Mallows processes and the expanded hypercube

TU/e

Benoît Corsini

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Tableau representation

Tableau representation

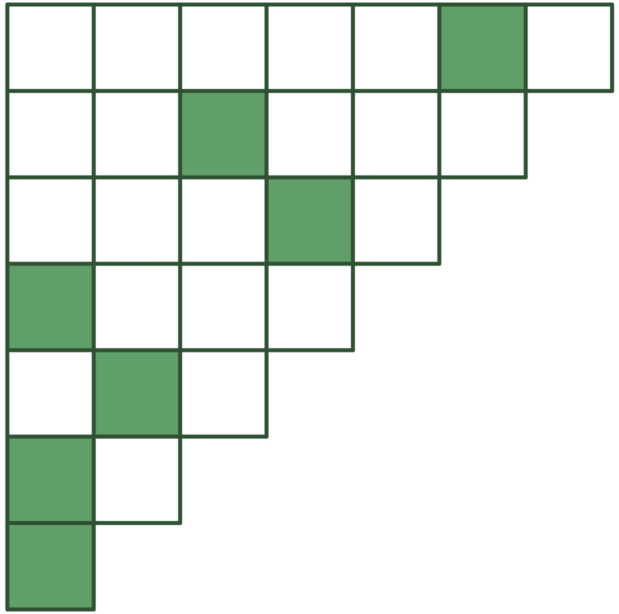


Tableau representation

$\mathrm{Inv}_7(\sigma) := |\{i < 7 : \sigma(i) > \sigma(7)\}|$

$\mathrm{Inv}_6(\sigma) := |\{i < 6 : \sigma(i) > \sigma(6)\}|$

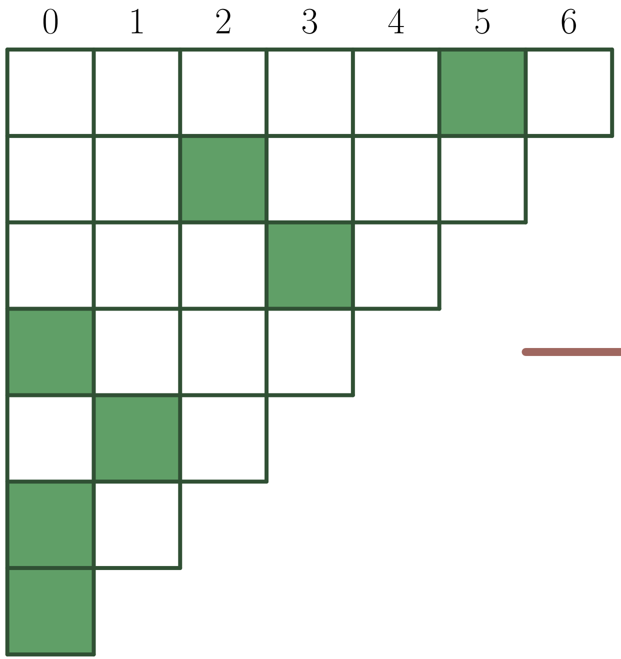
$\mathrm{Inv}_5(\sigma) := |\{i < 5 : \sigma(i) > \sigma(5)\}|$

$\mathrm{Inv}_4(\sigma) := |\{i < 4 : \sigma(i) > \sigma(4)\}|$

$\mathrm{Inv}_3(\sigma) := |\{i < 3 : \sigma(i) > \sigma(3)\}|$

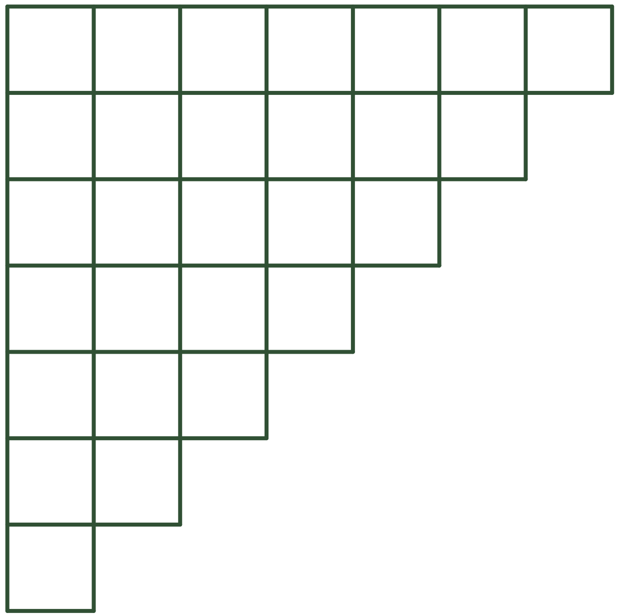
$\mathrm{Inv}_2(\sigma) := |\{i < 2 : \sigma(i) > \sigma(2)\}|$

