Fishburn trees

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The Fishburn numbers

$$\sum_{n\geq 0}\prod_{k=1}^{n}\left(1-(1-x)^k\right) = 1 + x + 2x^2 + 5x^3 + 15x^4 + 53x^5 + 217x^6 + \dots$$

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Milestone papers

(I) *(2+2)-free posets, ascent sequences and pattern avoiding permutations*,
M. Bousquet-Melou, A. Claesson, M. Dukes, S. Kitaev, 2010.

-Ascent sequences (and modified ascent sequences);

-Fishburn permutations;

-Unlabeled (**2**+**2**)-free posets;

-Stoimenow matchings.

The Fishburn numbers

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Milestone papers

(II) *Ascent sequences and upper triangular matrices containing non-negative integers*,
M. Dukes, R. Parviainen, 2010.

-Fishburn matrices and ascent sequences.

The Fishburn numbers

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Milestone papers

(iii) *Composition matrices, (2+2)-free posets and their specializations*, M. Dukes, V. Jelínek, M. Kubitzke, 2011.

-Fishburn matrices and (**2**+**2**)-free posets.

The Fishburn numbers

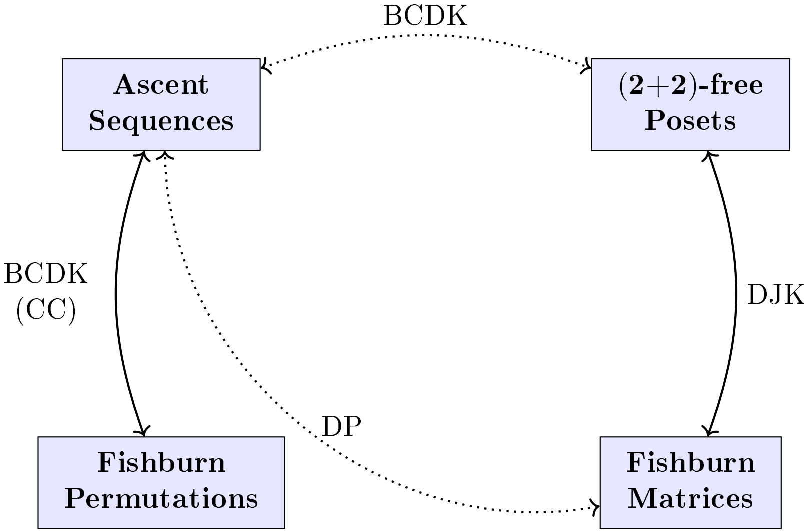
$$\sum_{n\geq 0}\prod_{k=1}^{n}\left(1-(1-x)^k\right) = 1 + x + 2x^2 + 5x^3 + 15x^4 + 53x^5 + 217x^6 + \dots$$

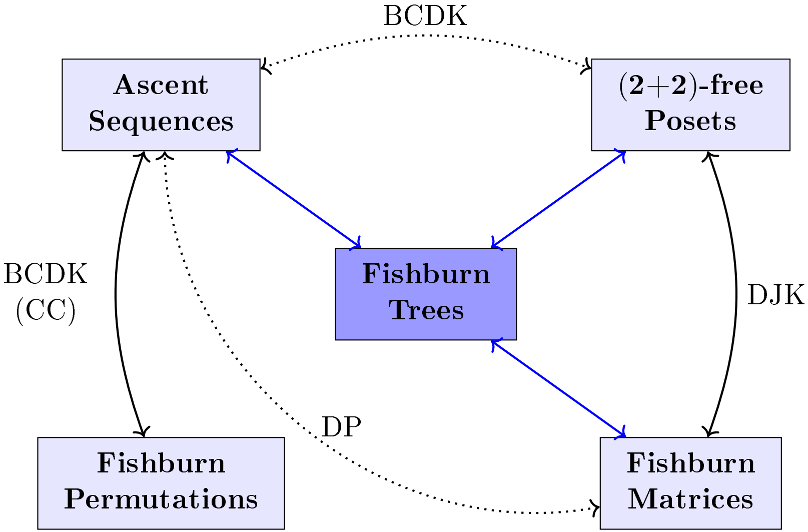
One more reference

(IV) *Transport of patterns by Burge transpose*,
C., Claesson, PP2022.

-Modified ascent sequences and Fishburn permutations;

-Transport of patterns based on the Burge transpose.


BCDK
Ascent Sequences
(2+2)-free Posets
BCDK
(CC)
DJK
DP
Fishburn Permutations
Fishburn Matrices


BCDK
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Modified Ascent Sequences

Combinatorial definition (C., Claesson, 2020)

A sequence $x = x_1 \cdots x_n$ is a **modified (ascent) sequence** if:

1. $x_1 = 1$;
2. $\{x_1, \ldots, x_n\}$ is an interval;
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$$\{x_1\} \cup \{\text{Ascent tops}\} = \{\text{Leftmost copies}\}$$

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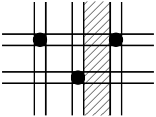
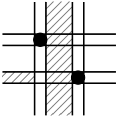
Cayley permutations avoiding the Cayley-mesh patterns

 and 

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Example

$$\mathrm{x} = \underline{1}\ 1\ \underline{4}\ 1\ 1\ 1\ \underline{2}\ 2\ \underline{5}\ 2\ 2\ \underline{3}$$

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Non-example

$$\mathrm{x} = 1\ 2\ 1\ \boxed{2}$$

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Another non-example: why?

$$\mathrm{x} = 1\ 2\ 1\ 4\ 3$$

FROM MODIFIED SEQUENCES TO FISHBURN TREES

- Let x_m be the leftmost copy of $\max(x)$ in x;
- The **MAX-DECOMPOSITION** of $x = x_1 \dots x_n$ is

$$\boxed{x \;=\; \mathrm{pref}(x)\, x_m\, \mathrm{suff}(x)}$$

From modified sequences to Fishburn trees

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- The **max-decomposition** of $x = x_1 \ldots x_n$ is

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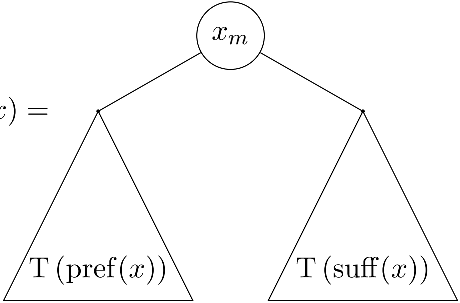
The **rooted**, **labeled**, **binary** tree $\mathrm{T}(x)$ is defined by $\mathrm{T}(\emptyset) = \emptyset$ and:

