Permutations avoiding bipartite partially ordered patterns have a regular insertion encoding

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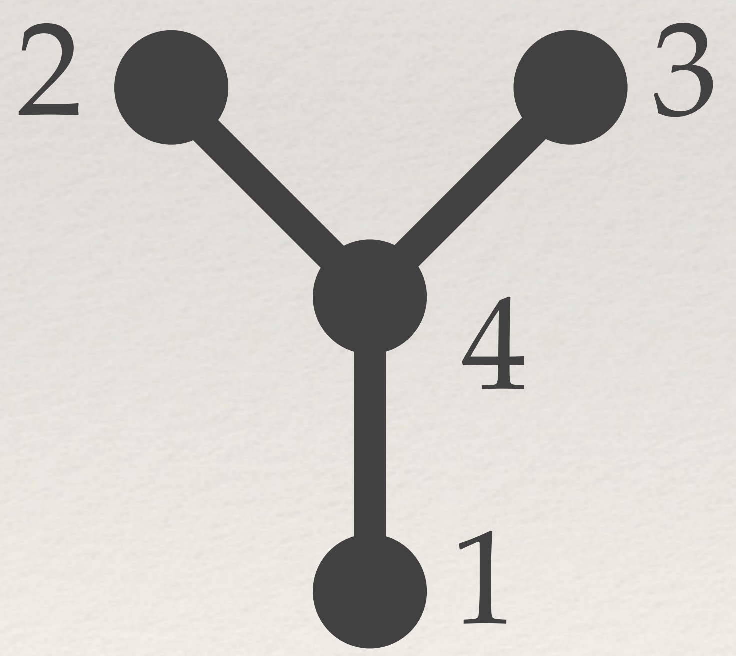
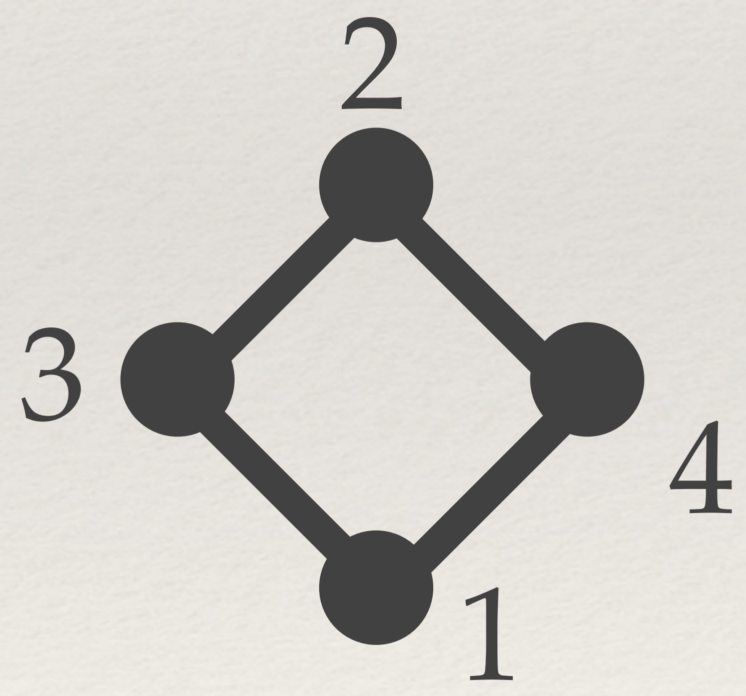
joint work with Émile Nadeau, Jay Pantone, and Henning Ulfarsson



Av(1342, 1432)



Av(1423, 1432)

