Enumerating grid signed permutation classes

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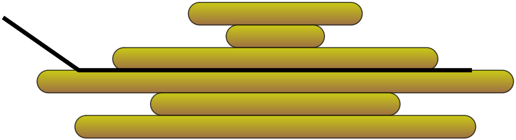
University of Burgundy
Dijon, France

The pancake problem

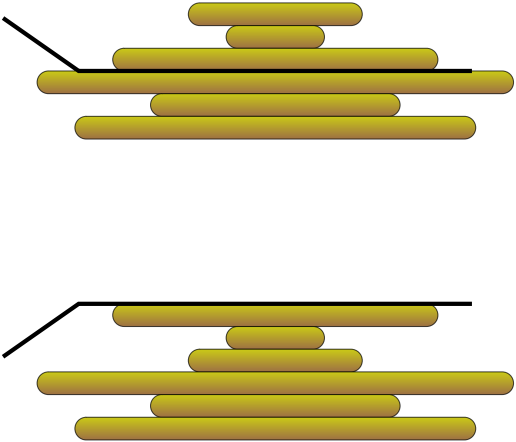
The original statement

The chef in our place is sloppy, and when he prepares a stack of pancakes they come out all different sizes. Therefore, when I deliver them to a customer, on the way to the table I rearrange them (so that the smallest winds up on top, and so on, down to the largest on the bottom) by grabbing several from the top and flipping them over, repeating this (varying the number I flip) as many times as necessary. If there are n pancakes, what is the maximum number of flips (as a function of n) that I will ever have to use to rearrange them?
-Harry Dweighter, December 1975. American Math. Monthly

Pancake flip illustration



Pancake flip illustration



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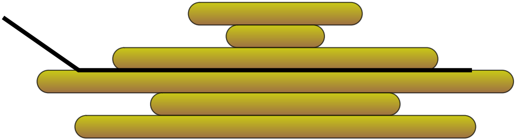

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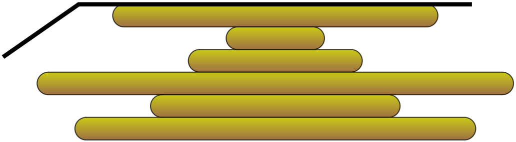

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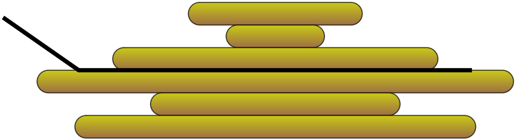

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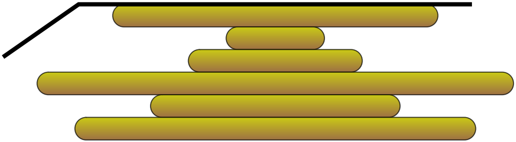
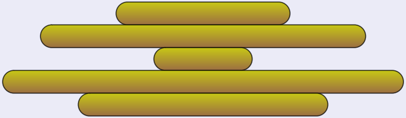


Figure: 412635

Largest pancake first algorithm

Example



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Largest pancake first algorithm

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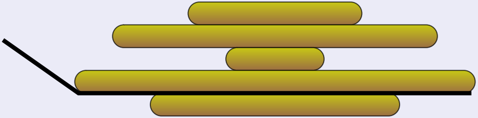

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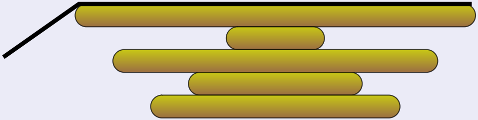

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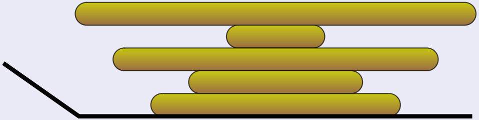

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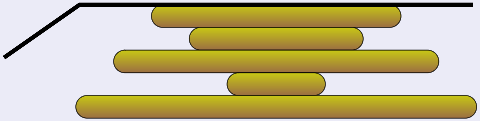