$\mathrm{T}(x) =$



$$x = \quad 1 \quad 1 \quad 4 \quad 1 \quad 1 \quad 1 \quad 2 \quad 2 \quad 5 \quad 2 \quad 2 \quad 3$$

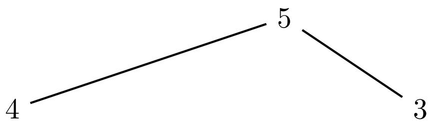
$$\mathrm{T}(x) =$$

$x =$ 1 1 4 1 1 1 2 2 5 2 2 3

5

$\mathrm{T}(x) =$

$x =$ 1 1 4 1 1 1 2 2 5 2 2 3



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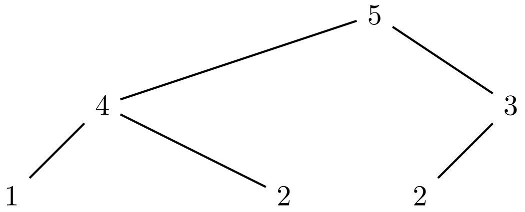
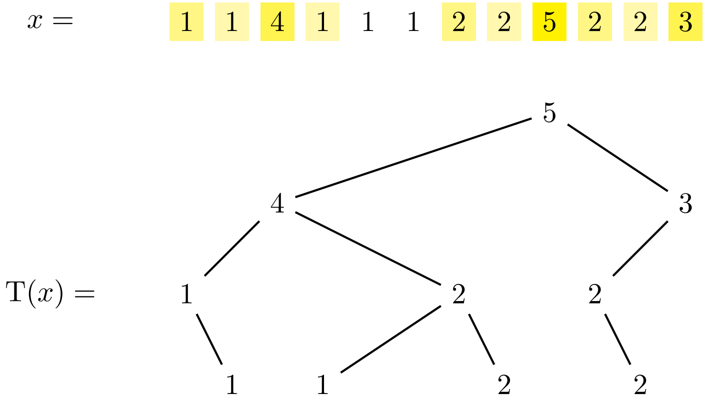
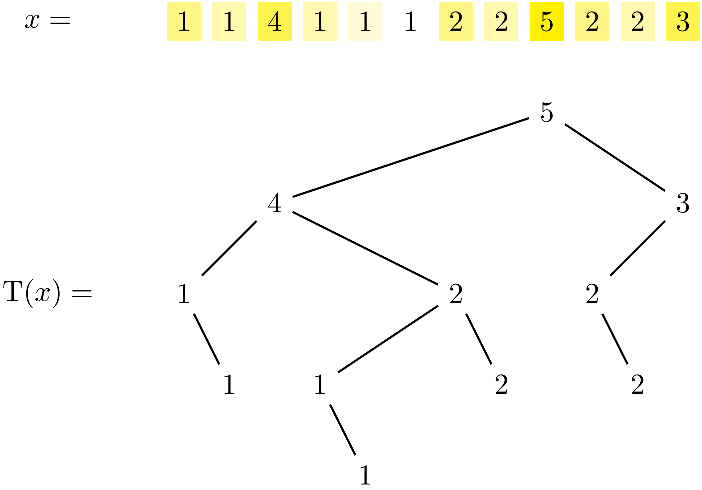
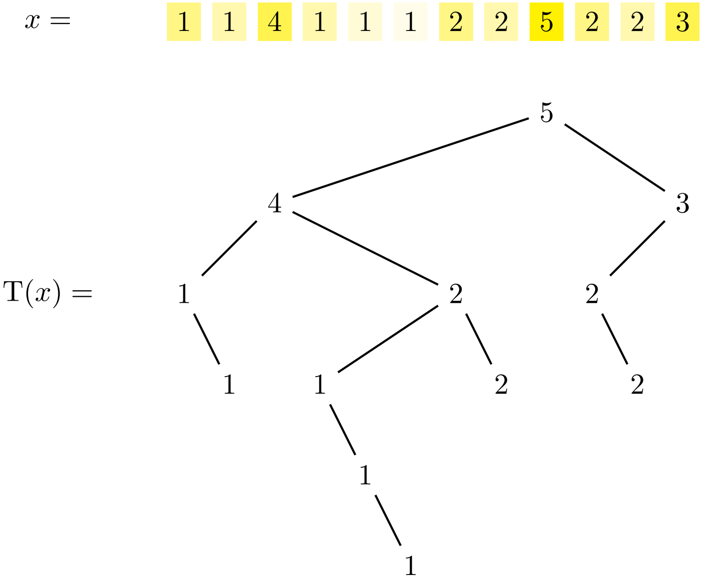




x = 1 1 4 1 1 1 2 2 5 2 2 3
T(x) =
5
4 3
1 2 2
1 1 2 2



x =
1 1 4 1 1 1 2 2 5 2 2 3
T(x) =
5
4 3
1 2 2
1 1 2 2
1



x =
1 1 4 1 1 1 2 2 5 2 2 3
T(x) =
5
4 3
1 2 2
1 1 2 2
1
1

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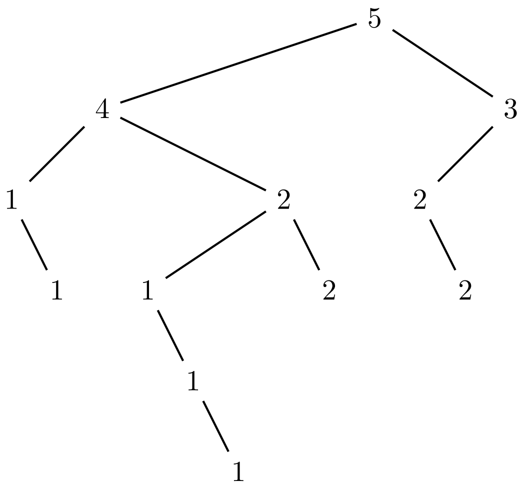
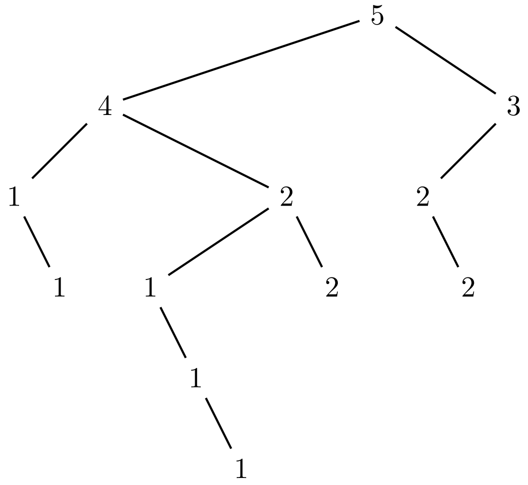
$\mathrm{T}(x) =$



Can you guess how to determine x from the tree $\mathrm{T}(x)$?