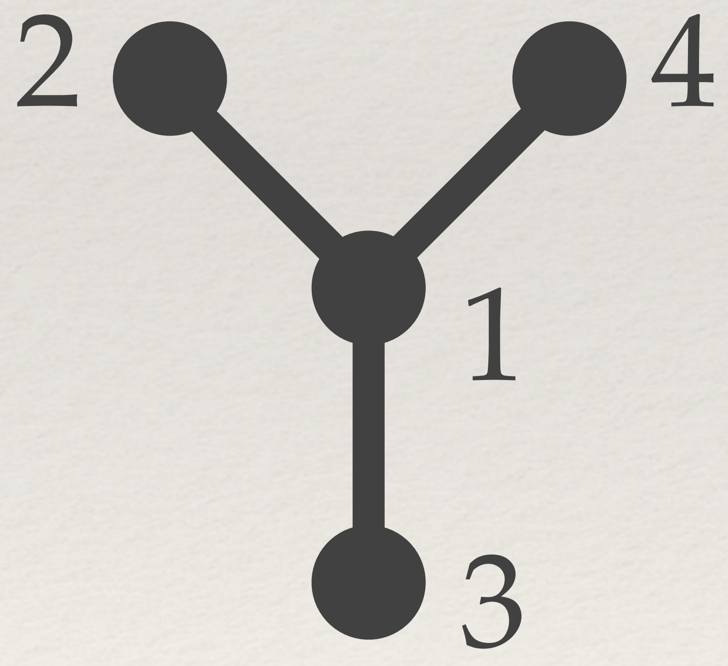
Partially ordered patterns (POPs)

A *partially ordered pattern* is a poset on the labels $\{1,2,\ldots,k\}$

A permutation π *contains* a partially ordered pattern P if there is a subword

$$\pi(i_1)\pi(i_2)\cdots\pi(i_k) \quad \text{with} \quad i_1 < i_2 < \cdots < i_k \quad \text{such that} \quad \pi(i_\ell) < \pi(i_m) \quad \text{if} \quad \ell < m \quad \text{in} \quad P$$

Example



$\pi(i_3)$ must be the lowest, then $\pi(i_1)$

$\pi(i_2)$ and $\pi(i_4)$ are incomparable

This implies avoidance of the patterns 2314 and 2413

POP classes

A POP can be defined by avoidance of classical patterns

Theorem 1.1. For a POP P, the basis of the permutation class $\mathrm{Av}(P)$ is the set of permutations that are the (group-theoretic) inverse of a linear extension of P.

We call Av(P) a *POP class*. Not every permutation class is a POP class

Code for checking:

https://gist.github.com/christianbean/e1df518db2b776b4ff2254b0a739cbd5

Symmetric equivalence

There are $2^{n!}$ permutation classes avoiding patterns of size n

Two permutation classes are symmetrically equivalent if they can be obtained by applying a symmetry of the square

The number of (finitely based) permutation classes avoiding size n patterns gives the sequence

$$1, 2, 3, 21, 2139264, \dots$$

A277086 Irregular triangle read by rows: T(n,k) = number of size k subsets of S_n with respect to the symmetries of the square.

The number of POP classes of size n gives the sequence

$$1, 3, 19, 219, 4231, 130023, 6129859, \ldots$$

A001035 Number of partially ordered sets ("posets") with n labeled elements (or labeled acyclic transitive digraphs).

