Mesh patterns in random permutations

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Work in progress

This is joint work with
Jason Smith
(Nottingham Trent University).

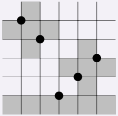
This is work-in-progress.

(All claims are subject to correction.)

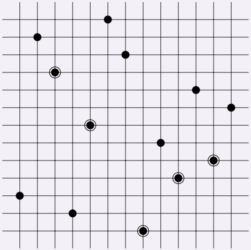
Mesh patterns

A mesh pattern: permutation with shaded cells

The permutation must occur, with the **shaded regions empty**.



An occurrence in a permutation



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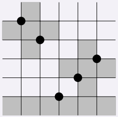

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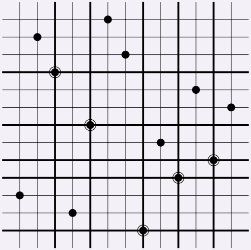

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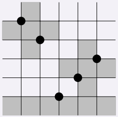

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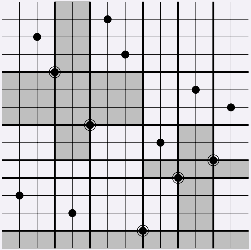

Likelihood

Likelihood of pattern p:
Asymptotic probability that a permutation contains a copy of p.

$$\kappa(p) \;=\; \lim_{n\to\infty} \mathbb{P}\big[\boldsymbol{\sigma}_n \text{ contains } p\big],$$

where $\boldsymbol{\sigma}_n$ is a random n-permutation.

$$\big|\mathsf{Av}_n(p)\big| \;\sim\; \big(1-\kappa(p)\big)n!$$

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$$\big|\mathsf{Av}_n(p)\big| \;\sim\; \big(1-\kappa(p)\big)n!$$

Observation (monotonicity)

The addition of shading never increases the likelihood.

If q is formed from p by shading some unshaded cells of p, then

$$\kappa(q) \;\leqslant\; \kappa(p).$$

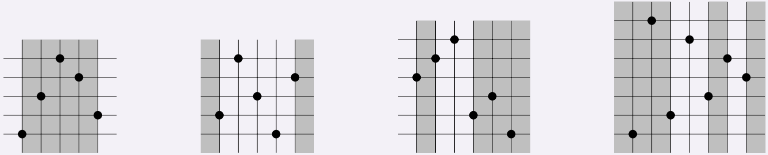
Vincular patterns

Vincular pattern: each column fully shaded or fully unshaded

Specifies adjacency requirements

- between points for medial columns, and
- to the start or end for peripheral columns.

Consecutive pattern:
all medial columns shaded, peripheral columns unshaded



Vincular patterns and beyond

Proposition

If p is any pattern with peripheral columns unshaded, then $\kappa(p) = 1$.

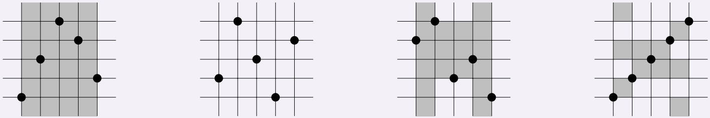
Proof

It suffices to consider consecutive patterns. Suppose $|p| = k$.

The probability of p occurring at a given position is $1/k!$.

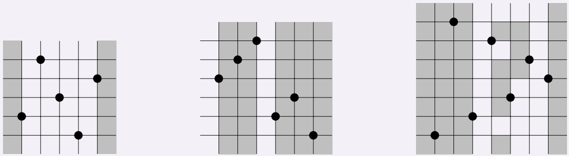
Occurrences of p at non-overlapping locations are independent.

Hence, $\mathbb{P}[\boldsymbol{\sigma}_n \text{ avoids } p] \leqslant (1 - 1/k!)^{\lfloor n/k \rfloor} \to 0$.



Vincular patterns and beyond

Margins:
shaded columns at the left and right adjacent to an unshaded column

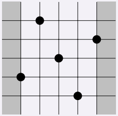


Vincular patterns and beyond

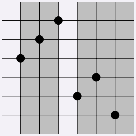
Margins:
shaded columns at the left and right adjacent to an unshaded column

Proposition

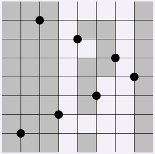
If p is any pattern with margins of width w, then $\kappa(p) = 1/w!$.