$x = \; 1 \; 1 \; 4 \; 1 \; 1 \; 1 \; 2 \; 2 \; 5 \; 2 \; 2 \; 3$

$\mathrm{T}(x) =$



Can you guess how to determine x from the tree $\mathrm{T}(x)$?

We use the in-order traversal!

Let $T = (L, r, R)$ be the rooted, labeled, binary tree with:

- Left subtree L;
- Root r with label $\mathfrak{l}(r)$;
- Right subtree R

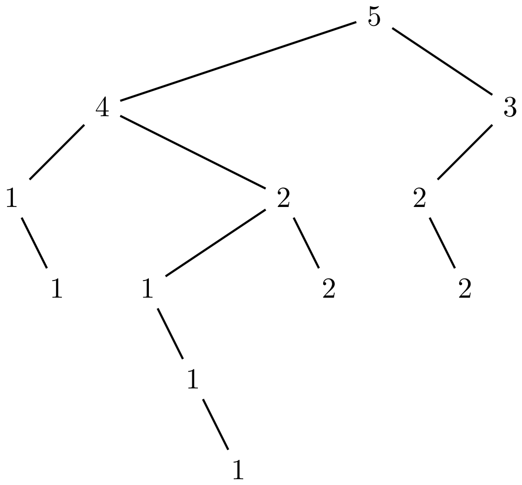
In-order sequence

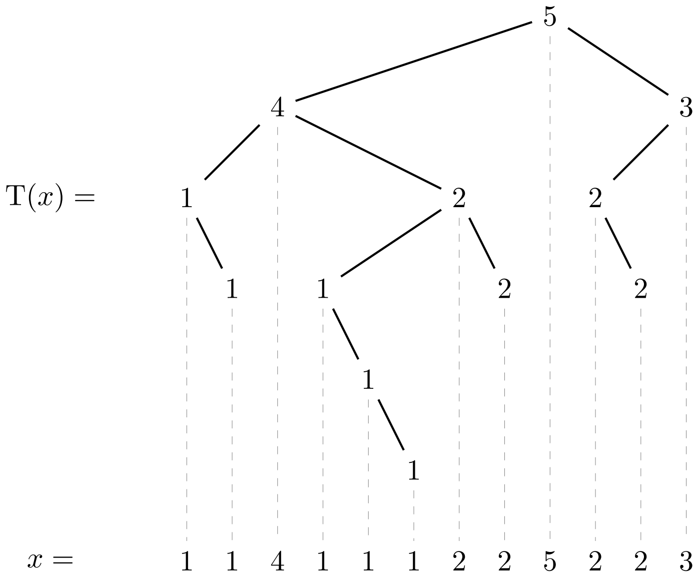
The in-order sequence of T is defined recursively by:

$$x(\emptyset) = \emptyset \qquad \text{and} \qquad x(T) \;=\; x(L)\;\mathfrak{l}(r)\;x(R).$$

"In-order traverse T and write down the label of each visited node".

$\mathrm{T}(x) =$




T(x) =
x = 1 1 4 1 1 1 2 2 5 2 2 3
5
4 3
1 2 2
1 1 2 2
1
1

Properties of $\mathrm{T}(x)$

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Properties of $T(x)$

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2. **BINARY**: Each node has either:
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A **Fishburn tree** is a tree that satisfies all the above properties.

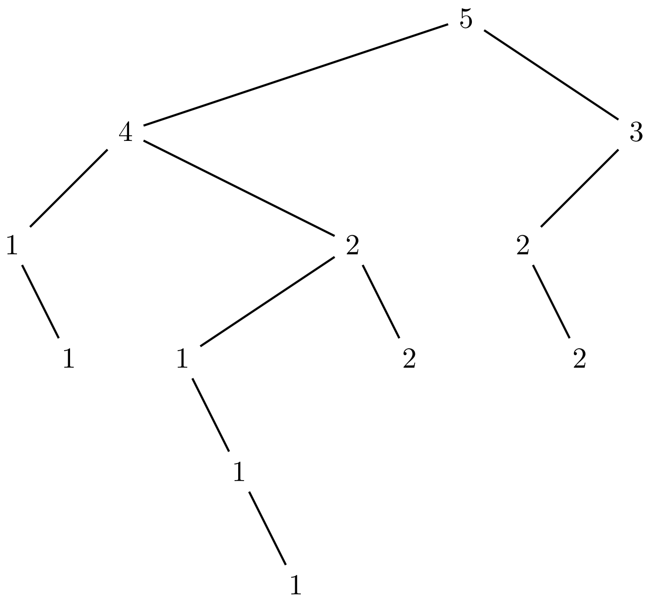
Maximal-right-path decomposition

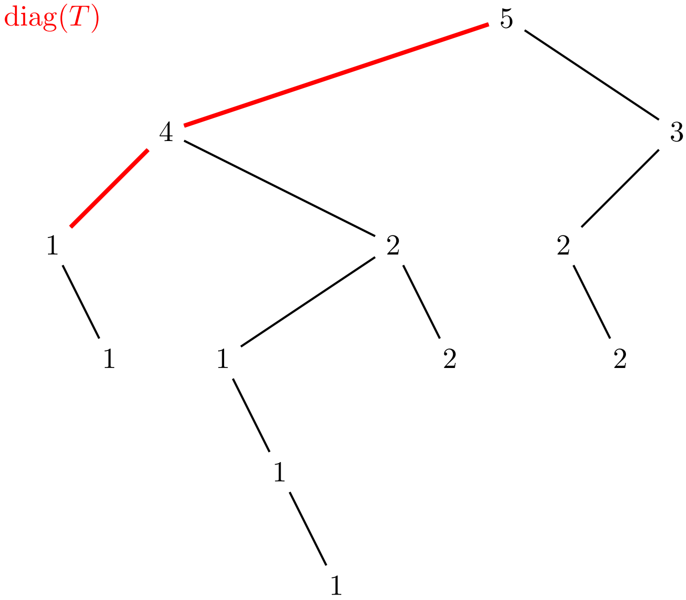
The **diagonal** of T is the path from the root to the leftmost node.

