produces $k = \pi_1 - 1 = h - 1$ sons at level $n+1$ by inserting the element $\ell$, with $\ell = 1, 2, \dots, h-1$, before $\pi$.

**Proposition:** There exists a bijection between the classes $\mathcal{S}_n^{(h,2)}(312)$ and $\mathcal{D}_n^{(h,2)}$, which is the restriction $\varphi\big|_{D_n^{(h,2)}}$.

## Enumeration

If $h = 2$, then $\Omega_2 : \begin{cases} (1) & \\ (1) & \rightsquigarrow\ (2) \\ (2) & \rightsquigarrow\ (1)(2) \end{cases}$ and the production matrix is $P_2 = \begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix}$. For $h \ge 3$, the production matrix is $P_h = \begin{pmatrix} 0 & u^t \\ 0 & P_{h-1} + eu^t \end{pmatrix}$ where $u^t$ is the row vector $(1\ 0\ 0 \dots)$ and $e$ is the column vector $(1\ 1\ 1\ \dots)^t$.

For the generating function we have:

- $f_h(x) = \frac{1}{1 - x f_{h-1}(x)}$ for $h \ge 2$ with $f_1(x) = 1 + x$ ;
- $f_h(x) = \frac{p_h(x)}{q_h(x)} = \frac{q_{h-1}(x)}{q_{h-1}(x) - x q_{h-2}(x)}$ ;
- $q_h(x) = a_{h,0} - a_{h,1}x - \dots - a_{h,j}x^j$ with $j = \left\lceil \frac{h+1}{2} \right\rceil$.

**Proposition:** For $h \ge 2$ and for $j = 1, 2, \dots, \left\lceil \frac{h+1}{2} \right\rceil$ we have:

$$a_{h,j} = \frac{3j - h - 2}{j} \binom{h - j + 1}{j - 1} (-1)^j$$

**Proposition:** Denoting $|\mathcal{D}_n^{(h,2)}|$ by $D_n^{(h,2)}$ and the Catalan numbers by $C_n$, and observing that $D_n^{(n+\alpha,2)} = C_n$, where $\alpha \ge 0$ is integer, we have:

$$D_n^{(h,2)} = \begin{cases} 1 & \text{for } n = 0\,; \\ \displaystyle\sum_{j=1}^{\lceil \frac{h+1}{2} \rceil} D_{n-j}^{(h,2)} a_{h,j} - \frac{3n - h - 1}{n} \binom{h - n}{n - 1} (-1)^n & \text{for } n \ge 1\,. \end{cases}$$

$$C_n = \sum_{j=1}^{n} C_{n-j} \frac{3j - n - \alpha - 2}{j} \binom{n + \alpha - j + 1}{j - 1} (-1)^j - \frac{2n - \alpha - 1}{n} \binom{\alpha}{n - 1} (-1)^n\,.$$