$\kappa = \frac{1}{2}$



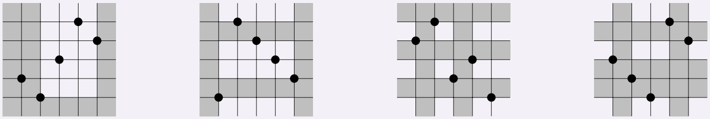
$\kappa = \frac{1}{6}$



$\kappa = \frac{1}{24}$

Bivincular patterns

Bivincular pattern: each shaded cell is in a shaded row or column



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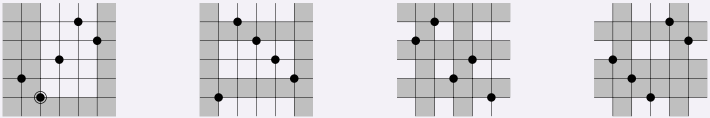

Double anchor: a point with all columns to the left or all columns to the right shaded and all rows above or all rows below shaded

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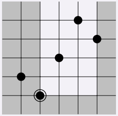
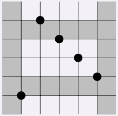
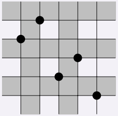
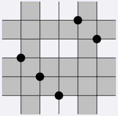








Double anchor: a point with all columns to the left or all columns to the right shaded and all rows above or all rows below shaded

Proposition

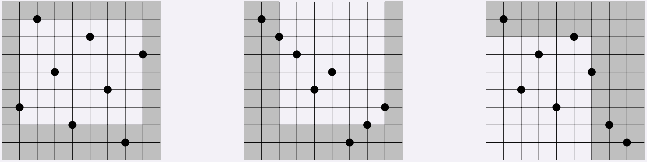
If p has a double anchor, then $\kappa(p) = 0$.

Proof

The probability of the double anchor occurring is $1/n$, which tends to 0.

Frames

Frame: Only columns at left/right and rows at top/bottom shaded.



$\kappa = \frac{1}{12}$

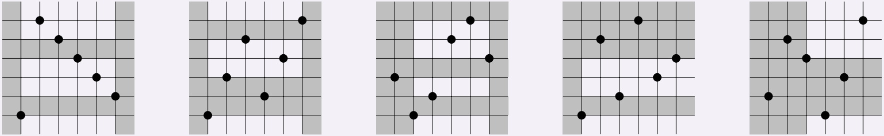
Proposition

If p is a frame with no double anchors, and with c shaded columns and r shaded rows, then

$$\kappa(p) \;=\; \frac{1}{c!r!}.$$

Ladders

Ladder: shaded uprights and rungs



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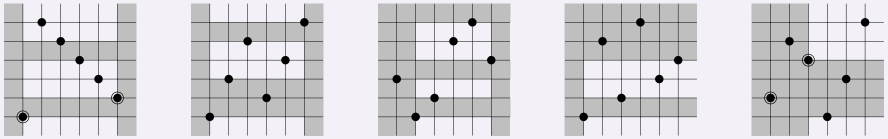
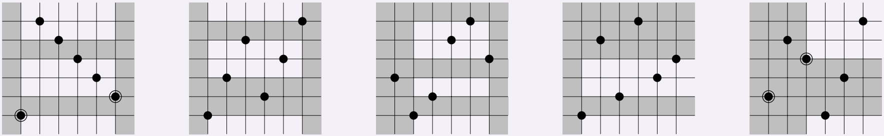


Fixed rung: two points touching both the rung and the uprights

Ladders

Ladder: shaded uprights and rungs



Fixed rung: two points touching both the rung and the uprights

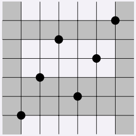
Proposition

If p has a fixed rung, then $\kappa(p) = 0$.

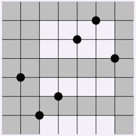
Proof

The probability of two points that fix the rung occurring is $1/n$.

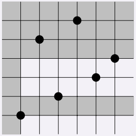
Ladders



$\kappa = \frac{1}{12}$



$\kappa = \frac{1}{36}$



$\kappa = \frac{1}{24}$

Proposition

If p is a ladder with no double anchors or fixed rungs, and with c shaded columns and r shaded rows, then

$$\kappa(p) \;=\; \frac{1}{c!r!}.$$

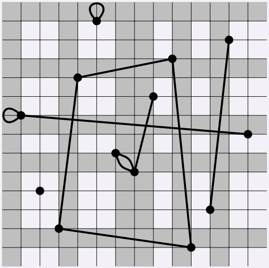
Anchor graphs

Anchor graph G_p of bivincular pattern p:
Points of p as vertices, and one edge for each shaded row or column,

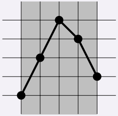
Anchor graphs

Anchor graph G_p of bivincular pattern p:
Points of p as vertices, and one edge for each shaded row or column,

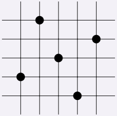
- joining the two adjacent points, if the row or column is medial, or
- forming a loop (anchor) on the adjacent point, if it is peripheral.