The number of POP classes of size n up to symmetric equivalence is

$$1, 2, 7, 64, 1068, 32561, \ldots$$

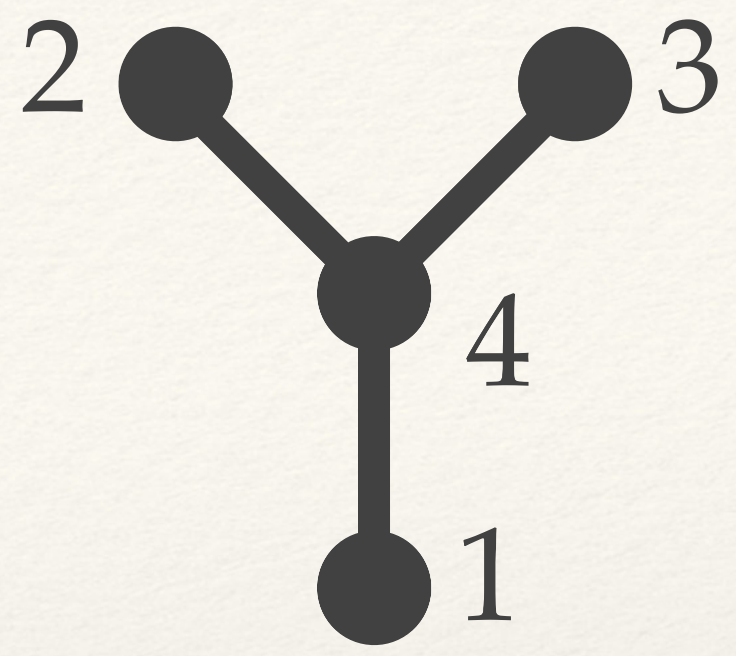
Brute force

Gao and Kitaev gave two operations

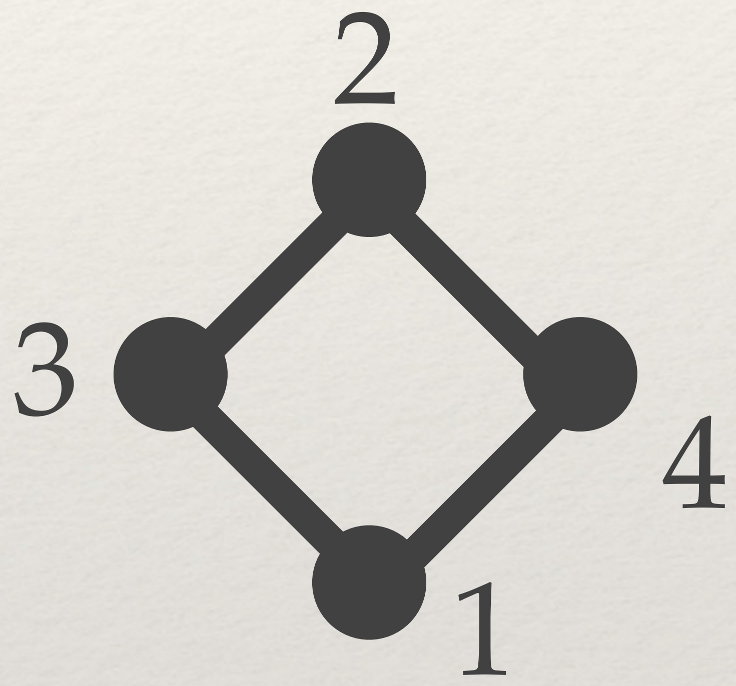
- complementing labels of the poset corresponds to the reverse of the permutation class
- reversing the relations corresponds to the complement of the permutation class

Inverse of a POP class

The permutation class Av(1342, 1432) is the POP class avoiding



The inverse $\text{Av}^{-1}(1342, 1432)$ is Av(1423, 1432) which is the POP class avoiding



The permutation class Av(123, 132, 231) is a POP class but its inverse Av(123, 132, 312) is not.

Question 3.1. When is the inverse of a POP class also a POP class? When it is, is there an operation on posets that can be defined in a uniform way to transform the poset of one class into the poset of its inverse?

Literature so far

On partially ordered patterns of length 4 and 5 in permutations

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Abstract

Partially ordered patterns (POPs) generalize the notion of classical patterns studied widely in the literature in the context of permutations, words, compositions and partitions. In an occurrence of a POP, the relative order of some of the elements is not important. Thus, any POP of length k is defined by a partially ordered set on k elements, and classical patterns correspond to k-element chains. The notion of a POP provides a convenient language to deal with larger sets of permutation patterns.

This paper contributes to a long line of research on classical permutation patterns

On Permutations Avoiding Partially Ordered Patterns Defined by Bipartite Graphs

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Abstract

Partially ordered patterns (POPs) generalize the notion of classical patterns studied in the literature in the context of permutations, words, compositions and partitions. In this paper, we give a number of general, and specific enumerative results for POPs in permutations defined by bipartite graphs, substantially extending the list of known results in this direction. In particular, we completely characterize the Wilf-equivalence for patterns defined by the N-shape posets.

Mathematics Subject Classifications: 05A05, 05A15

1 Introduction

An occurrence of a (classical) permutation pattern $p = p_1 \cdots p_k$ in a permutation $\pi = \pi_1 \cdots \pi_n$ is a subsequence $\pi_{i_1} \cdots \pi_{i_k}$, where $1 \leqslant i_1 < \cdots < i_k \leqslant n$, such that $\pi_{i_j} < \pi_{i_m}$ if and only if $p_j < p_m$. For example, the permutation 364125 has two occurrences of the

Bipartite $\iff$ regular insertion encoding

We show that the generating function of any bipartite POP class can be computed algorithmically

Theorem

A POP class has a regular insertion encoding if and only if it is bipartite

Insertion encoding

A language for encoding permutations

$\diamond \mapsto \diamond n \diamond$	represented by **m** (for middle)
$\diamond \mapsto n \diamond$	represented by **l** (for left)
$\diamond \mapsto \diamond n$	represented by **r** (for right)
$\diamond \mapsto n$	represented by **f** (for fill)

