Uncountably many well-quasi-ordered permutation classes

Robert Brignall

Joint work with Vince Vatter (U. Florida)

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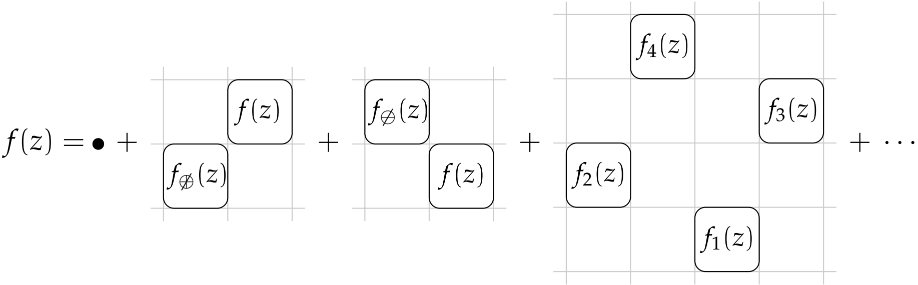
The Open University

Given a class $\mathcal{C}$, what properties guarantee a ‘tame’ enumeration?

Finitely many simple permutations

Theorem (Albert and Atkinson 2005)

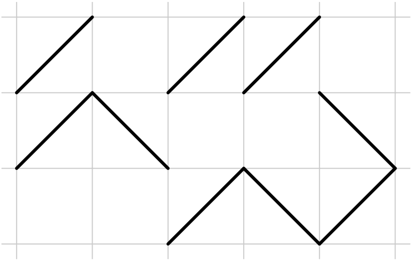
If a class $\mathcal{C}$ contains only finitely many simple permutations, then it has an algebraic generating function and is finitely based.



Geometrically griddable classes

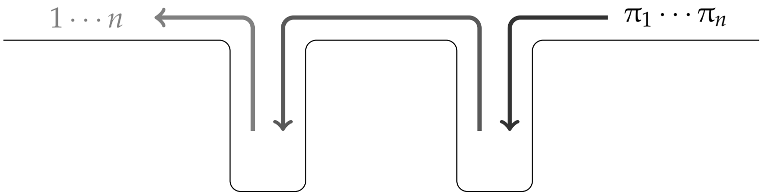
Theorem (Albert, Atkinson, Bouvel, Ruškuc and Vatter 2013)

If a class $\mathcal{C}$ is geometrically griddable, then it has a rational generating function and is finitely based.



$\longleftrightarrow$ regular language over finite alphabet

Non-example: two stacks in series



Pierrot & Rossin (2017) Membership is polynomial time

Elvey Price & Guttman (2017) Exact enumeration to length 20
Generating function $\sim A(1 - \mu \cdot z)^{\gamma}$

Murphy (2003) Not finitely based

Finitely based classes

All classes that have finitely many simples, or that are geometrically griddable are finitely based. Two-stacks are not.

Conjecture (Noonan, Zeilberger, 1996)

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Theorem (Garrabrant, Pak, 2015)

Zeilberger is right: Noonan-Zeilberger is false.

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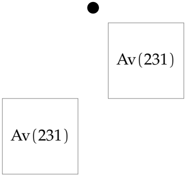
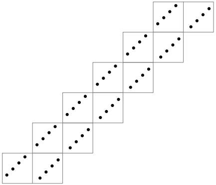
Conjecture

Every finitely based class with growth rate < 4 *has a rational generating function.*

Given a class $\mathcal{C}$, what properties guarantee a 'tame' enumeration?

	Av(231)	Av(321)
Growth rate	4	4
Generating function	$\frac{1-\sqrt{1-4z}}{2z}$	$\frac{1-\sqrt{1-4z}}{2z}$
Basis	231	321
'Look like'		

Subclasses of Av(231), Av(321)

	$\mathcal{C} \subsetneq \mathrm{Av}(231)$	$\mathcal{D} \subsetneq \mathrm{Av}(321)$
Growth rate	Countably many possibilities	Includes $[2.36, 2.48]$ (Bevan, 2018)
Generating function	Rational (Albert, Atkinson, 2005)	Could be anything
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Infinite antichains?	No	Yes:  …

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Every WQO or finitely based subclass of Av(321) has a rational generating function.