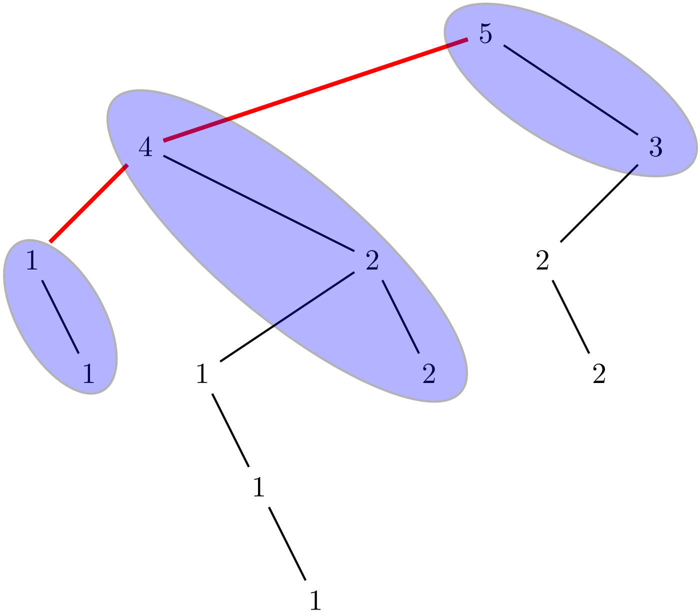
Maximal-right-path decomposition

The **diagonal** of T is the path from the root to the leftmost node.

- Decompose T in maximal right paths.
- In each path, the topmost node v is either:
  (i) on the diagonal;
  (ii) or, the left child of a node that is not on the diagonal.
- Number each maximal right path by the label of:
  (i) its top node, if the top node is on the diagonal;
  (ii) the father of its top node, otherwise.


5
4
3
1
2
2
1
1
2
2
1
1


diag(T)
5
4
3
1
2
2
1
1
2
2
1
1


5
4
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2
1
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2
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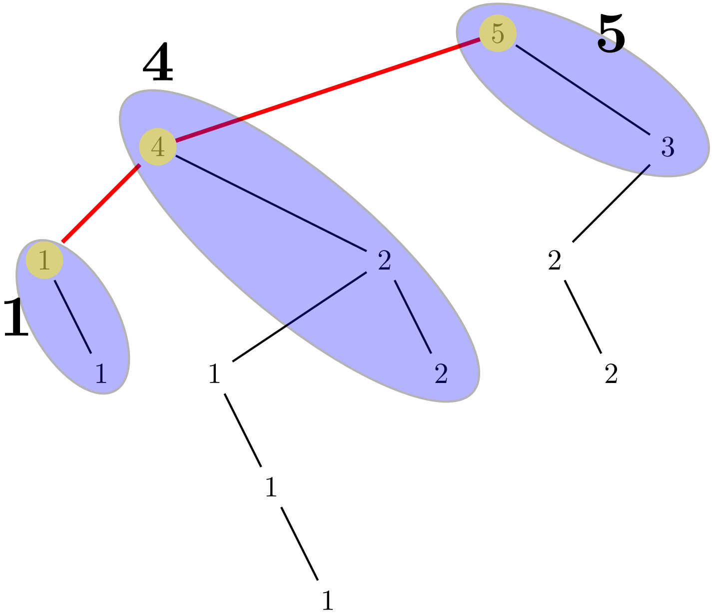
1



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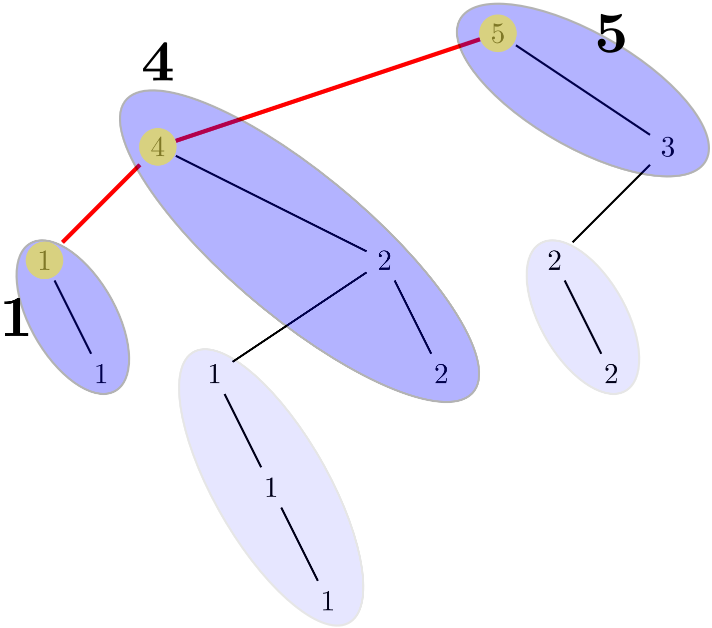
2
1
2
1
1


5
4
1
5
3
4
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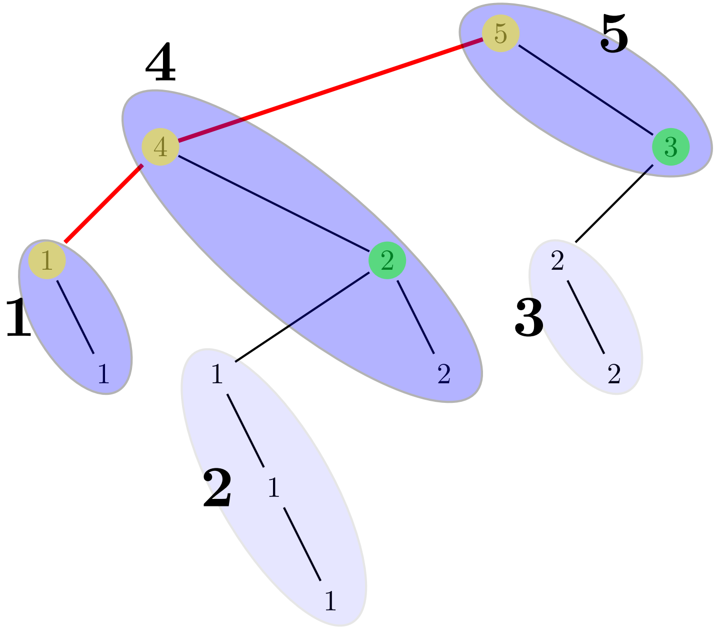

4
5
5
4
3
1
2
1
3
1
1
2
2
2
1
2
1
1

Maximal-right-path decomposition

Theorem

Let k be the maximal value of a label in T.

