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Spherical Dyck paths

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Abstract

We study the *partition Schubert* varieties that are *spherical* ones via *Dyck paths*. Specifically, among the Schubert varieties whose associated permutation are 312-avoiding, we determine which ones are *spherical* varieties by this combinatorial object. We call these lattice paths *spherical Dyck paths*, and we find a recursive formula to count them. On the other hand, a spherical **G**-variety **Y** is *nearly toric variety* if the general codimension of *torus* in **Y** is one. We identify the nearly toric partition Schubert varieties and all *singular* nearly toric Schubert varieties. Moreover, at computing their cardinalities, the *Fibonacci* numbers pop up surprisingly (see [2] for more details).

Algebraic-Geometric Scene

Notation. The algebraic groups and representations are defined over $\mathbb{C}$.

$\mathbf{G}$: connected reductive group	$\mathbf{T}$: maximal torus in $\mathbf{B}$
$\mathbf{B}$: Borel subgroup of $\mathbf{G}$	$\mathbf{W}$: Weyl group of $(\mathbf{G}, \mathbf{T})$
S : Coxeter generators of $(\mathbf{G}, \mathbf{B}, \mathbf{T})$	
$\mathbf{P_I}$: parabolic subgroup generated by $\mathbf{I} \subseteq \mathrm{S}$ and $\mathbf{B}$	
$\mathbf{L_I}$: Levi subgroup of $\mathbf{P_I}$ containing $\mathbf{T}$	$w_0(\mathbf{I})$: longest element of $\mathbf{P_I}$

Definition 1. An irreducible normal $\mathbf{G}$-variety $\mathbf{Y}$ is **spherical** if a Borel subgroup $\mathbf{B}$ of $\mathbf{G}$ has an open orbit in $\mathbf{Y}$.

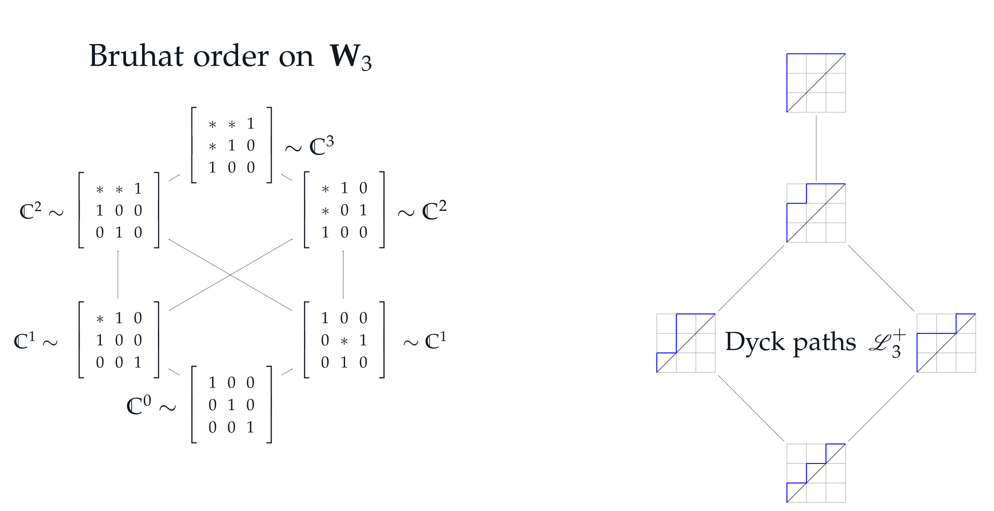
Definition 2. Let $\mathbf{Y}$ be a spherical variety. The $\mathbf{T}$-complexity of $\mathbf{Y}$, denoted by $c_{\mathbf{T}}(\mathbf{Y})$, is the codimension of the maximal torus $\mathbf{T}$ in $\mathbf{Y}$. If the $\mathbf{T}$-complexity of $\mathbf{T}$ is 1, we call $\mathbf{Y}$ a **nearly toric variety**.