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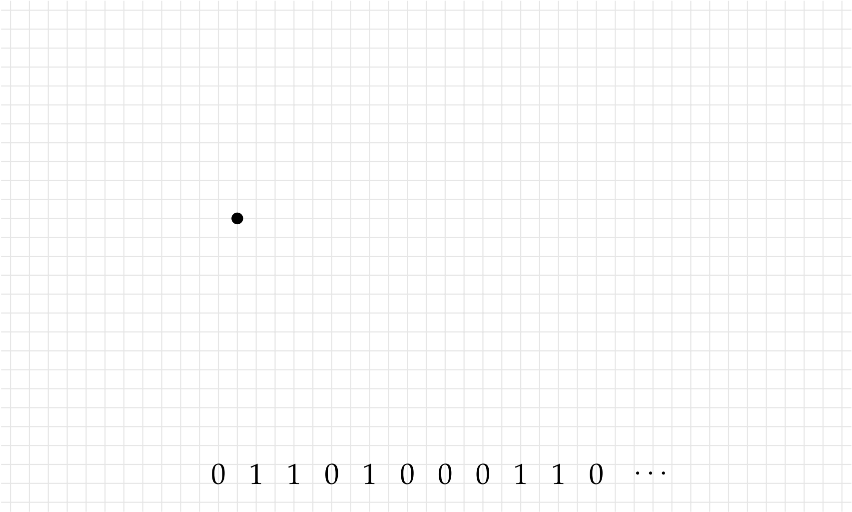
FALSE

$0 \quad 1 \quad 1 \quad 0 \quad 1 \quad 0 \quad 0 \quad 0 \quad 1 \quad 1 \quad 0 \quad \cdots$

Binary word $w \longrightarrow$ permutation π_w



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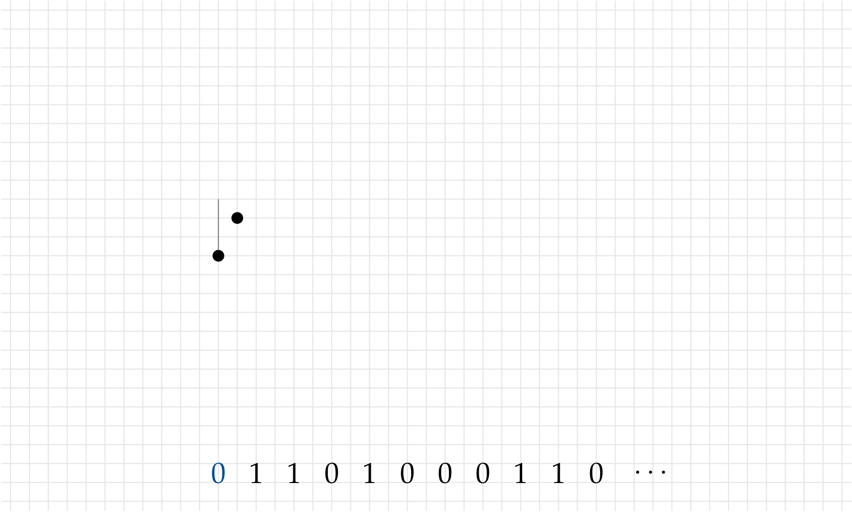