- There are k maximal-right-paths.
- Each one corresponds to a unique number in $\{1, 2, \ldots, k\}$.

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Theorem

$$\mathfrak{l}(v) \leq \mathfrak{b}(u).$$

Proof. A Fishburn tree is decreasing.

Fishburn matrices

- Lower triangular;
- Non-negative integer entries;
- Every row and column contains at least one non-zero entry.

- The size of a Fishburn matrix is the sum of its entries.

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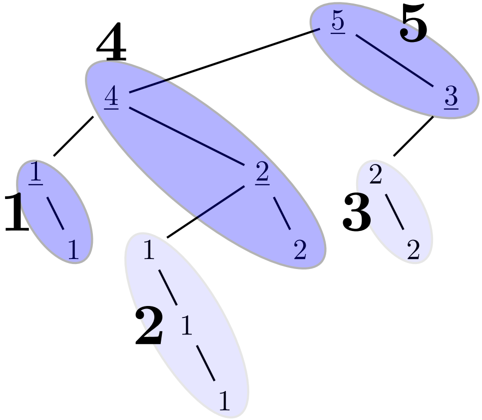
Fishburn matrices of size 3

$$[3]\,,\begin{bmatrix}2 & \\ \cdot & 1\end{bmatrix},\begin{bmatrix}1 & \\ \cdot & 2\end{bmatrix},\begin{bmatrix}1 & \\ 1 & 1\end{bmatrix},\begin{bmatrix}1 & & \\ \cdot & 1 & \\ \cdot & \cdot & 1\end{bmatrix}$$

The Fishburn matrix $A = (a_{ij})$ associated with T is:

$$\begin{aligned} a_{ij} &= |\{\text{nodes with label } j \text{ contained in the } i\text{th path of } T\}| \\ &= |\{v : \mathfrak{l}(v) = i, \mathfrak{b}(v) = j\}|. \end{aligned}$$



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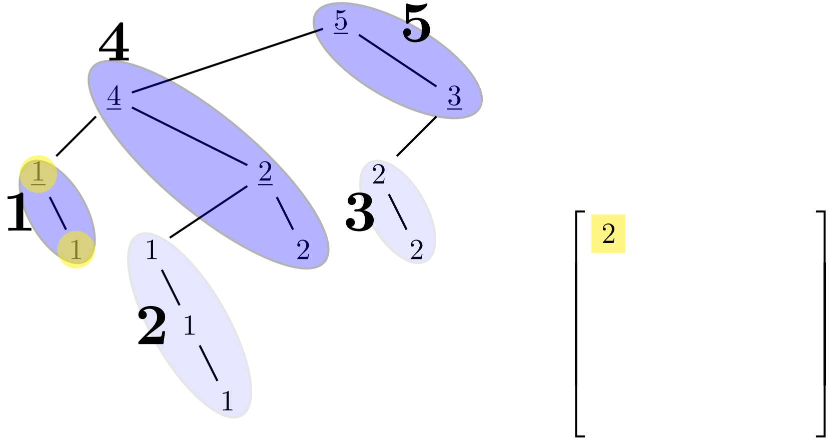

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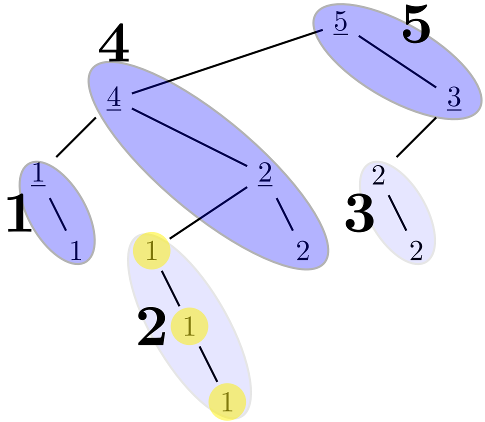

$$\begin{bmatrix} 2 & & & & \\ 3 & & & & \\ & & & & \\ & & & & \end{bmatrix}$$

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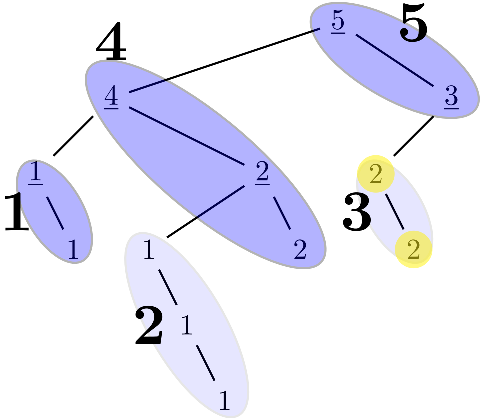

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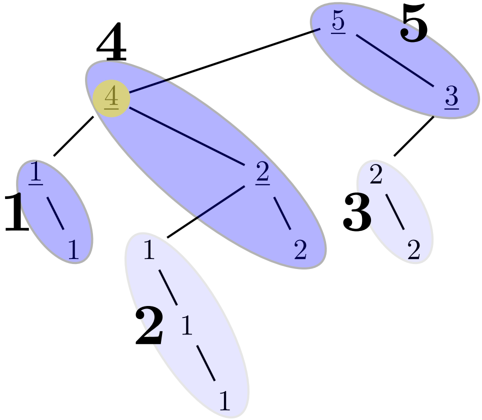

$$\begin{bmatrix} 2 & & & & \\ 3 & & & & \\ & 2 & & & \\ & & & 1 & \\ & & & & \end{bmatrix}$$

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$$
\begin{aligned}
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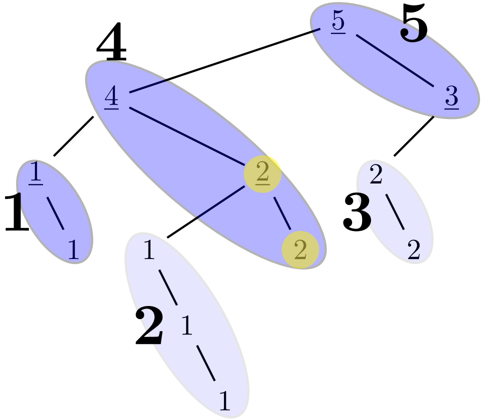
\end{aligned}
$$



$$
\begin{bmatrix}
2 & & & & \\
3 & & & & \\
 & 2 & & & \\
 & 2 & & 1 & \\
 & & & &
\end{bmatrix}
$$