$\mathrm{Inv}_1(\sigma) := |\{i < 1 : \sigma(i) > \sigma(1)\}|$



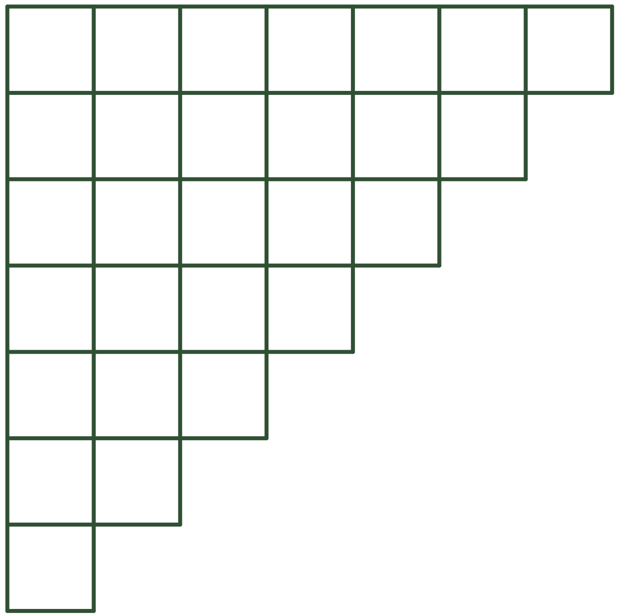
$$\longrightarrow \quad \sigma = (1, 7, 5, 3, 6, 2, 4)$$

Random geometric tableau



Random geometric tableau

Fix $q \in [0, \infty)$



Random geometric tableau

Fix $q \in [0, \infty)$

$\perp\!\!\!\perp$

1	q	q^2	q^3	q^4	q^5	q^6
1	q	q^2	q^3	q^4	q^5	
1	q	q^2	q^3	q^4		
1	q	q^2	q^3			
1	q	q^2				
1	q					
1						

Random geometric tableau

Fix $q \in [0, \infty)$

$\perp\!\!\!\perp$

1	q	q^2	q^3	q^4	q^5	q^6	$/(1+\ldots+q^6)$
1	q	q^2	q^3	q^4	q^5	$/(1+\ldots+q^5)$	
1	q	q^2	q^3	q^4	$/(1+\ldots+q^4)$		
1	q	q^2	q^3	$/(1+\ldots+q^3)$			
1	q	q^2	$/(1+q+q^2)$				
1	q	$/(1+q)$					
1	$/1$						

Fix $q \in [0, \infty)$

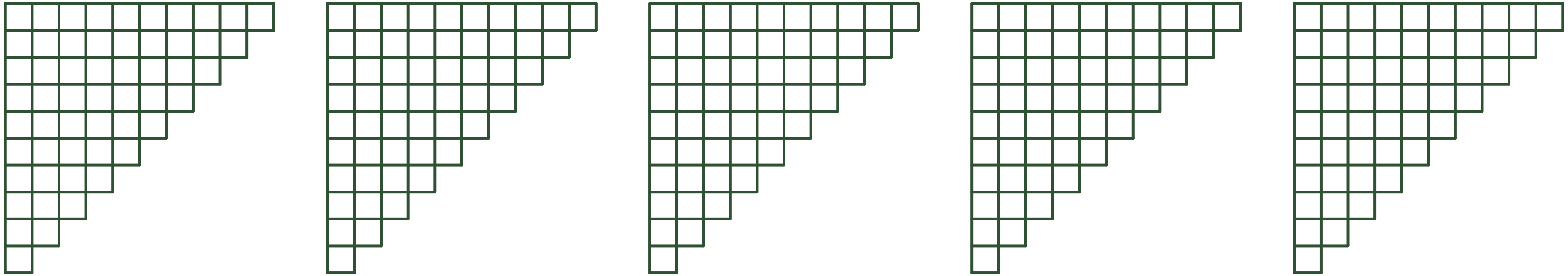
Independent rows ($\perp\!\!\!\perp$):

1	q	q^2	q^3	q^4	q^5	q^6	$/(1+\ldots+q^6)$
1	q	q^2	q^3	q^4	q^5	$/(1+\ldots+q^5)$	
1	q	q^2	q^3	q^4	$/(1+\ldots+q^4)$		
1	q	q^2	q^3	$/(1+\ldots+q^3)$			
1	q	q^2	$/(1+q+q^2)$				
1	q	$/(1+q)$					
1	$/1$						

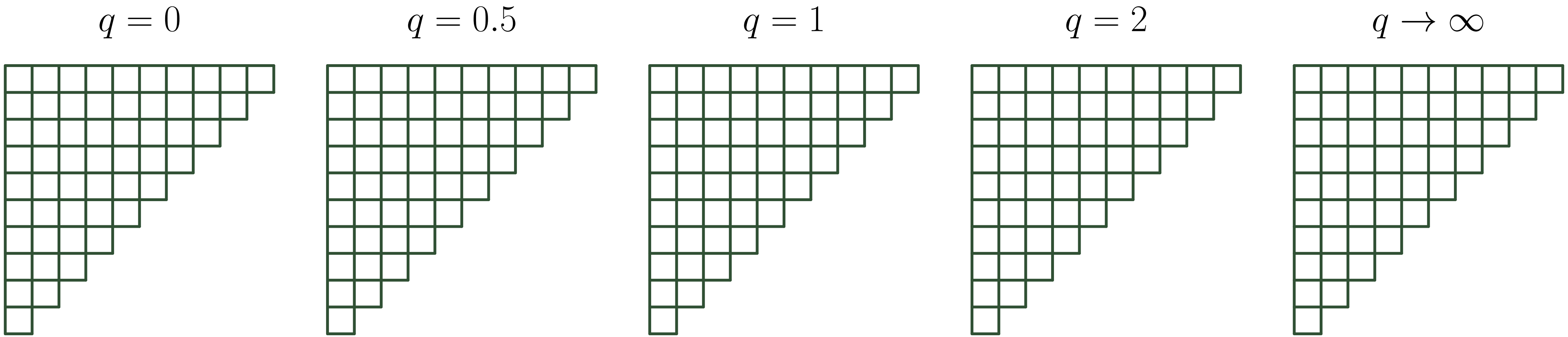
$$\longrightarrow \quad \sigma \sim \textsc{Mallows}(n, q)$$