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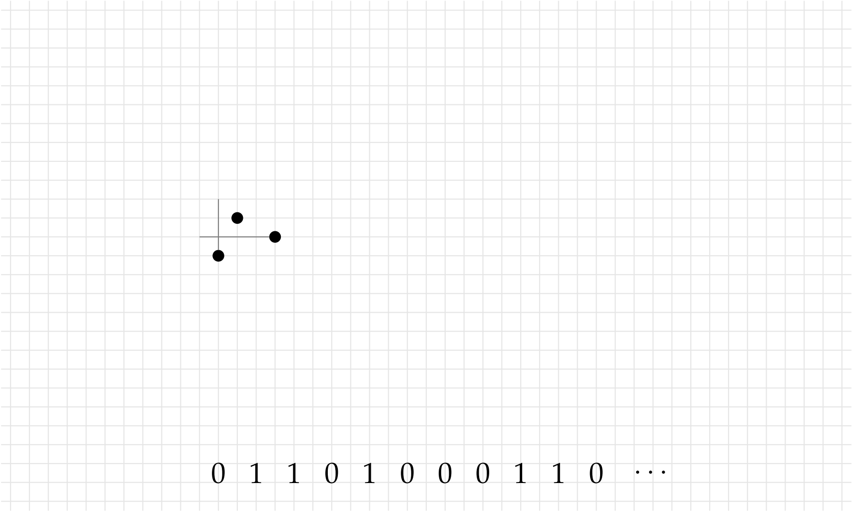

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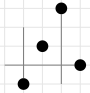

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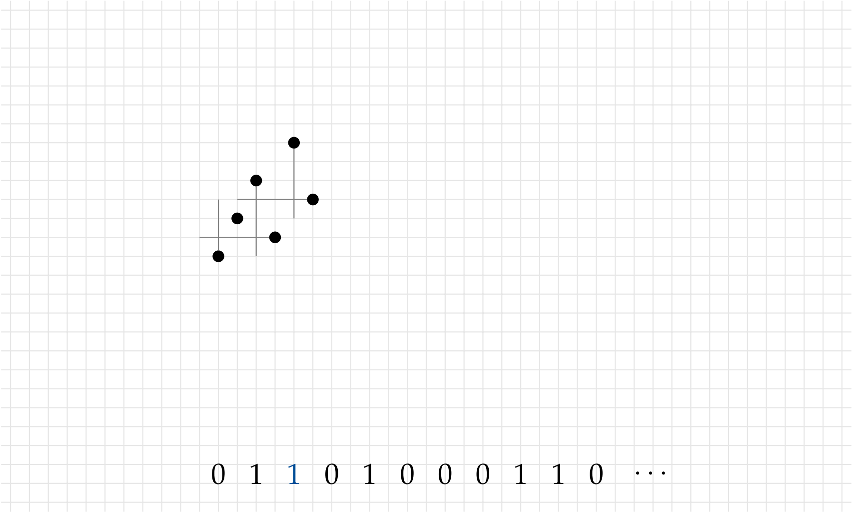

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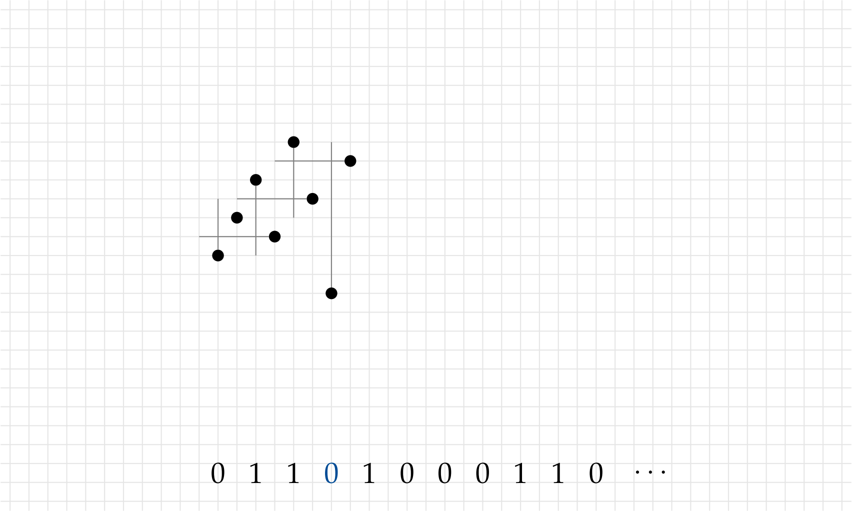