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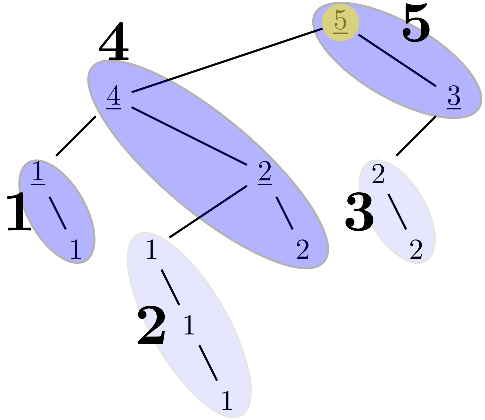

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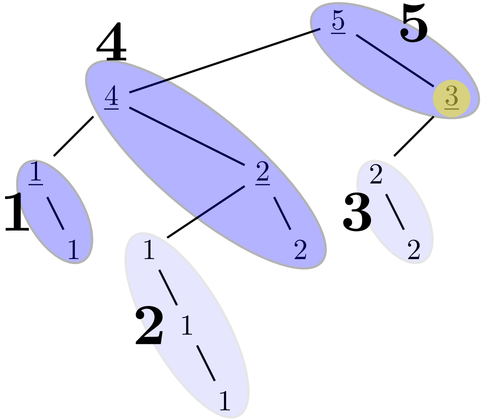

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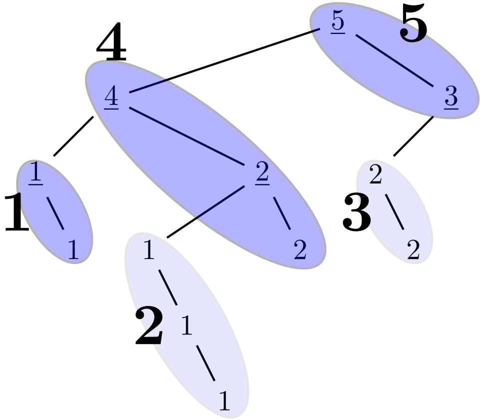

$$\begin{bmatrix} 2 & & & & \\ 3 & \cdot & & & \\ \cdot & 2 & \cdot & & \\ \cdot & 2 & \cdot & 1 & \\ \cdot & \cdot & 1 & \cdot & 1 \end{bmatrix}$$

From Fishburn matrices to Fishburn trees

$$\begin{bmatrix} 2 & & & & \\ 3 & \cdot & & & \\ \cdot & 2 & \cdot & & \\ \cdot & 2 & \cdot & 1 & \\ \cdot & \cdot & 1 & \cdot & 1 \end{bmatrix}$$

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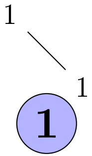

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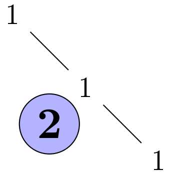

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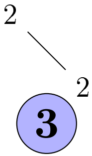

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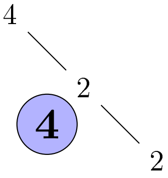

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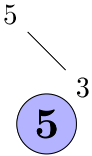

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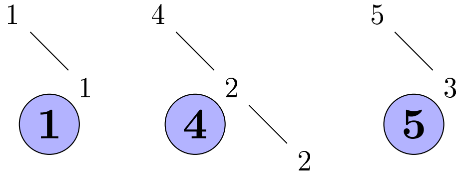

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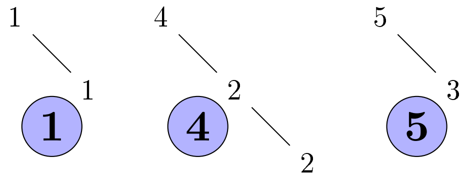

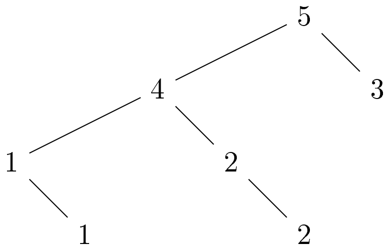

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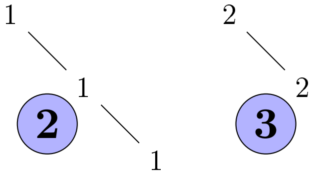

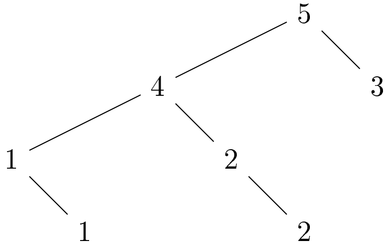

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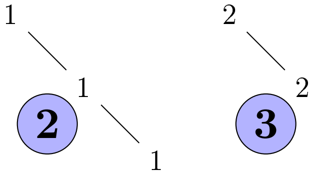

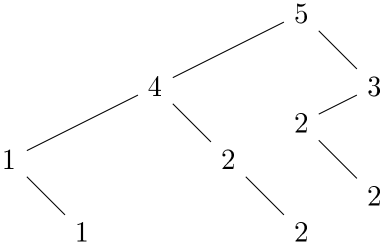

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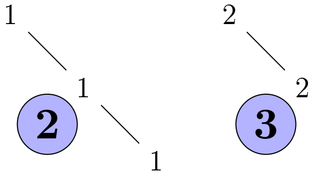
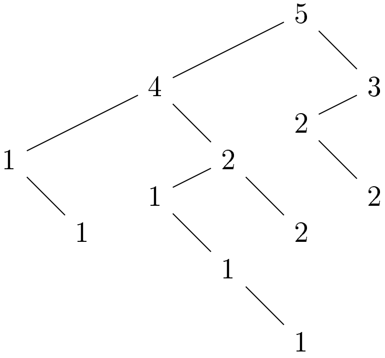
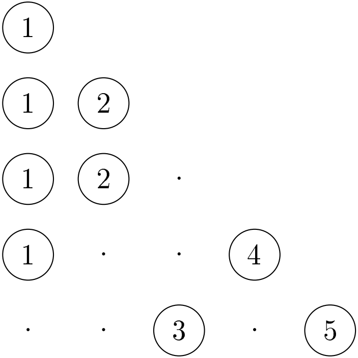