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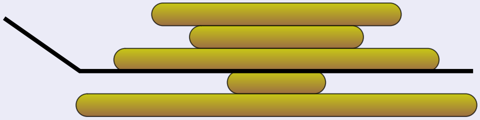

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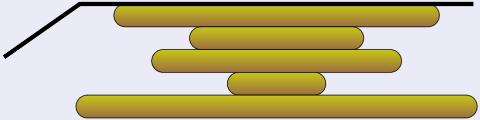

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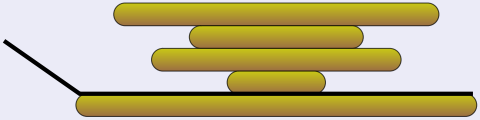

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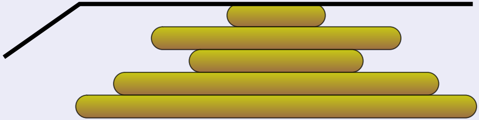

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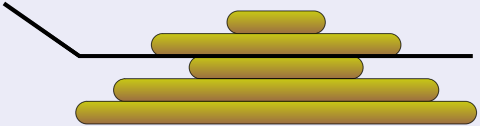

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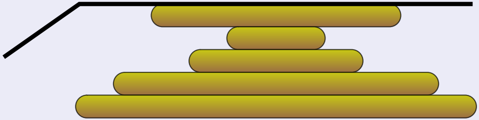

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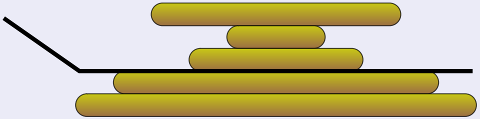

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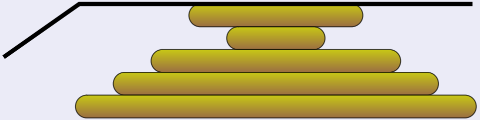

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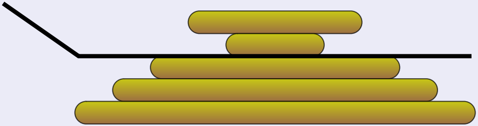

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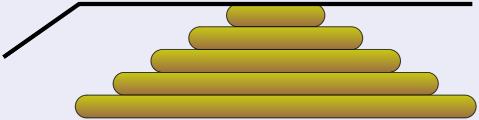

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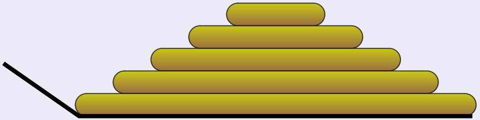

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This took **7** flips!

There is a shorter sequence

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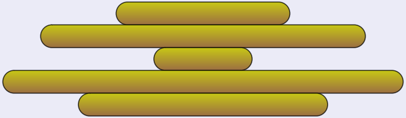

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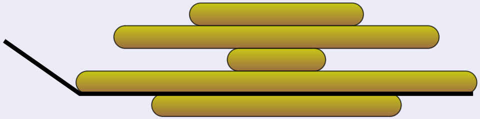

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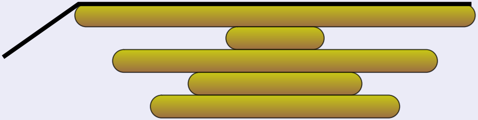

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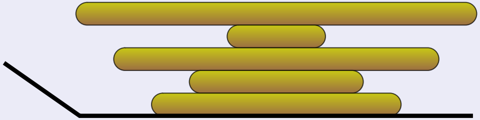

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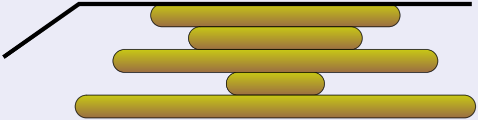