$\diamond$

$\diamond\ 1\ \diamond$

$\diamond\ 2\ \diamond\ 1\ \diamond$

$3\ 2\ \diamond\ 1\ \diamond$

$3\ 2\ \diamond\ 1\ 4\ \diamond$

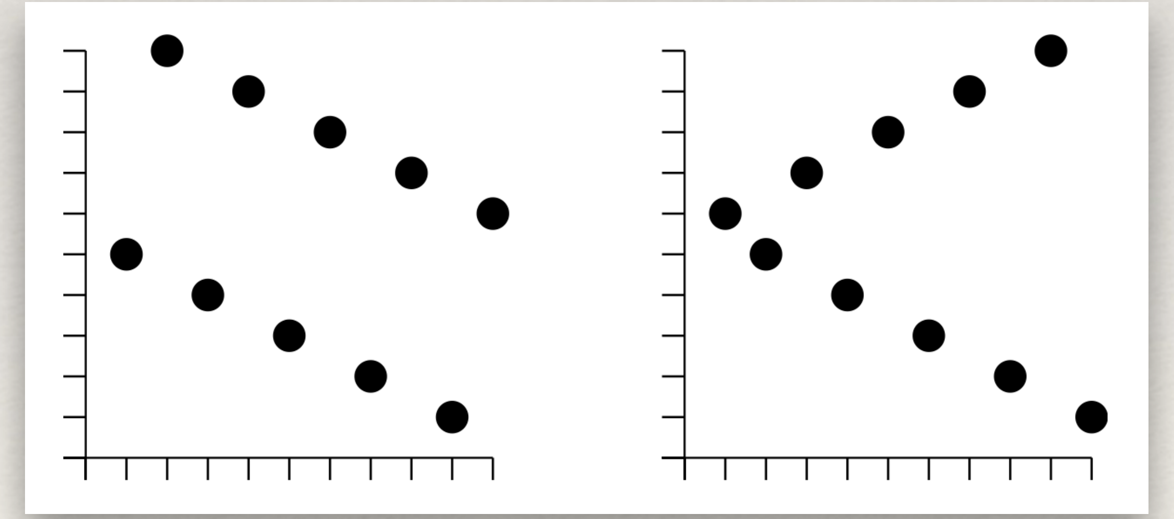
$3\ 2\ 5\ 1\ 4\ \diamond$

$3\ 2\ 5\ 1\ 4\ 6$

The insertion encoding of 325146 is $\mathbf{m_1 m_1 f_1 l_2 f_1 f_1}$

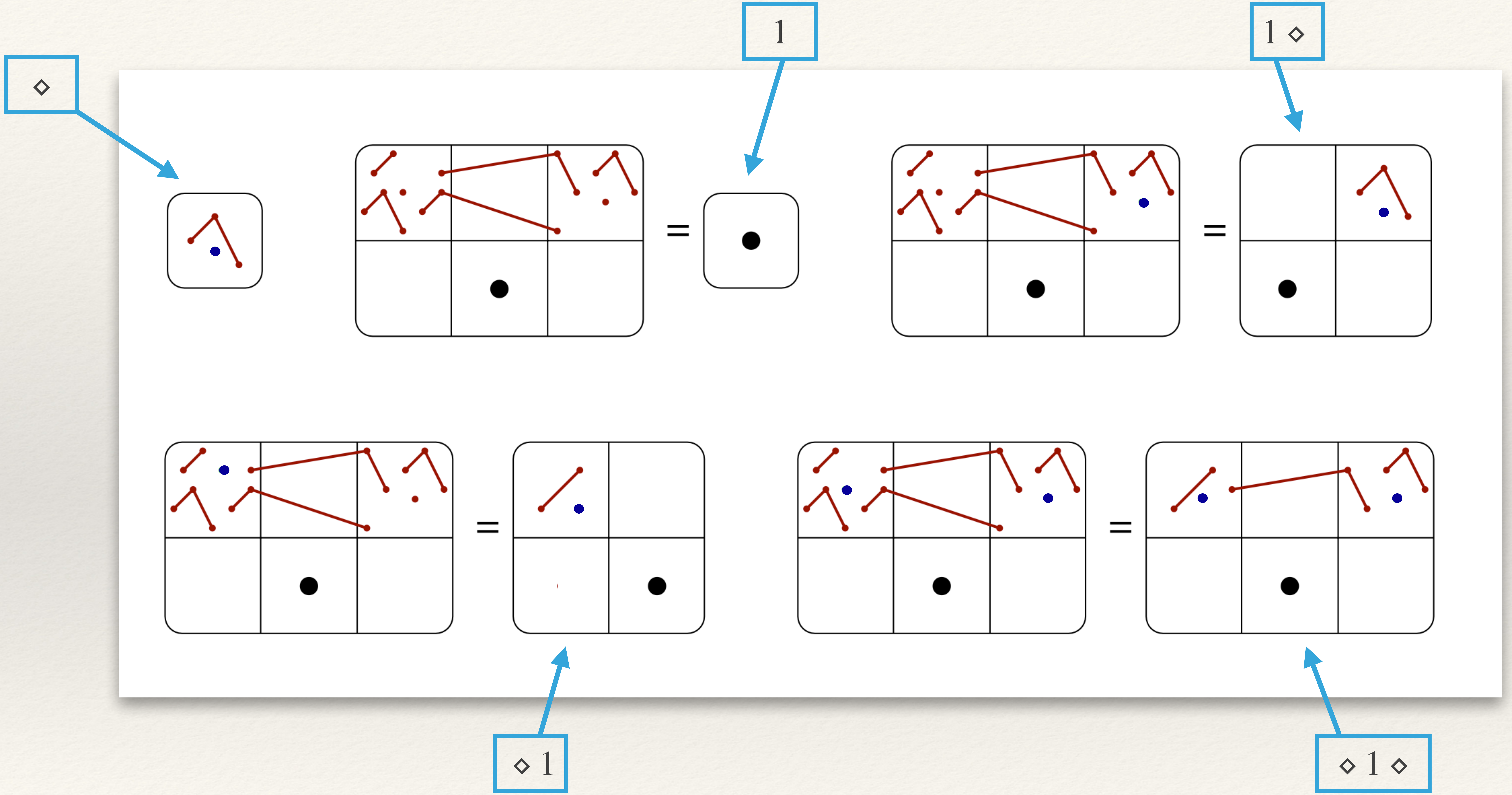
Let $\mathscr{L}(\mathscr{C})$ be the language formed by the insertion encodings of a permutation class $\mathscr{C}$

Theorem [Albert, Linton, and Ruškuc (2005), Vatter (2012)]
For a permutation class Av(B), the following are equivalent



1. The language $\mathscr{L}(\mathrm{Av}(B))$ is regular
2. There are at most k slots in any evolution
3. The set B contains at least one permutation in each of Av(132, 312), Av(213, 231), Av(123, 3142, 3412), and Av(321, 2143, 2413)