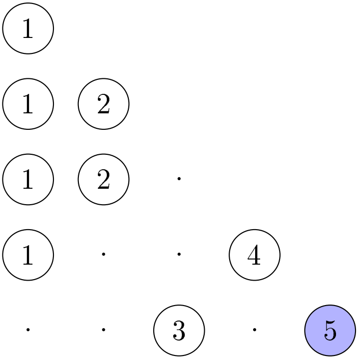
“Embedding” of trees on binary matrices

1				
1	1			
1	1	·		
1	·	·	1	
·	·	1	·	1

1				
1	1			
1	1	·		
1	·	·	1	
·	·	1	·	1

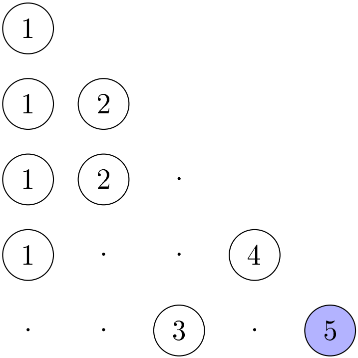

1
1 2
1 2
1 4
3 5

"Embedding" of trees on binary matrices



-Root: bottom-right node

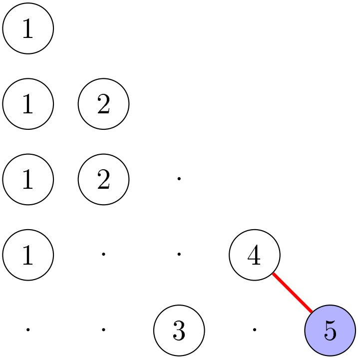
“Embedding” of trees on binary matrices



-Root: bottom-right node

-Left children: go up the diagonal

"Embedding" of trees on binary matrices



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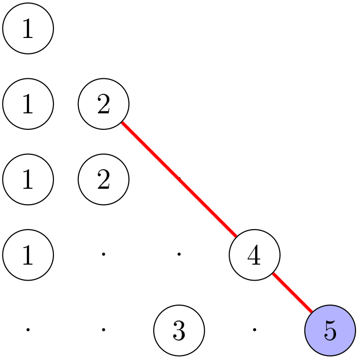

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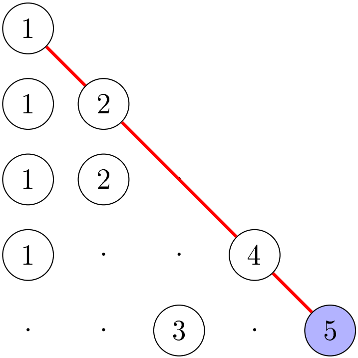

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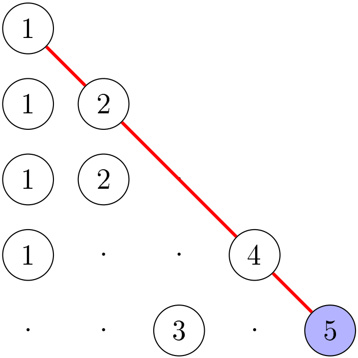
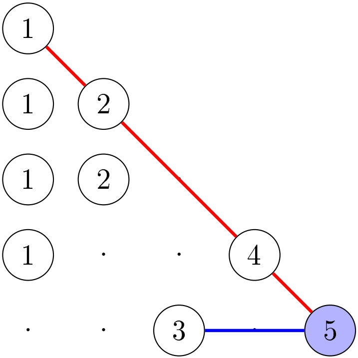


-Root: bottom-right node

-Left children: go up the diagonal

-Right children: go back on each row



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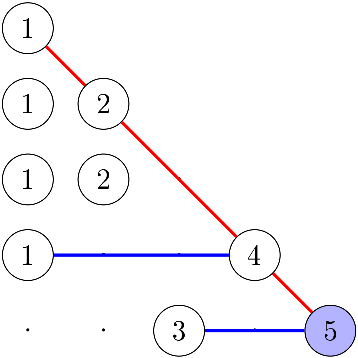

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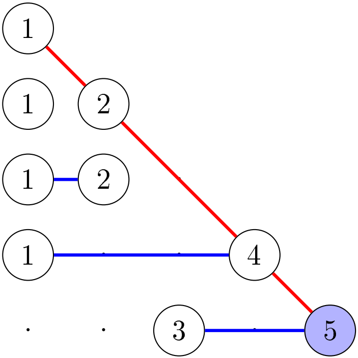

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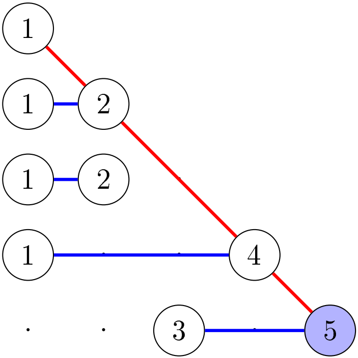

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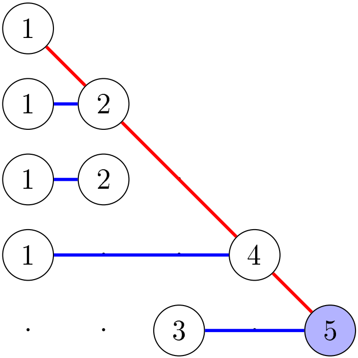
"Embedding" of trees on binary matrices



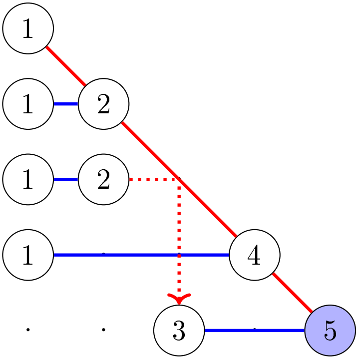
-Root: bottom-right node

-Left children: go up the diagonal

-Right children: go back on each row

-Left children: bounce off the diagonal

“Embedding” of trees on binary matrices



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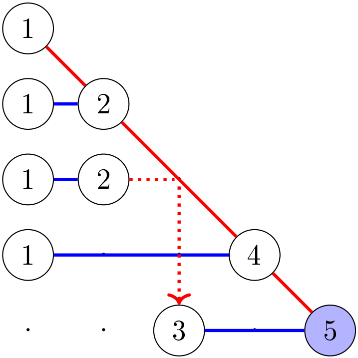
"Embedding" of trees on binary matrices