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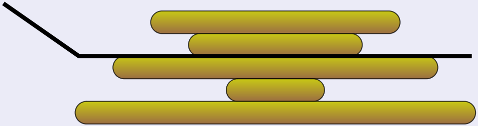

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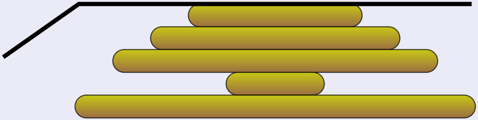

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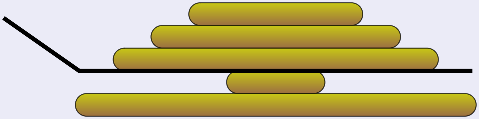

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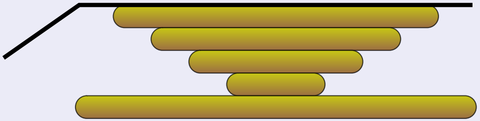

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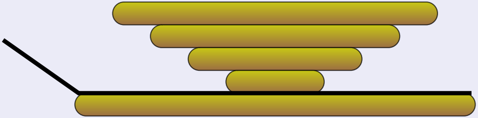

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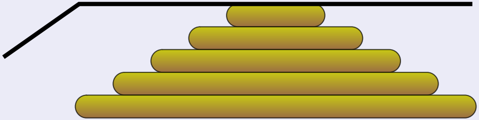

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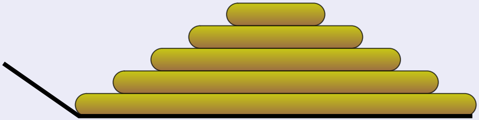

24153 51423 32415 23415 43215 **12345**

This took **5** flips!

$f(n)$, some of what is known

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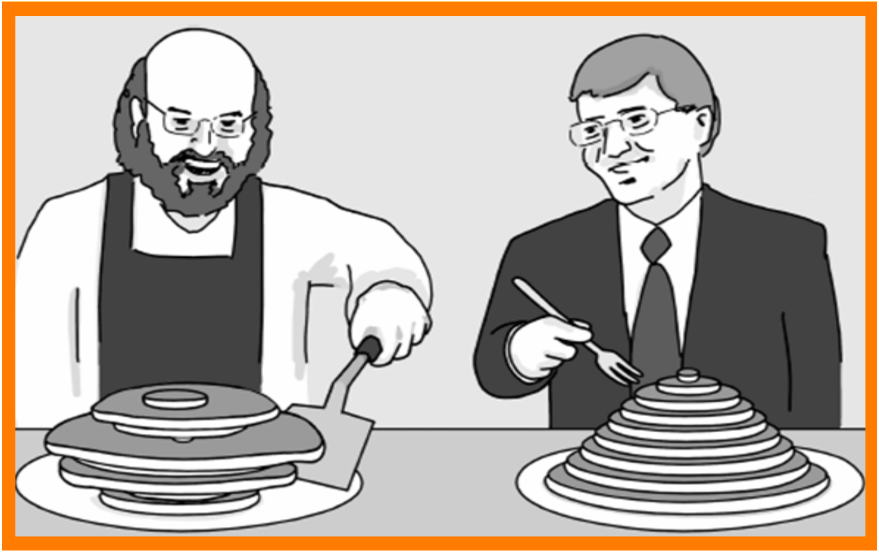
- Pancake sorting is NP-hard: Laurent Bulteau, et al. (2015). To be specific, finding the optimal sequence of flips to sort a stack is NP-hard



Source: Neil Jones and Pavel Pevzner,
2004 “Introduction to BioInformatics Algorithms”