Insertion encoding as tilings



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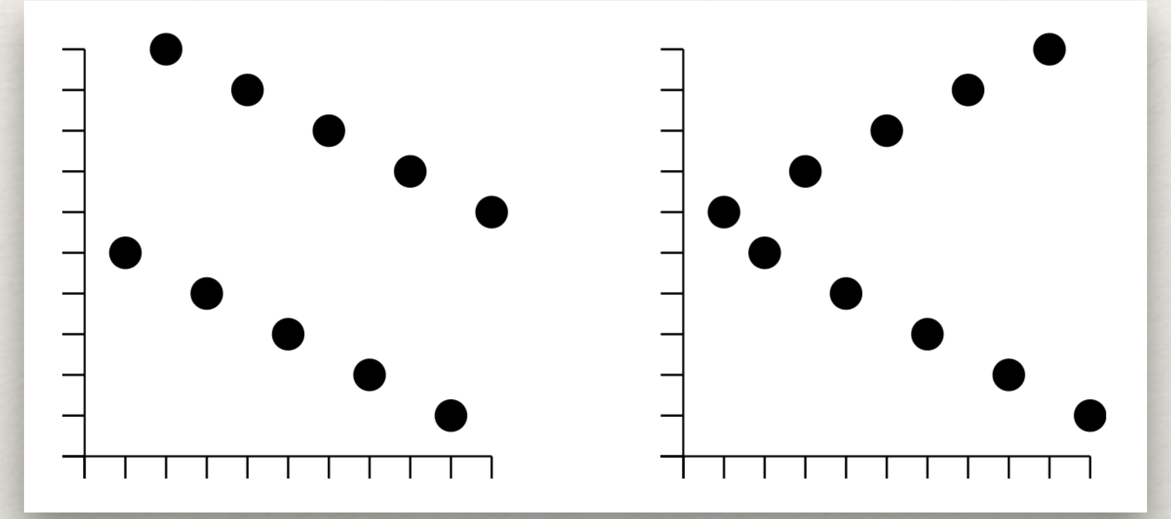
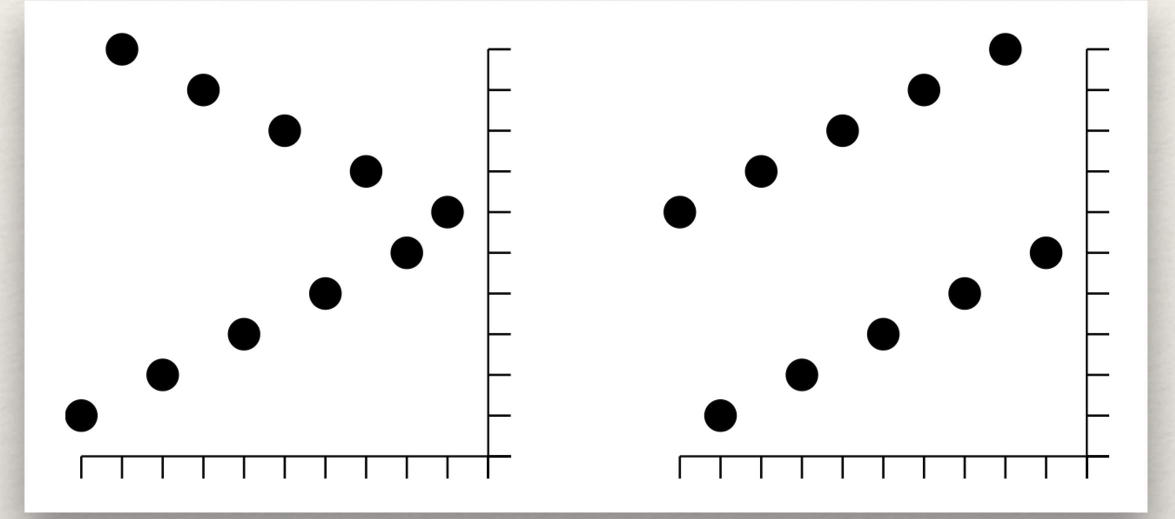




Theorem

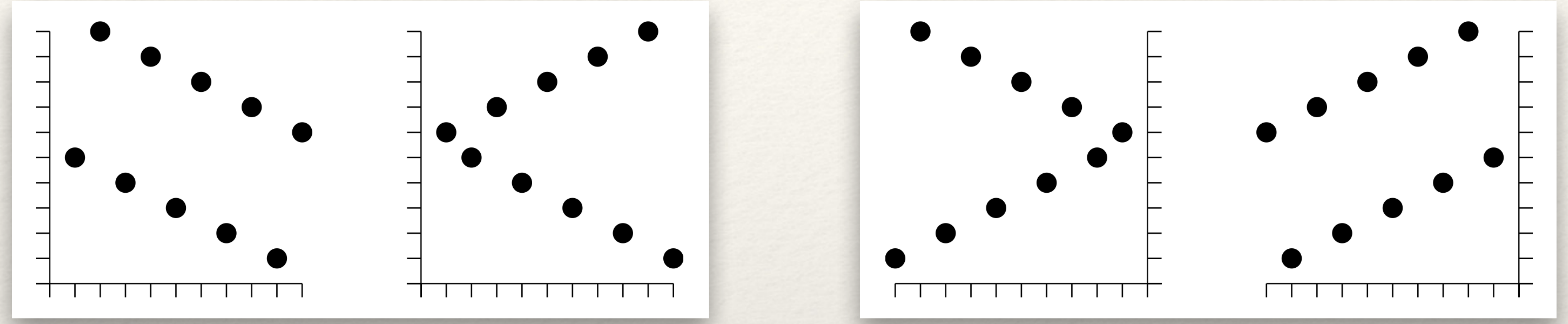
A POP class has a regular insertion encoding if and only if it is bipartite

Proof ($\Longrightarrow$)

A non-bipartite POP contains a chain with $a < b < c$, which implies every basis element contains some size three pattern, so can't meet condition 3.