Example 1. If $\mathbf{G}$ is the **general linear group** GL_n, the Borel subgroup and maximal torus are the *upper triangular matrices* and the *diagonal matrices* respectively. By the Bruhat-Chevalley decomposition, we obtain the **full flag variety**

$$\mathrm{GL}_n \,/\, \mathbf{B} = \bigsqcup_{w \in \mathbf{W}_n} \mathbf{B}\, w\, \mathbf{B} \,/\, \mathbf{B}$$

where $\mathbf{W}_n$ is the *symmetric group*. In particular, the $\mathbf{B}$-orbit $\mathbf{B}\, w_0\, \mathbf{B} \,/\, \mathbf{B}$ is open in $\mathrm{GL}_n \,/\, \mathbf{B}$. Hence, $\mathrm{GL}_n \,/\, \mathbf{B}$ is a spherical variety.

Definition 3. Let w be in $\mathbf{W}_n$. The **Schubert variety associated** with w is the $\mathbf{B}$-orbit (Zariski) closure $X_{w\mathbf{B}} := \overline{\mathbf{B}\, w\, \mathbf{B} \,/\, \mathbf{B}}$ in $\mathrm{GL}_n \,/\, \mathbf{B}$. Moreover, $X_{w\mathbf{B}}$ is said to be a **partition** Schubert variety if w is a 312-avoiding permutation. Let $\mathbf{W}_n^{312}$ denote the set of all 312-avoiding permutations.

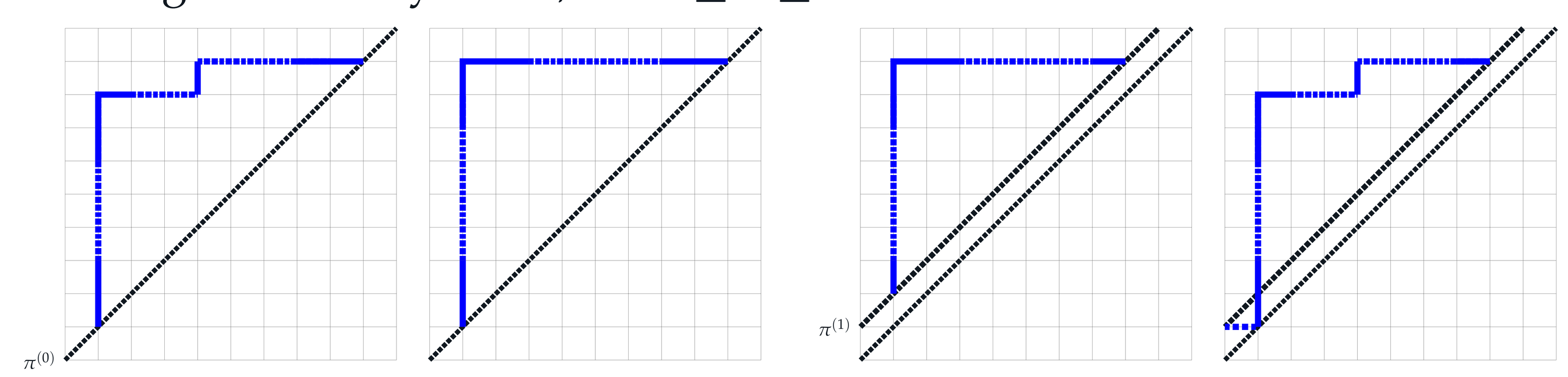


Definition 4. Let $\mathbf{B_L}$ be Borel subgroup of $\mathbf{L}$ containing $\mathbf{T}$. The Schubert variety $X_{w\mathbf{B}}$ is spherical if $\mathbf{B_L}$ has only finitely many orbits in $X_{w\mathbf{B}}$.

X-ray: Combinatorics

Definition 5. A *Dyck path* π is an **elbow** if its *Dyck word* has the form NN...NEE...E, where the number of N's and E's are equal. A Dyck path π is an **ledge** if its Dyck word has the form NN...NE...ENE....EE starting with $(n-1)$-N steps followed by n-E steps, a unique N step, and ends with at least two E steps.

Definition 6. Let $\pi = a_1 a_2 ... a_r$ be a Dyck word. We say that a Dyck path π' is a E_+ *extension* of π if $\pi' = \mathsf{E}\,\pi$. A portion τ of $\pi^{(r)}$ is said to be a **connected component** if τ starts and ends at the r-th diagonal, and it intersects the r-th diagonal exactly twice, for $0 \le r \le n-1$.



(a) Ledge and elbow of $\pi^{(0)}$ (b) Elbow and ledge of $\pi^{(1)}$

Definition 7. A Dyck path π is called **spherical** if every connected component on the first diagonal $\pi^{(0)}$ is either an elbow or a ledge as depicted in (a), or every connected component of $\pi^{(1)}$ is an elbow, or a ledge whose E_+ extension is the final step of a connected component of $\pi^{(0)}$ as shown in (b).