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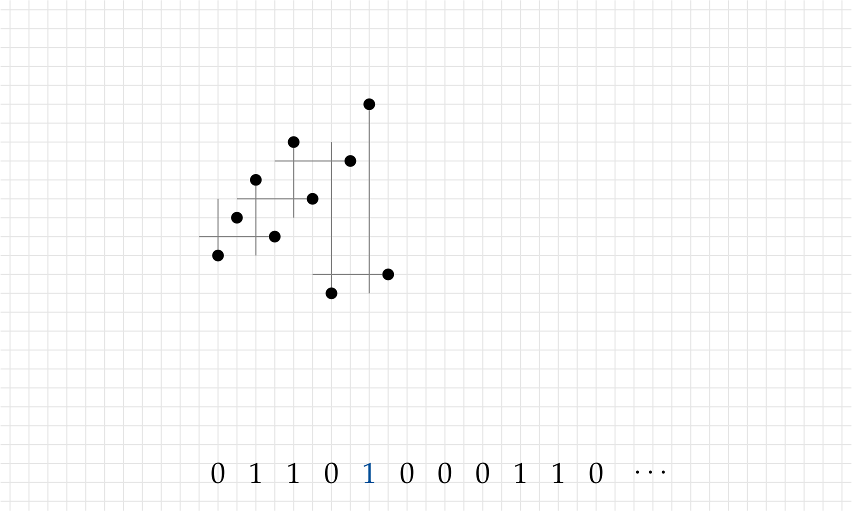

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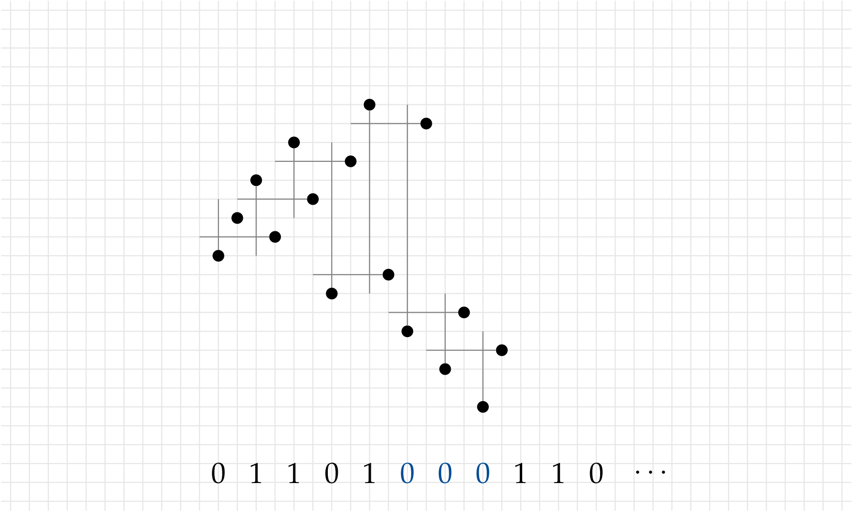

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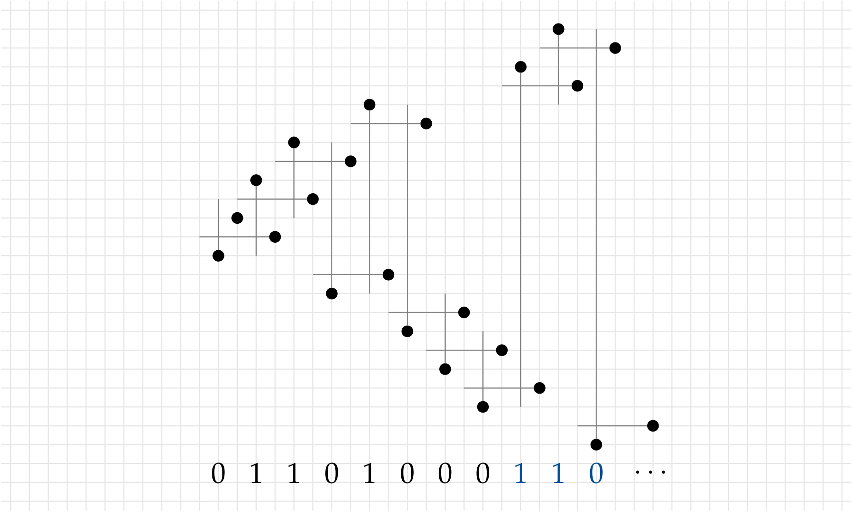