Anchor graphs



$\kappa = 1$



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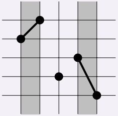

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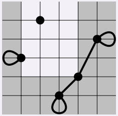

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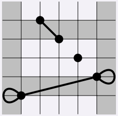

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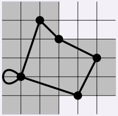

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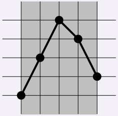

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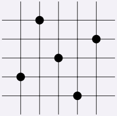

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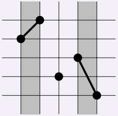

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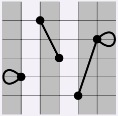
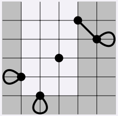


$\kappa = \frac{1}{6}$



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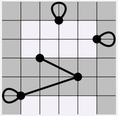

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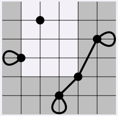

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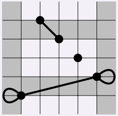

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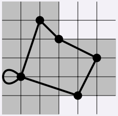
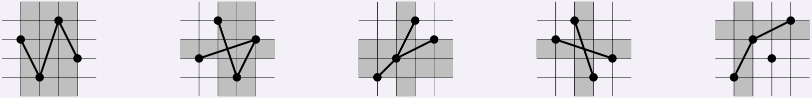


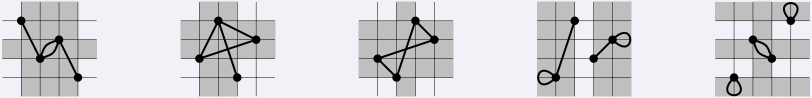
$\kappa = 0$

Anchor graphs

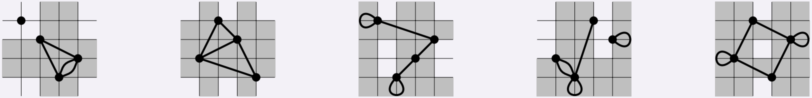
acyclic: no cycles



pseudoforest: not acyclic; no component has more than one cycle



polycyclic: some component has more than one cycle



The Trichotomy

If p is bivincular, then *it would be nice if it were the case that*

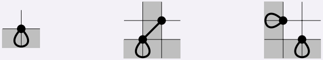
$$\begin{aligned} \kappa(p) &= 0 && \text{if } G_p \text{ is polycyclic} \\ \kappa(p) &\in (0,1) && \text{if } G_p \text{ is a pseudoforest} \\ \kappa(p) &= 1 && \text{if } G_p \text{ is acyclic} \end{aligned}$$

The Small Anchors Theorem

If p is bivincular, then

$$\begin{aligned} \kappa(p) &= 0 && \text{if } G_p \text{ is polycyclic,} \\ \kappa(p) &\in (0,1) && \text{if } G_p \text{ is a pseudoforest } \textbf{but not small anchored}, \\ \kappa(p) &= 1 && \text{if } G_p \text{ is acyclic } \textbf{or small anchored}. \end{aligned}$$

The three (or 16) small anchor graphs ($\kappa = 1$)



G_p is small anchored if it is a pseudoforest whose unicyclic components form one of the small anchor graphs.

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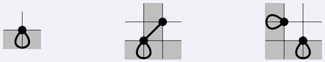
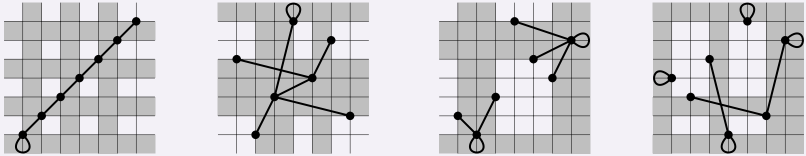


G_p is small anchored if it is a pseudoforest whose unicyclic components form one of the small anchor graphs.