Random geometric tableaux

Random geometric tableaux



Random geometric tableaux



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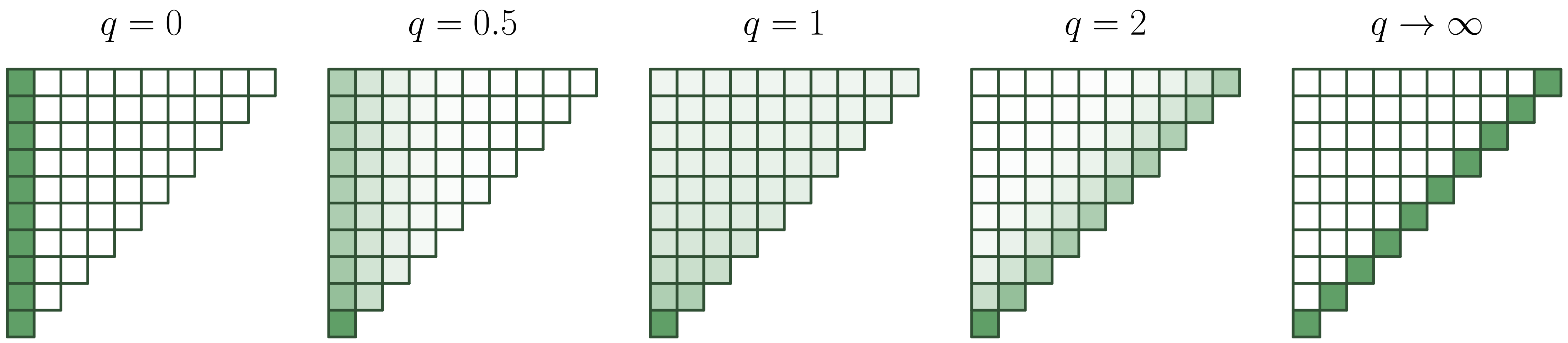

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Mallows processes

Idea:

If Mallows permutations are defined with $n \in \mathbb{N}$ and $q \in [0, \infty)$, can we define a family of **interesting** stochastic processes $\mathcal{M}^n = (\mathcal{M}^n_t)_{t\in[0,\infty)}$ such that, for any $t \in [0, \infty)$, $\mathcal{M}^n_t$ is a Mallows permutation with parameters n and t?

Properties of Mallows processes

$\perp\!\!\!\perp$

1	t	t^2	t^3	t^4	t^5	t^6
1	t	t^2	t^3	t^4	t^5	
1	t	t^2	t^3	t^4		
1	t	t^2	t^3			
1	t	t^2				
1	t					
1						