Some applications of pancake sorting

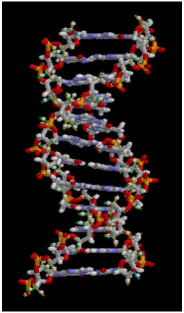


Figure: Genome rearrangements

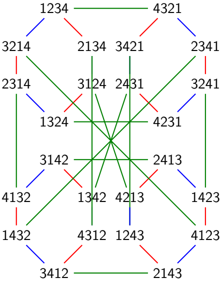


Figure: Parallel processing

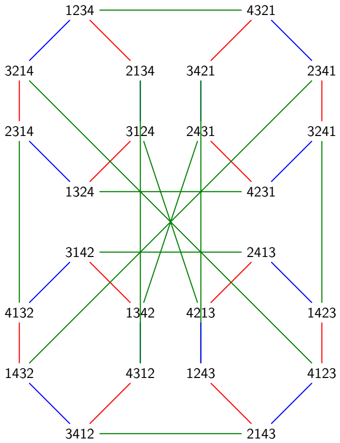
Pancake graphs

We can think of these pancake flips as *prefix-reversal generators* of the symmetric group S_n

$$r_i = i\ (i-1) \cdots 1\ (i+1)\ (i+2) \cdots n,$$

with $2 \leq i \leq n$.

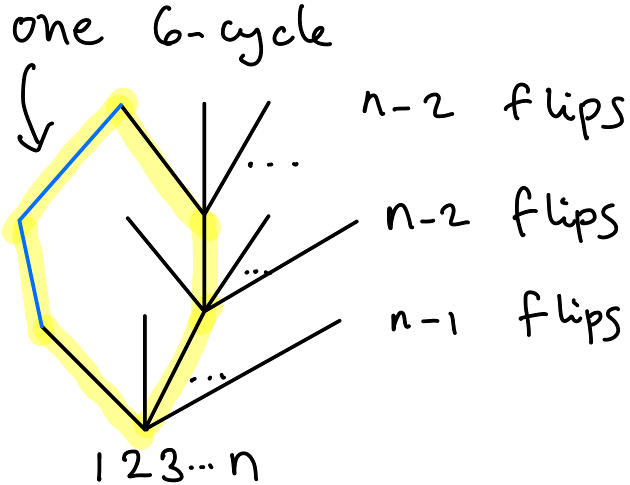
Pancake graph P_4



How many stacks of n pancakes take exactly k flips to be sorted?

A great deal is known about the structure of cycles in P_n. In particular, there is only one type of 6-cycle, and this is the shortest cycle that it exists.

1. If $k = 0$, then 1.
2. If $k = 1$, then $n - 1$.
3. If $k = 2$, then $(n-1)(n-2)$.
4. If $k = 3$, then $(n-1)(n-2)^2 - 1$ (the "-1" comes from the one 6-cycle)



There are $(n-1)(n-2)(n-2)-1$ permutations that require exactly 3 flips to be sorted.

How many stacks of n pancakes take exactly 4 flips to be sorted?

Theorem

(B., Buehrle, and Patidar 2019) If $n \geq 3$, there are

$$\frac{1}{2}(2n^4 - 15n^3 + 29n^2 + 6n - 34)$$

stacks of n pancakes that take exactly 4 flips to be sorted.

The proof used elementary methods like the our classification of 7- and 8-cylces in P_n and PIE. It was a lot of book keeping.

There's gotta to be a better way!

The Homberger–Vatter Algorithm (2016)

Vince Vatter emailed us to tell us about their 2016 paper

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On the effective and automatic enumeration of polynomial permutation classes

Cheyne Homberger [a], Vincent Vatter [b,1]

Permutation classes that are eventually polynomial

The *Fibonacci Dichotomy* of Kaiser and Klazar (2003) : If $\mathcal{C}$ is a permutation class with $|\mathcal{C} \cap S_n| < F_n$ for **some** n, then $|\mathcal{C} \cap S_n|$ is given by a polynomial for *sufficiently large* n.

The HVA algorithm finds this polynomial.

It turns out that the permutations that take *at most* k flips to be sorted is a permutation class which can be *eventually* enumerated by polynomials (it might miss the first few values of n.)

Burnt pancakes

Defined by Gates and Papadimitriou: Suppose pancakes now have an **orientation**. Our goal is to sort them by size **and** orientation.