Corollary

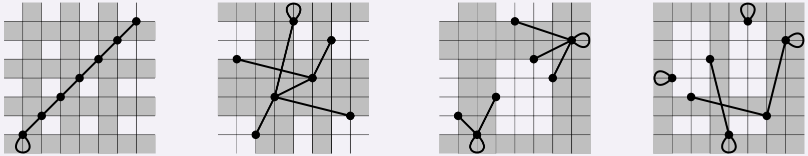
If p has more shaded rows and columns than points, then $\kappa(p) = 0$.

Anchored trees



Anchored tree: bivincular pattern having a connected unicyclic (a.k.a. pseudotree) anchor graph with a loop

Anchored trees

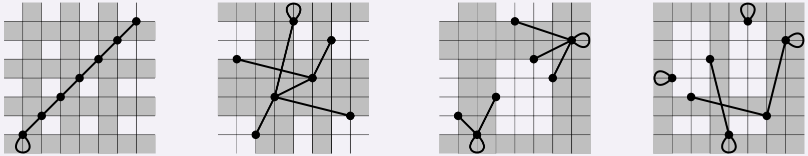


Anchored tree: bivincular pattern having a connected unicyclic (a.k.a. pseudotree) anchor graph with a loop

Proposition

If p is a forest of anchored trees with c shaded columns and r shaded rows, then $\kappa(p) = 1/c!r!$.

Anchored trees



Anchored tree: bivincular pattern having a connected unicyclic (a.k.a. pseudotree) anchor graph with a loop

Proposition

If p is a forest of anchored trees with c shaded columns and r shaded rows, then $\kappa(p) = 1/c!r!$.

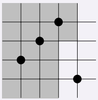
Proposition

Acyclic components don't affect the likelihood.

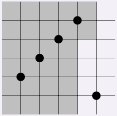
Rational likelihoods

$\mathcal{Q}$: Is there a ***bivincular*** pattern whose likelihood is rational, but is not the reciprocal of a product of two factorials?

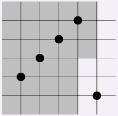
Some non-bivincular patterns with other rational likelihoods



$\kappa = \frac{1}{18}$



$\kappa = \frac{1}{72}$



$\kappa = \frac{1}{96}$

$\mathcal{Q}$: Which rational numbers are likelihoods?

Small steps

Small ascent and small descent

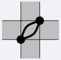
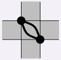
Proposition (known since the 1940s)

If p is either the small ascent or the small descent, then

$$\kappa(p) \;=\; 1 - e^{-1} \;\approx\; 0.63212.$$

Small steps

Small ascent and small descent

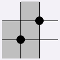
Proposition (known since the 1940s)

If p is either the small ascent or the small descent, then

$$\kappa(p) \;=\; 1 - e^{-1} \;\approx\; 0.63212.$$

$\mathcal{Q}$: Are the small steps the only ***bivincular*** patterns with likelihood strictly between $\frac{1}{2}$ and 1?

$\mathcal{Q}$: What is the greatest likelihood less than 1?

 $\kappa = 1 - J_0(2) \approx 0.77611$

The Chen–Stein Method

The essence of the Chen–Stein Method

If events are

- not too far from being independent, and
- asymptotically, the expected number that occur is constant, then
- the number that occur is asymptotically Poisson.

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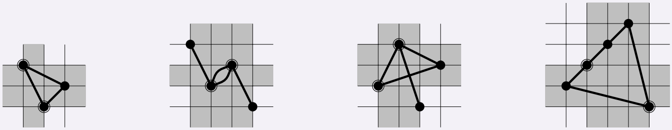
Small ascents

- Small ascents in $\boldsymbol{\sigma}_n$ at different positions are sufficiently close to being independent.
- The expected number of small ascents in $\boldsymbol{\sigma}_n$ equals $1 - 1/n$.
- So the probability of avoiding a small ascent tends to e^{-1}.

Larger steps

(c, r)-step: $c + r$ points,
c columns and r rows shaded, two points touching their intersection



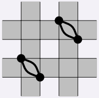
Proposition

If p is a (c, r)-step, then $\kappa(p) = 1 - e^{-1/c!\,r!}$.

Proof

The expected number of occurrences of p is asymptotically $1/c!r!$.

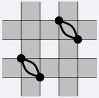
Multiple steps



$$\kappa \;=\; \sum_{k=2}^{\infty} \left(1 - \frac{1}{k!}\right) \frac{e^{-1}}{k!} \;=\; 1 - e^{-1} I_0(2) \;\approx\; 0.16139,$$

where I_0 is a modified Bessel function of the first kind.