Properties of Mallows processes

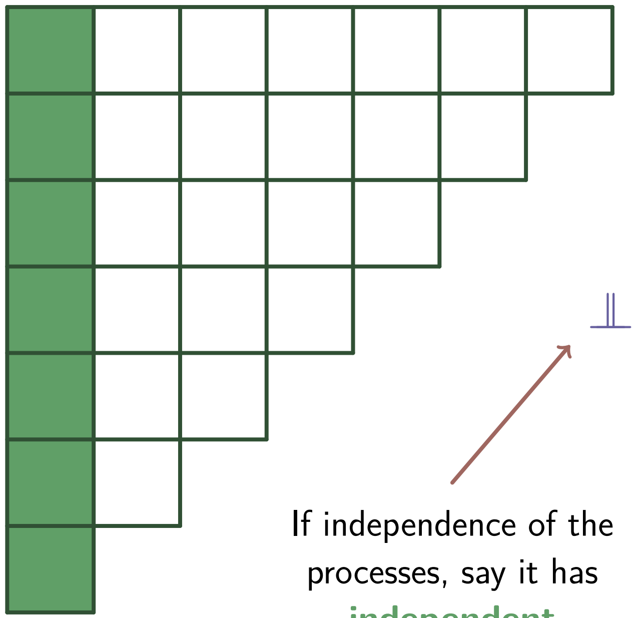
$\perp\!\!\!\perp$

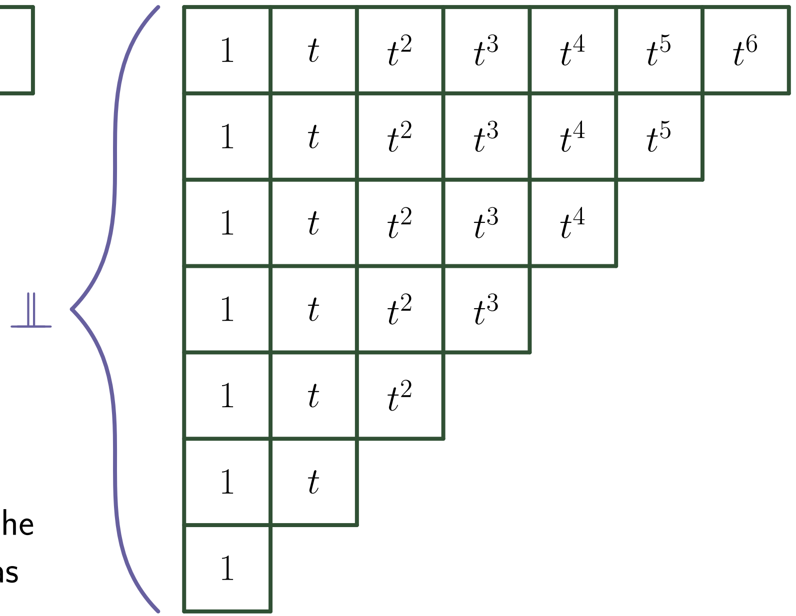
1	t	t^2	t^3	t^4	t^5	t^6
1	t	t^2	t^3	t^4	t^5	
1	t	t^2	t^3	t^4		
1	t	t^2	t^3			
1	t	t^2				
1	t					
1						

If independence of the processes, say it has **independent inversions**.

Properties of Mallows processes

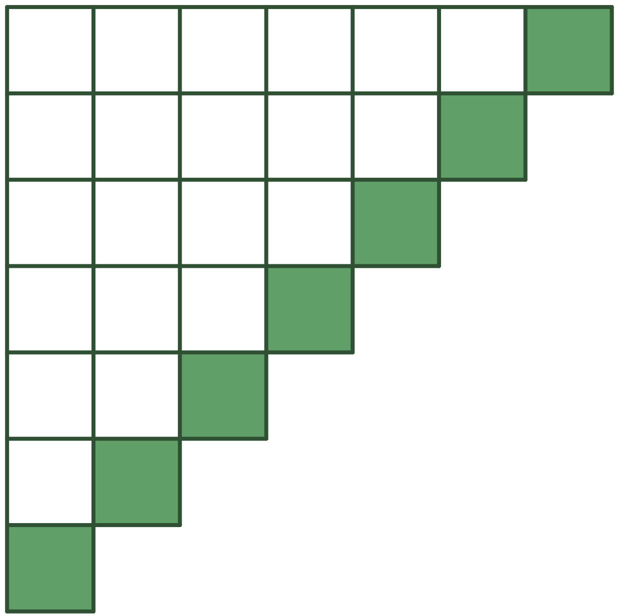
$t = 0$



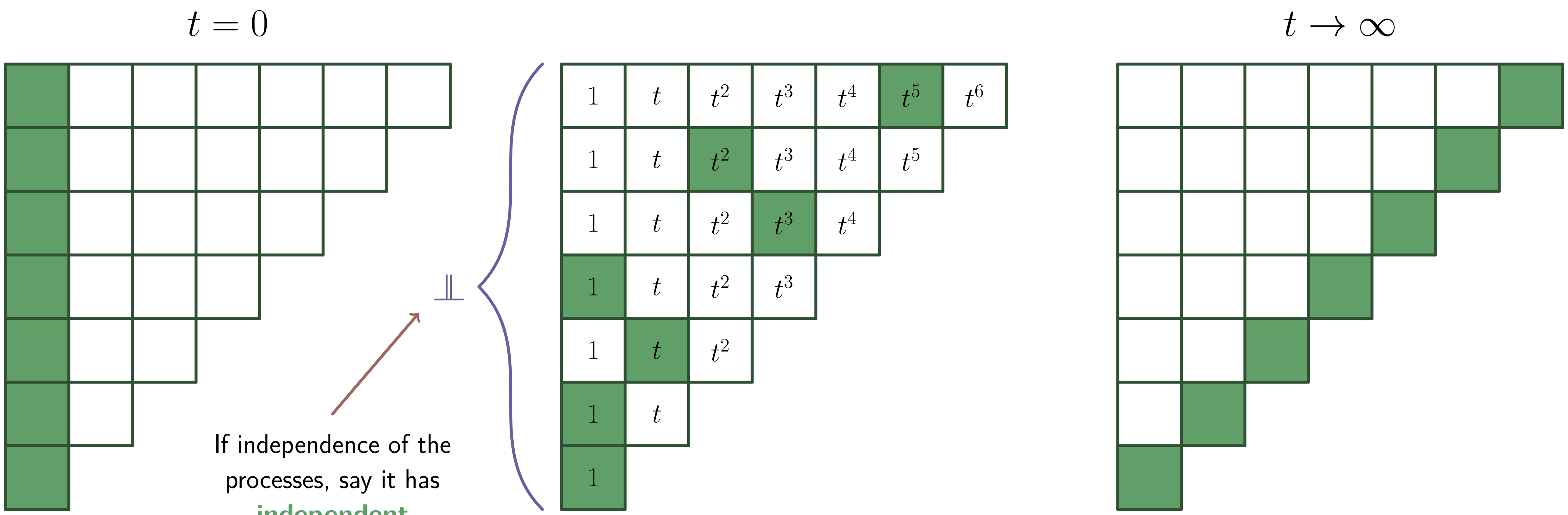


If independence of the processes, say it has **independent inversions**.