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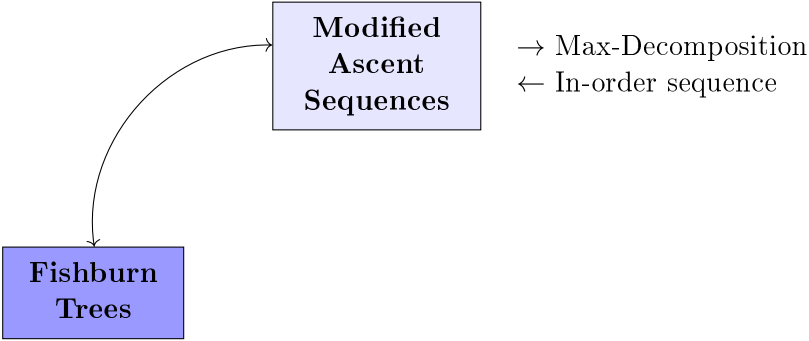
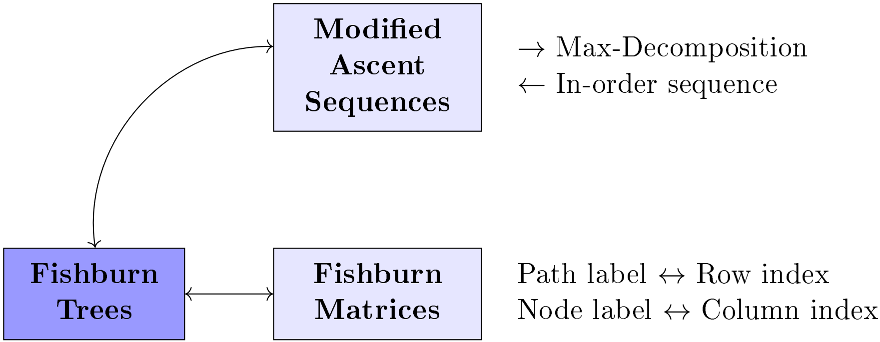
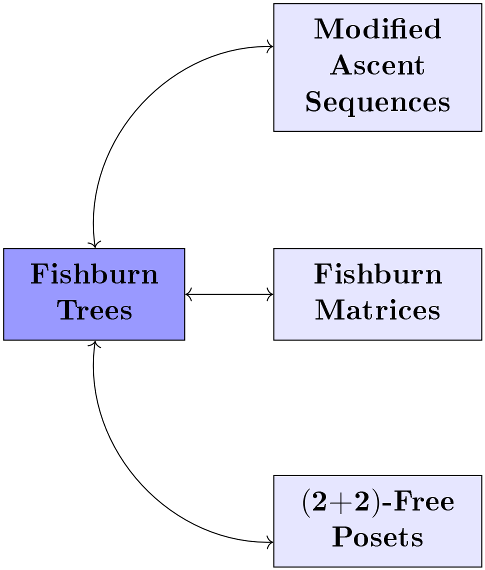
Rotate the matrix by 90° to obtain the Fishburn tree!

Fishburn Trees


Modified Ascent Sequences
→ Max-Decomposition
← In-order sequence
Fishburn Trees


Modified Ascent Sequences
→ Max-Decomposition
← In-order sequence
Fishburn Trees
Fishburn Matrices
Path label ↔ Row index
Node label ↔ Column index



→ Max-Decomposition
← In-order sequence

Path label ↔ Row index
Node label ↔ Column index

Path label ↔ Down-sets
Node label ↔ Levels

Applications: flip and sum

flip and sum

- Duality (`flip`) acts as an involution on Fishburn posets.
- On Fishburn matrices, the `flip` corresponds to the reflection of a matrix in its antidiagonal.
- The `sum` of two Fishburn matrices is a Fishburn matrix.

Applications: `flip` and `sum`

`flip` and `sum`

- Duality (`flip`) acts as an involution on Fishburn posets.
- On Fishburn matrices, the `flip` corresponds to the reflection of a matrix in its antidiagonal.
- The `sum` of two Fishburn matrices is a Fishburn matrix.

1. How do `flip` and `sum` act on the corresponding ascent sequences?
2. How do `flip` and `sum` act on Fishburn trees?
3. Is there a natural involution on the set of Fishburn trees?

Applications: Subfamilies

Fishburn trees	**Modified seq.**	**Fishburn mat.**	**(2+2)-free posets**
Strictly decreasing	Primitive	Binary	Primitive
Comb-shaped	Self-modified	Positive diagonal	∃ Chain of max length
(⋆)	$\hat{\mathcal{A}}(212, 312)$	NW-free	N-free (Series-parallel)
(†)	$\hat{\mathcal{A}}(231)$	SW-free	(**3**+**1**)-free (Semiorders)
By intersection of the above two rows	$\hat{\mathcal{A}}(212, 312, 231)$	(NW,SW)-free	N- and (**3**+**1**)-free

$$(\star)\ \nexists\, u, v : \mathfrak{l}(u) < \mathfrak{l}(v) \leq \mathfrak{b}(u) < \mathfrak{b}(v)$$

$$(\dagger)\ \nexists\, u, v : \mathfrak{b}(u) < \mathfrak{b}(v),\ \mathfrak{l}(u) > \mathfrak{l}(v)$$

Applications: Statistics

Fishburn trees	Modified seq.	Fishburn mat.	(2+2)-free posets
# nodes	Length	Sum of entries	# elements
Biggest node label	Maximum value	# rows/cols	# levels
$\lvert\{u : \mathfrak{l}(u) = j\}\rvert$	# copies of j	$\sum_i a_{i,j}$	$\lvert L_j\rvert$
$\lvert\{u : \mathfrak{b}(u) = \mathfrak{l}(u)\}\rvert$	# weak ltr-max	Trace	$\sum_i \lvert L_i \cap D_{i+1}\rvert$
$\lvert\{u : \mathfrak{l}(u) = 1\}\rvert$	# copies of 1	$\sum_i a_{i,1}$	# minimal elements
$\lvert\text{rpath}(r)\rvert$	# weak rtl-max	$\sum_j a_{k,j}$	# maximal elements
$\mathfrak{l}(v_n)$	x_n	index	Minimal level of a maximal element

Applications: Statistics

Fishburn trees	Modified seq.	Fishburn mat.	(2+2)-free posets
# nodes	Length	Sum of entries	# elements
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$\lvert\{u : \mathfrak{l}(u) = j\}\rvert$	# copies of j	$\sum_i a_{i,j}$	$\lvert L_j \rvert$
$\lvert\{u : \mathfrak{b}(u) = \mathfrak{l}(u)\}\rvert$	# weak ltr-max	Trace	$\sum_i \lvert L_i \cap D_{i+1}\rvert$
$\lvert\{u : \mathfrak{l}(u) = 1\}\rvert$	# copies of 1	$\sum_i a_{i,1}$	# minimal elements
$\lvert\text{rpath}(r)\rvert$	# weak rtl-max	$\sum_j a_{k,j}$	# maximal elements
$\mathfrak{l}(v_n)$	x_n	index	Minimal level of a maximal element

Thanks!