Multiple steps



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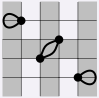
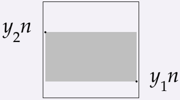
Proposition

If p is a pattern consisting of small steps arranged to form the classical pattern π, then

$$\kappa(p) = 1 - e^{-1} \sum_{k=0}^{\infty} \frac{|\mathsf{Av}_k(\pi)|}{k!^2}.$$

Steps and anchors

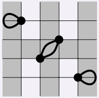


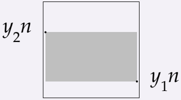


The number of small ascents in the region is asymptotically Poisson with mean $y_2 - y_1$, so

$$\kappa \;=\; \int_0^1 \int_0^{y_2} 1 - e^{-(y_2 - y_1)}\, dy_1\, dy_2 \;=\; \tfrac{1}{2} - e^{-1} \;\approx\; 0.13212.$$

Steps and anchors





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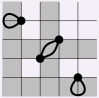
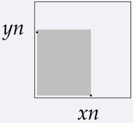




$$\kappa \;=\; \int_0^1 \int_0^1 1 - e^{-xy}\, dx\, dy \;=\; 1 - \gamma + \mathrm{Ei}(-1) \;\approx\; 0.20340,$$

where γ is Euler's constant, and Ei is the exponential integral function.

Lots of special functions

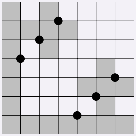
Pattern p	$\kappa(p)$
$(0,1)$-gridded 1-step	e^{-1}
$(0,2)$-gridded 1-step	$\frac{1}{2} - e^{-1}$
$(1,1)$-gridded 1-step	$1 - \gamma + \mathrm{Ei}(-1)$
$(1,2)$-gridded 1-step	$\frac{1}{2} + e^{-1} - \gamma + \mathrm{Ei}(-1)$
$(2,2)$-gridded 1-step	$\frac{5}{4} + e^{-1} - 2\gamma + 2\mathrm{Ei}(-1)$
$(1,1)$-gridded pair of 1-steps	$1 - 2\gamma + 2\mathrm{Ei}(-1) + 2e^{-1/2}\mathrm{Shi}(\frac{1}{2})$
$(0,1)$-gridded split 2-step	$1 - e^{-1/4}\sqrt{\pi}\,\mathrm{erfi}(\frac{1}{2})$
two 1-steps	$1 - e^{-1}I_0(2)$
three 1-steps	$1 - e^{-1}\,{}_1F_2(\frac{1}{2}; 1, 2; 4)$
$(0,1)$-gridded pair of 1-steps	$e^{-1}\big(2 + I_1(2) - I_2(2)\big) - 1$

Beyond bivincular: Nesting

Proposition

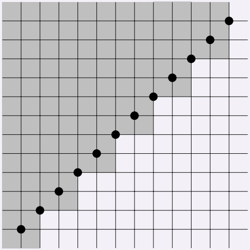
Suppose q is bivincular and $q_1, \ldots, q_k$ are patterns for which $\kappa(q_i)$ is defined. If p is formed by substituting $q_1, \ldots, q_k$ into k of q's unshaded cells no pair of which share a row or column, then

$$\kappa(p) \;=\; \kappa(q) \sum_{i=1}^{k} \kappa(q_i).$$

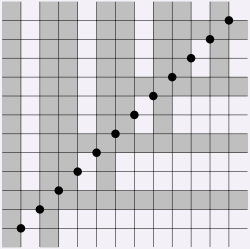


$$\kappa \;=\; (1 - e^{-1})^2 \;\approx\; 0.3995764$$

Iterated nesting



If $|p| = 2k$, then $\kappa(p) = 2^{-k}$



If $|p| = 3k$, then $\kappa(p) = e^{-k}$

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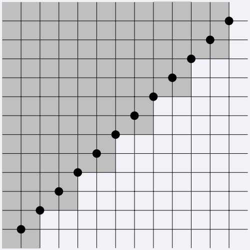

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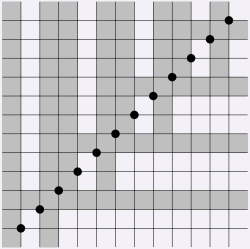

If $|p| = 3k$, then $\kappa(p) = e^{-k}$

Thanks for listening!

Some references

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