Definition 8. The **Bruhat–Chevalley** on $\mathbf{W}$ is the partial order defined by

$$v \le w \iff X_{v\mathbf{B}} \subseteq X_{w\mathbf{B}}, \qquad \ell(w) := \dim X_{w\mathbf{B}}.$$

Definition 9. Let $\mathrm{J}(w) := \{s \in \mathrm{S} : \ell(sw) < \ell(w)\}$ denote the **left descent** set of w. The Levi factor $\mathbf{L_I}$ of $\mathbf{P_I}$ is given by $\mathbf{I} = \mathrm{J}(w)$. A **standard Coxeter element** $\mathbf{c}$ in $\mathbf{W_I}$ is any product of the elements of $\mathbf{I}$ sorted out in some order.

Example 2. Let $w = 2\,3\,1\,8\,7\,6\,9\,5\,4\,10$ be in $\mathbf{W}_{10}$. We parse

$$w \in \mathbf{W}_{10}^{312}, \quad \mathrm{J}(w) = \{s_2, s_4, s_5, s_6, s_7\}, \quad w_0(\mathrm{J}(w)) = s_1 s_4 s_5 s_4 s_6 s_5 s_4 s_7 s_6 s_5 s_4.$$

Classification

Gao-Hodges-Yong [5]. A Schubert variety $X_{w\mathbf{B}}$ is spherical if and only if $w_0(\mathrm{J}(w))w$ is a standard Coxeter element (*Boolean*).

$$w = 2\,3\,1\,8\,7\,6\,9\,5\,4\,10 \rightsquigarrow w_0(\mathrm{J}(w))w = s_2 s_8 s_7 = \mathbf{c}.$$

Gaetz [4]. A Schubert variety $X_{w\mathbf{B}}$ is spherical if and only if w avoids the following 21 patterns

$$\mathscr{P} := \begin{Bmatrix} 24531 & 25314 & 25341 & 34512 & 34521 & 35412 & 35421 \\ 42531 & 45123 & 45213 & 45231 & 45312 & 52314 & 52341 \\ 53124 & 53142 & 53412 & 53421 & 54123 & 54213 & 54231 \end{Bmatrix}.$$

Can-Diaz [2]. Let w be in $\mathbf{W}_n^{312}$. Let π denote the Dyck path of size n corresponding to w. Then $X_{w\mathbf{B}}$ is a spherical Schubert variety if and only if π is spherical Dyck path.

Lee-Masuda-Park [6]. $c_{\mathbf{T}}(X_{w\mathbf{B}}) = 1$ and smooth $\iff$ w contains the pattern 321 exactly once and avoids 3412 $\iff$ there exists a reduced word of w containing $s_i s_{i+1} s_i$ as a factor and no other repetitions. Moreover, $c_{\mathbf{T}}(X_{w\mathbf{B}}) = 1$ and singular $\iff$ w contains the pattern 3412 exactly once and avoids the pattern 321.

Corollary 1 (Can-Diaz). If $c(X_{w\mathbf{B}}) = 1$ and w in $\mathbf{W}^{312}$, then $X_{w\mathbf{B}}$ is **nearly toric variety**. Moreover, its cardinality is $2^{n-3}(n-2)$ for $n \ge 4$.

Can-Diaz [2]. Let $X_{w\mathbf{B}}$ be a singular Schubert variety of $\mathbf{T}$-complexity 1. Then $X_{w\mathbf{B}}$ is nearly toric variety (There is a geometric proof in [3]). Furthermore, let b_n be the cardinality of this family. Then the generating series of b_n is given by A001871-OEIS.

Bankston-Diaz. Let $\mathscr{S}_n$ be the set of spherical Dyck paths.

$$|\mathscr{S}_n| = \begin{cases} 1 & n = 1 \\ \sum_{k=2}^{n-1} |\mathscr{S}_{n-k}| \pi_k^{(1)} + \pi_n^{(1)} + |\mathscr{S}_{n-1}| & n \ge 2 \end{cases}, \quad \pi_n^{(1)} = \begin{cases} 1 & 1 \le n \le 2 \\ 3 \cdot 2^{n-3} - 1 & n \ge 3 \end{cases}.$$

$\pi_n^{(1)}$ counts the independence number of *n-Mylcielski graph* based on A266550-OIES.