$t \to \infty$



Properties of Mallows processes



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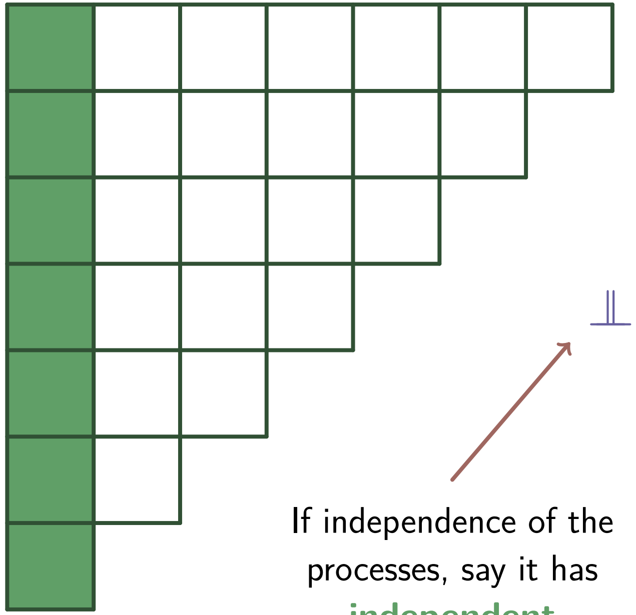
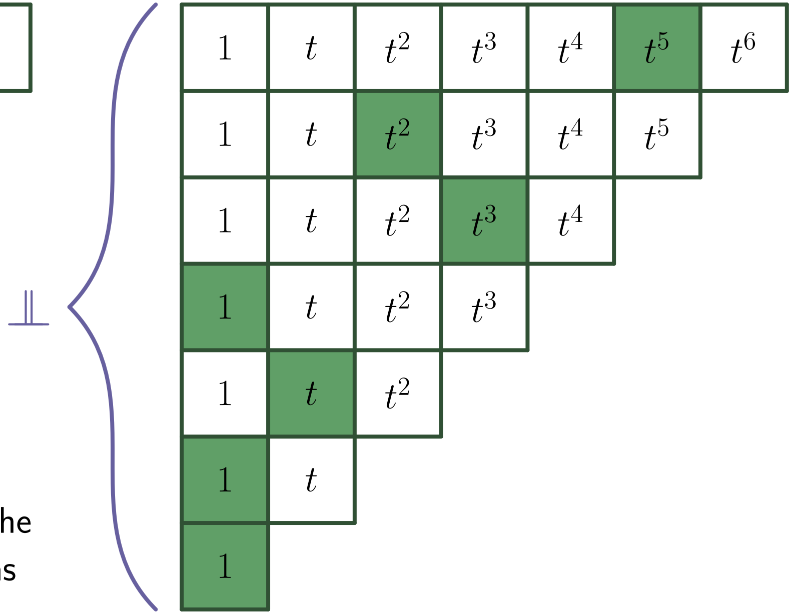
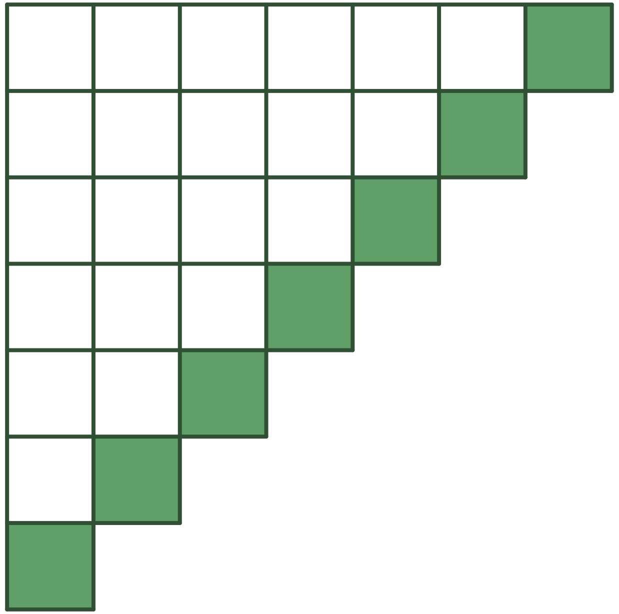
Properties of Mallows processes