A *signed permutation* is defined as a permutation w of the set

$$\{-n, -(n-1), \ldots, -1, 1, 2, \ldots, n\}$$

satisfying $w(-i) = -w(i)$ for all i. Usually we overline negative entries and write $\bar{i}$ instead of $-i$.

Normally one uses *window notation* and denotes w by $[w(1), w(2), \ldots, w(n)]$.

In this case, *prefix reversals* are

$$r_i = [\bar{i}, \overline{i-1}, \ldots, \bar{1}, i+1, i+2, \ldots, n]$$

for $1 \leq i \leq n$.

Burnt pancake graph BP_n

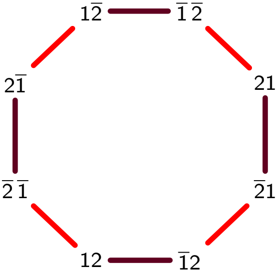
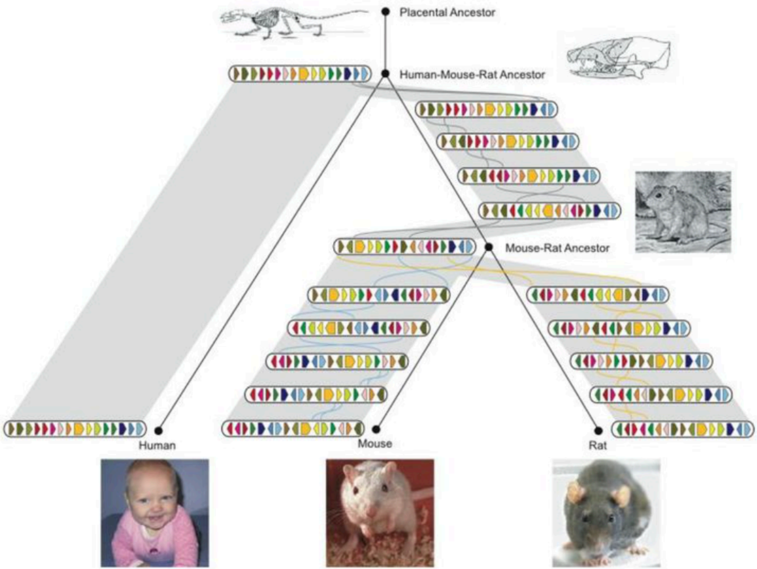


Figure: BP_2 is an 8-cycle.



Rat Consortium, *Nature*, 2004

How many stacks of n burnt pancakes take exactly k flips to be sorted?

A great deal is known about the structure of cycles in BP_n. In particular, we know what 8-cycles are possible (this was an earlier result by our "group.")

1. If $k = 0$, then 1.
2. If $k = 1$, then n.
3. If $k = 2$, then $n(n-1)$.
4. If $k = 3$, then $n(n-1)^2$.

How many stacks of n burnt pancakes take exactly 4 flips to be sorted?

Theorem

(B., Buehrle, and Patidar 2019) If $n \geq 1$, there are

$$\frac{1}{2}n(n-1)^2(2n-3)$$

stacks of n burnt pancakes that take exactly 4 flips to be sorted.

The proof used elementary methods like the our classification of and 8 and 9-cylces in BP_n and PIE. It was a lot of book keeping.

There'a gotta to be a better way!

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Extending HVA to signed permutations

Theorem ((B. and Skora), very informal version)

There is an algorithm to enumerate certain signed permutation classes (grid signed permutation classes) that include the class of stacks of burnt pancakes that take at most k flips to be sorted.

The algorithm has three steps: Completion, compacting, and enumeration.

Inflating a signed permutation

Let $\pi \in B_k$ (group of all signed permutations of length k) and $\mathbf{v} \in (\mathbb{Z}_{\geq 0})^k$ denote the set of all k-dimensional vectors with positive integers components.