Conjuncture. If $w = 25314$, we found out that $c_{\mathbf{T}}(X_{w\mathbf{B}}) = 1$ is smooth, yet $c_{\mathbf{B_L}}(X_{w\mathbf{B}}) \neq 0$. By using [7], the sequence

n	1	2	3	4	5	6	7	8	9
r_n	0	0	1	6	24	84	275	864	2639

$\rightsquigarrow r_{n+2} = n \cdot \mathscr{F}_{2n}, \quad n \ge 0$

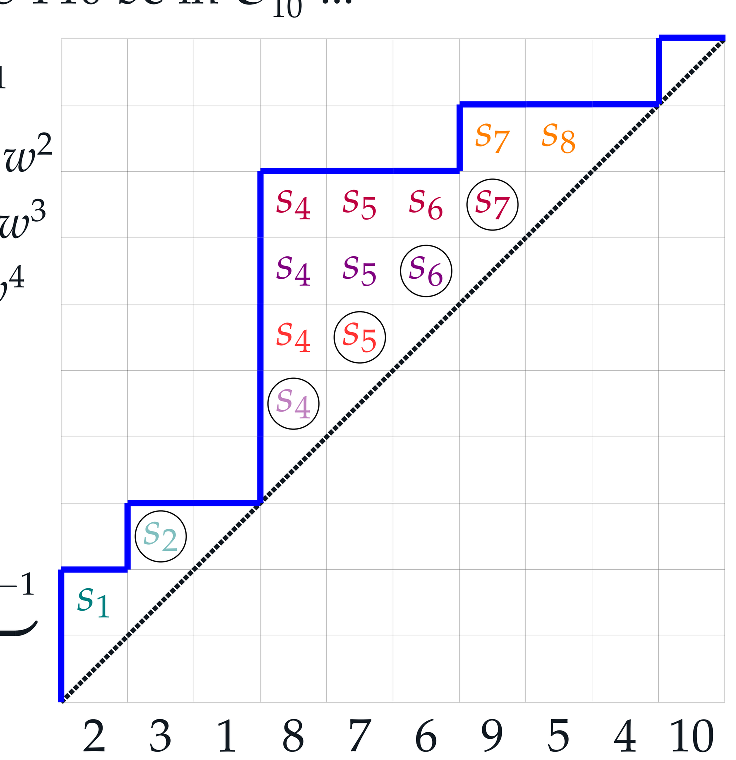
depicted in A317408-OEIS.

Sketchy Proof

Let $w = 2\,3\,1\,8\,7\,6\,9\,5\,4\,10$ be in $\mathfrak{S}_{10}^{312}$...

$$w s_7 s_8 = 2\,3\,1\,8\,7\,6\,5\,4\,\mathbf{9}\,10 := w^1$$
$$w^1 s_4 s_5 s_6 s_7 = 2\,3\,1\,7\,6\,5\,4\,\mathbf{8}\,9\,10 := w^2$$
$$w^2 s_4 s_5 s_6 = 2\,3\,1\,6\,5\,4\,\mathbf{7}\,8\,9\,10 := w^3$$
$$w^3 s_4 s_5 = 2\,3\,1\,5\,4\,\mathbf{6}\,7\,8\,9\,10 := w^4$$
$$w^4 s_4 = 2\,3\,1\,4\,\mathbf{5}\,6\,7\,8\,9\,10 := w^5$$
$$w^5 s_2 = 2\,1\,\mathbf{3}\,4\,5\,6\,7\,8\,9\,10 := w^6$$
$$w^6 s_1 = 1\,\mathbf{2}\,3\,4\,5\,6\,7\,8\,9\,10 := e$$

$$\underbrace{\left(s_7 s_8 | s_4 s_5 s_6 s_7 | s_4 s_5 s_6 | s_4 s_5 | s_4 | s_2 | s_1\right)^{-1}}_{\mathrm{Red}(w)}$$



This construction was developed by **Bandlow-Killpatric** in [1]

$$\mathbf{W}_n^{312} \underset{\phi}{\overset{\psi}{\rightleftarrows}} \mathscr{L}_n^+ \; ; \; \ell(w) \longmapsto \mathrm{area}(\psi(w)) := \pi.$$

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