$t = 0$





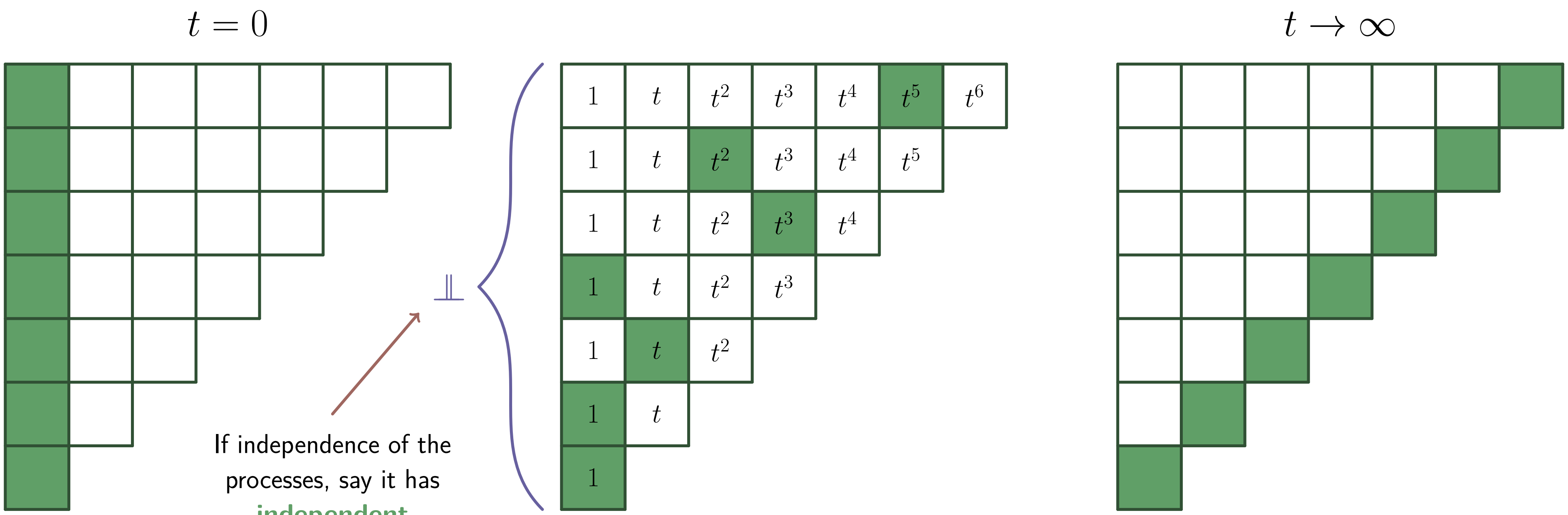
$t \to \infty$



If independence of the processes, say it has **independent inversions**.

Say it is **strictly monotone** if the blocks move from left to right.

Properties of Mallows processes



If independence of the processes, say it has **independent inversions**.

Say it is **strictly monotone** if the blocks move from left to right.

Say it is **smooth** if no block moves by more than one step at a time and no two blocks move at the same time.

Constructing a regular Mallows process

Constructing a regular Mallows process

1	t	t^2	t^3	t^4	**t^5**	t^6
1	t	**t^2**	t^3	t^4	t^5	
1	t	t^2	**t^3**	t^4		
1	t	t^2	t^3			
1	**t**	t^2				
1	t					
1						

Constructing a regular Mallows process

$I_t^7 = \left\lfloor \frac{\log(1-U_7(1-t^7))}{\log t} \right\rfloor$	1	t	t^2	t^3	t^4	**t^5**	t^6
$I_t^6 = \left\lfloor \frac{\log(1-U_6(1-t^6))}{\log t} \right\rfloor$	1	t	**t^2**	t^3	t^4	t^5	
$I_t^5 = \left\lfloor \frac{\log(1-U_5(1-t^5))}{\log t} \right\rfloor$	1	t	t^2	**t^3**	t^4		
$I_t^4 = \left\lfloor \frac{\log(1-U_4(1-t^4))}{\log t} \right\rfloor$	**1**	t	t^2	t^3			
$I_t^3 = \left\lfloor \frac{\log(1-U_3(1-t^3))}{\log t} \right\rfloor$	1	**t**	t^2				
$I_t^2 = \left\lfloor \frac{\log(1-U_2(1-t^2))}{\log t} \right\rfloor$	**1**	t					
$I_t^1 = \left\lfloor \frac{\log(1-U_1(1-t^1))}{\log t} \right\rfloor$	**1**						

Constructing a regular Mallows process

$$I_t^7 = \left\lfloor \frac{\log(1-U_7(1-t^7))}{\log t} \right\rfloor$$
$$I_t^6 = \left\lfloor \frac{\log(1-U_6(1-t^6))}{\log t} \right\rfloor$$
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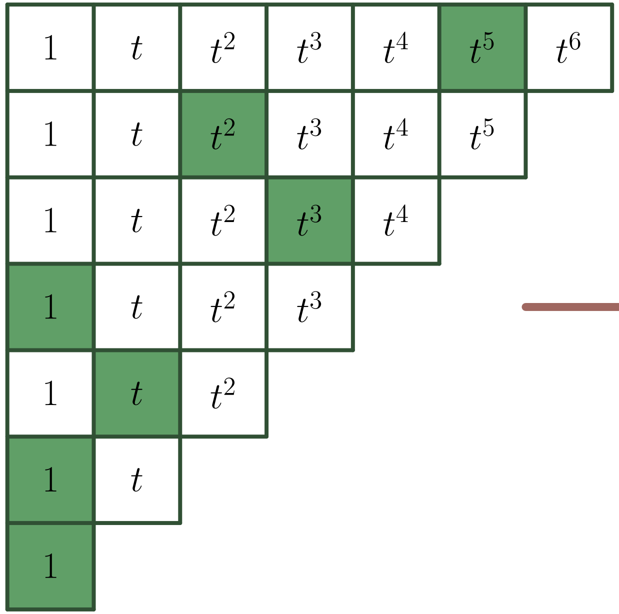
$$I_t^4 = \left\lfloor \frac{\log(1-U_4(1-t^4))}{\log t} \right\rfloor$$
$$I_t^3 = \left\lfloor \frac{\log(1-U_3(1-t^3))}{\log t} \right\rfloor$$
$$I_t^2 = \left\lfloor \frac{\log(1-U_2(1-t^2))}{\log t} \right\rfloor$$
$$I_t^1 = \left\lfloor \frac{\log(1-U_1(1-t^1))}{\log t} \right\rfloor$$