Theorem

A POP class has a regular insertion encoding if and only if it is bipartite



Proof ($\Longleftarrow$)

Let $A = \{a_1, a_2, \ldots, a_k\}$ be the lower elements of size two chains and $B = \{b_1, b_2, \ldots, b_{n-k}\}$ be the remaining elements

Any permutation in which the values at indices in A are $\{1, 2, \ldots, k\}$ and at indices in B are $\{k+1, k+2, \ldots, n\}$ are in the basis

Setting these as increasing or decreasing gives a permutation from each of the four classes

Moral of the story

If you see a lot of rational generating functions check if your permutation classes have a regular insertion encoding

If you're only interested in one permutation class, it is easy to check with Permuta (https://github.com/PermutaTriangle/Permuta) if it has a regular insertion encoding, using our command line tool

```
christianbean@my-macbook /Users/christianbean
% permtools insenc '31425, 31524, 32415, 32514, 41235, 41325, 42135, 42315, 43125, 43215'
Av(20314,20413,21304,21403,30124,30214,31024,31204,32014,32104) does not have a regular
insertion encoding
christianbean@my-macbook /Users/christianbean
% permtools insenc '123, 3142, 3412'
The class Av(012,2031,2301) has a regular rightmost insertion encoding
```

You can also check for finitely many simple permutations which implies another automatic method

```
christianbean@my-macbook /Users/christianbean
% permtools simple '123, 3142, 3412'
The class Av(012,2031,2301) has infinitely many simples
christianbean@my-macbook /Users/christianbean
% permtools simple '132, 312'
The class Av(021,201) has finitely many simples.
```

Counting size 5 POP classes

TileScope has enumerated all size 4 POP classes as these are defined by avoiding size 4 patterns.
The results can be found at https://permpal.com/

We applied TileScope to size 5 POPs, and collected all our results on PermPAL

We settled five of the six conjectures from Gao and Kitaev

This is Av(31425, 31524, 32415, 32514, 41235, 41325, 42135, 42315, 43125, 43215). The generating function satisfies

$$(4x-1)F(x)^4-(16x-6)F(x)^3+(x^2+24x-13)F(x)^2-(16x-12)F(x)+4x-4=0$$

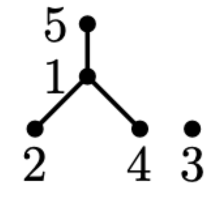
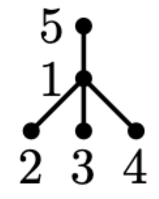
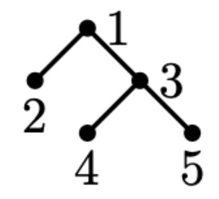
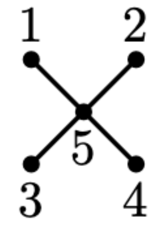
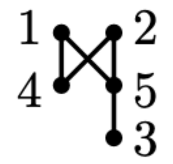
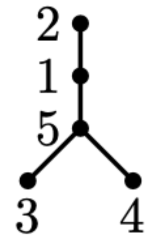
POP	Sequence (beginning with $n = 1$)	OEIS	
	1, 2, 6, 24, 110, 540, 2772, 14704, ...	A216879	✗
	1, 2, 6, 24, 114, 600, 3372, 19824, ...	A054872	✓
	1, 2, 6, 24, 112, 568, 3032, 16768, ...	A118376	✓
	1, 2, 6, 24, 116, 632, 3720, 23072, ...	A212198	✓
	1, 2, 6, 24, 114, 598, 3336, 19402, ...	A228907	✓
	1, 2, 6, 24, 118, 672, 4256, 29176, ...	A224295	?

Table 5: Conjectured connections for POPs of length 5. For the highlighted OEIS sequences no interpretation in terms of permutation patterns was known until this work. Enumeration for A054872 comes from [3]. Also, A212198 comes from [23, 26]. A224295 was obtained using the methods developed in [27].

End on a conjecture

Our data suggests the following

Conjecture
A POP class has a rational generating function if and only if it is bipartite

The proof of our theorem showed that every non-bipartite POP, there is a size three pattern σ such that Av(σ) is a subclass

This is not sufficient since every permutation class is contained in some rational permutation class