Let's say $\pi = \pi_1\pi_2 \cdots \pi_k$ and $\mathbf{v} = (v_1, v_2, \ldots, v_k)$. *Inflating* π by $\mathbf{v}$, denoted by $\pi[\mathbf{v}]$, means

- Replacing each character π_i by a *monotone* interval of length v_i.
- The interval must be increasing if π_i is positive and decreasing if π_i is negative.
- Each of the elements of the interval has the same sign as π_i.
- Finally, the order of these intervals must match the order of the entries of π.

Example

$\bar{1}2[(3,4)] = \bar{3}\,\bar{2}\,\bar{1}4567$, $21\bar{3}[(2,0,3)] = 12\bar{5}\,\bar{4}\,\bar{3}$.

Grid signed permutation classes

The grid class of a $\pi \in B_k$, written $\mathrm{Grid}(\pi)$, is the set of signed permutations

$$\{\pi[\mathbf{v}] : \mathbf{v} \in (\mathbb{Z}_{\geq 0})^k\}$$

If Π is a set of signed permutations (not necessarily of the same length) then $\mathrm{Grid}(\Pi)$ is given by

$$\{\mathrm{Grid}(\pi) : \pi \in \Pi\}.$$

It is easy to see that $\mathrm{Grid}(\Pi)$ is closed under containment (thus a permutation class).

We say that $\pi \in \Pi$ is *compact* if $\pi' < \pi$ implies that $\mathrm{Grid}(\pi') \subset \mathrm{Grid}(\pi)$ (strict inequality and inclusion). For example 12 is NOT compact since $1 < 12$ and $\mathrm{Grid}(\{1\}) = \mathrm{Grid}(\{12\})$.

Enumerating Grid(Π)

1. Completion: Add all $\sigma \leq \bar{2}13$. We get

$$\{\varepsilon, 1, \bar{1}, 12, \bar{1}2, \bar{2}1, \bar{2}13\}$$

2. Compacting: Remove non-compact signed permutations

$$\{\varepsilon, 1, \bar{1}, \cancel{12}, \bar{1}2, \bar{2}1, \bar{2}13\}$$

3. Enumeration: The generating function enumerating the vectors of length m with positive integers is $(\frac{x}{1-x})^m$. So
   - 1 for the empty permutation ε
   - $\frac{x}{1-x} = \sum_{n\geq 0} x^{n+1}$ for 1 and $\bar{1}$,
   - $\frac{x^2}{(1-x)^2} = \sum_{n\geq 0}(n+1)x^{n+2}$ for $\bar{1}2$ and $\bar{2}1$, and
   - $\frac{x^3}{(1-x)^3} = \sum_{n\geq 0}\binom{n+2}{2}x^{n+3}$ for $\bar{2}13$.

Enumerating Grid(Π)

Therefore, by isolating the coefficient of x^n with $n \geq 1$ in the expansion of

$$1 + \frac{2x}{1-x} + \frac{2x^2}{(x-1)^2} + \frac{x^3}{(1-x)^3},$$

we obtain the polynomial

$$P(n) = 2 + 2(n-1) + \binom{n-1}{2} = \frac{n^2}{2} + \frac{n}{2} + 1.$$

Stacks of burnt pancakes that take at most k flips to be sorted

- $\mathrm{Grid}(\{\bar{1}2\})$ is the class representing the stacks of burnt pancakes that take at most one flip to be sorted. The algorithm outputs $n+1$.
- $\mathrm{Grid}(\{2\bar{1}3, \bar{2}13\})$ is the class representing the stacks of burnt pancakes that take at most two flips to be sorted. The algorithm outputs n^2+1.
- There are
$$n^5 - \frac{29}{6}n^4 + \frac{17}{2}n^3 - \frac{26}{6}n^2 + \frac{1}{2}n + 1$$
stacks of burnt pancakes that require at most 5 flips to be sorted, etc...
- These polynomials work for all $n \geq 1$ (they don't miss any values like in the case of S_n.)

Thank you!