$$\longrightarrow \quad (\mathcal{M}_t^7)_{t\in[0,\infty)}$$

$I_t^7 = \left\lfloor \frac{\log(1-U_7(1-t^7))}{\log t} \right\rfloor$	1	t	t^2	t^3	t^4	**t^5**	t^6
$I_t^6 = \left\lfloor \frac{\log(1-U_6(1-t^6))}{\log t} \right\rfloor$	1	t	**t^2**	t^3	t^4	t^5	
$I_t^5 = \left\lfloor \frac{\log(1-U_5(1-t^5))}{\log t} \right\rfloor$	1	t	t^2	**t^3**	t^4		
$I_t^4 = \left\lfloor \frac{\log(1-U_4(1-t^4))}{\log t} \right\rfloor$	**1**	t	t^2	t^3			
$I_t^3 = \left\lfloor \frac{\log(1-U_3(1-t^3))}{\log t} \right\rfloor$	1	**t**	t^2				
$I_t^2 = \left\lfloor \frac{\log(1-U_2(1-t^2))}{\log t} \right\rfloor$	**1**	t					
$I_t^1 = \left\lfloor \frac{\log(1-U_1(1-t^1))}{\log t} \right\rfloor$	**1**						

$\longrightarrow (\mathcal{M}_t^7)_{t \in [0,\infty)}$

$$\mathbb{P}\left(\left\lfloor \frac{\log(1-U(1-t^j))}{\log t} \right\rfloor = k\right) = \frac{t^k(1-t)}{1-t^j} \propto t^k$$

Markov and Mallows

Theorem (🕴 2022)

There exists a unique Markovian regular Mallows process.

Jumping process

Jumping process

Definition

Given a smooth and strictly increasing Mallows process $\mathcal{M}^n$, let $\tilde{\mathcal{M}}^n = (\tilde{\mathcal{M}}^n_k)_{0\leq k\leq \binom{n}{2}}$ be the corresponding jumping process defined as the sequence of permutations taken by $\mathcal{M}^n$.

Definition

Given a smooth and strictly increasing Mallows process $\mathcal{M}^n$, let $\tilde{\mathcal{M}}^n = (\tilde{\mathcal{M}}^n_k)_{0 \le k \le \binom{n}{2}}$ be the corresponding jumping process defined as the sequence of permutations taken by $\mathcal{M}^n$.

- Note that $\mathrm{Inv}\left(\tilde{\mathcal{M}}^n_k\right) = k$ for all $0 \le k \le \binom{n}{2}$.

Theorem (2022)

There exists a unique Markovian regular Mallows process.

Markov and Mallows

Theorem (🚹 2022)

There exists a unique Markovian regular Mallows process.

Conjecture (🚹 2022)

Let $\mathcal{M}^n$ be the unique Markovian regular Mallows process. Then the corresponding jumping process $\tilde{\mathcal{M}}^n$ is **not** a Markov chain.