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Prouhet-Thue-Morse

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w_1 &= 01 \\
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$$\mathcal{P} = \{\text{permutations } \pi \text{ contained in } \pi_{w_i} \text{ for some } i\}$$

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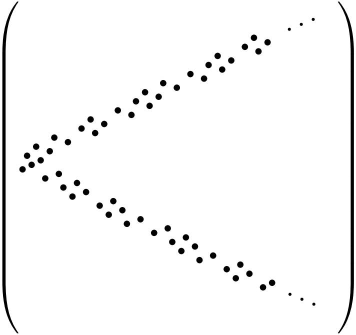

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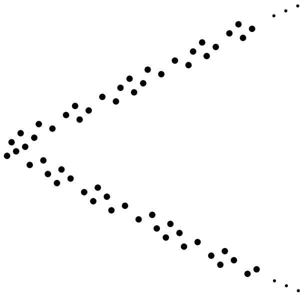
$= \text{Sub}\left(\right.$  $\left.\right)$

Prouhet-Thue-Morse is uniformly recurrent $\Longrightarrow \mathcal{P}$ is WQO.

Prouhet-Thue-Morse '$(1, 1, 1, \ldots)$'

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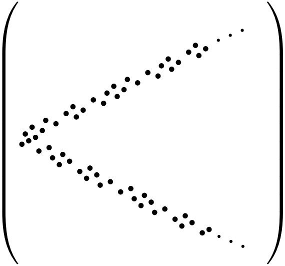
$\mathcal{P} = \mathrm{Sub}$ 

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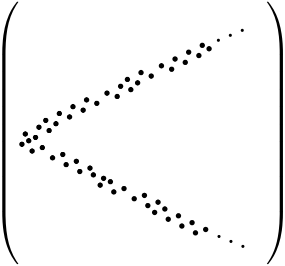
$\mathcal{Q} = \mathrm{Sub}$ 

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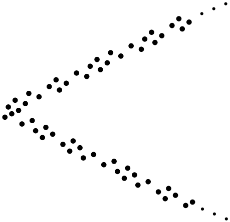
$\mathcal{P} = \mathrm{Sub}\Bigg($  $\Bigg)$

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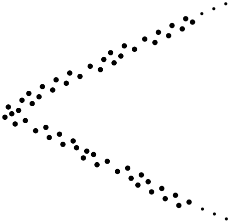
$\mathcal{Q} = \mathrm{Sub}\Bigg($  $\Bigg)$

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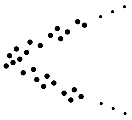

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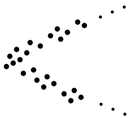

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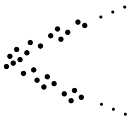
$$01101001\cdots \to \text{Sub}\Bigg($$  $$\Bigg)$$

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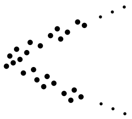

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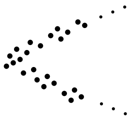

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Corollary

There exist WQO permutation classes that do not have algebraic (or even D-finite) generating functions.

Concluding remarks

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In search of 'tame' enumeration:

- Is *labelled* WQO enough to guarantee algebraic g.f.s?
- (Note: LWQO $\implies$ WQO + finitely based.)

Merci!