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The expanded hypercube

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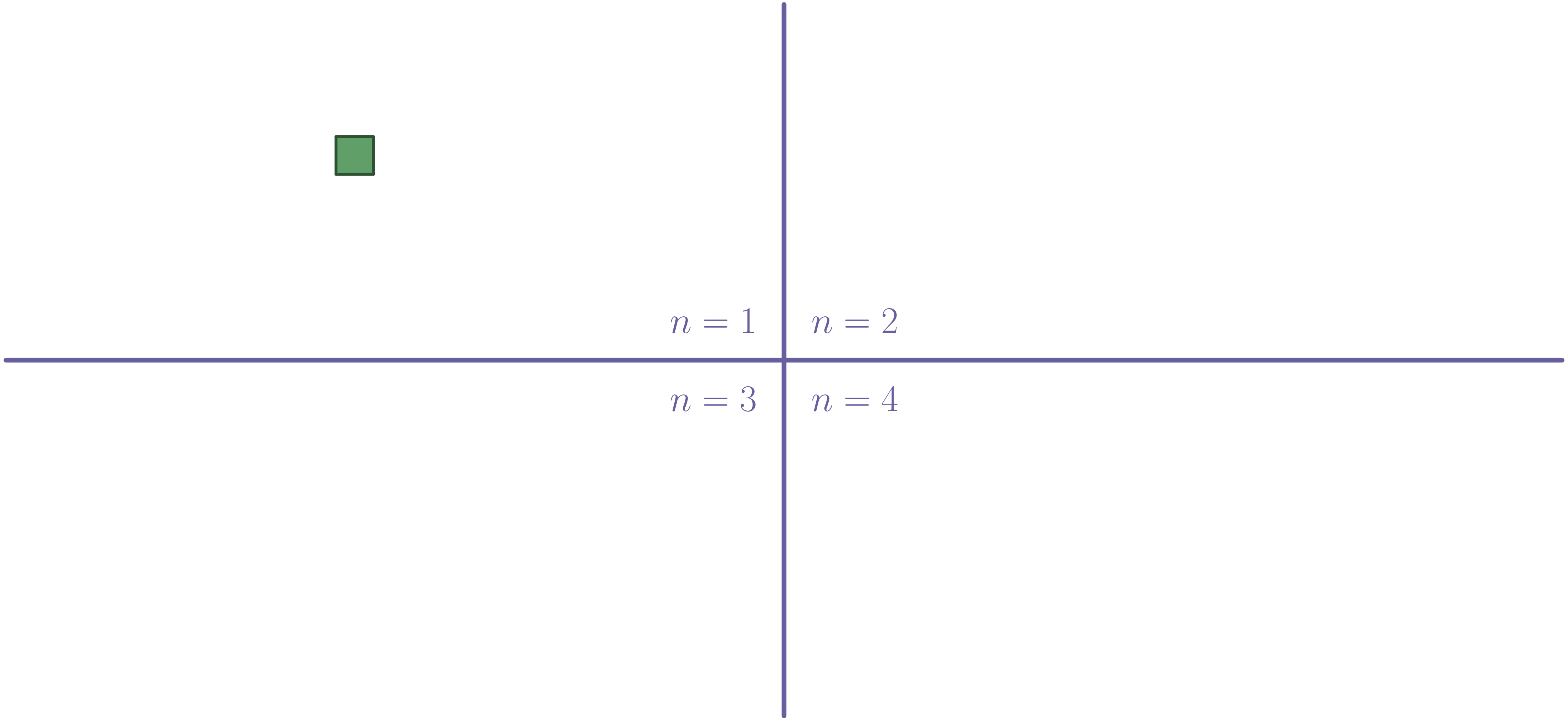
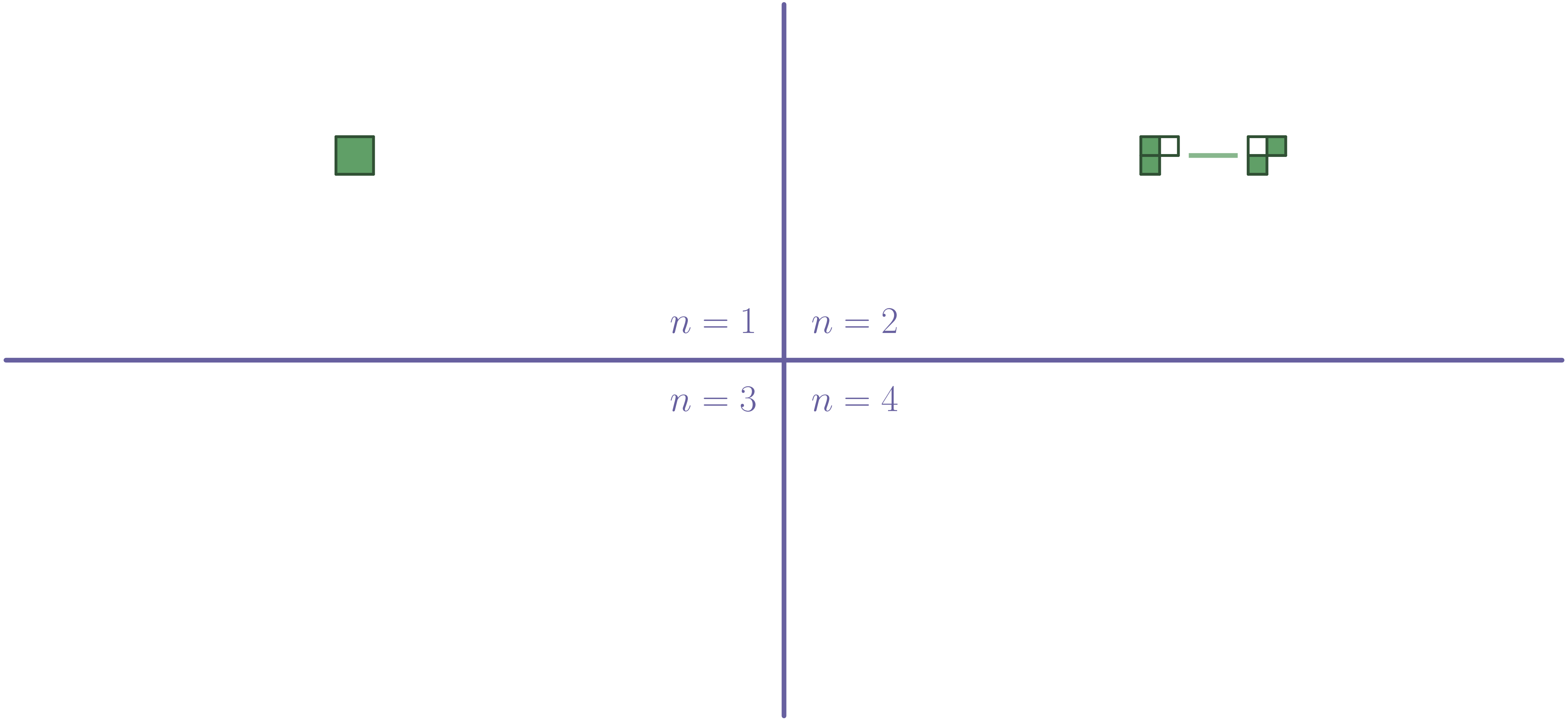
$n = 1$ $n = 2$

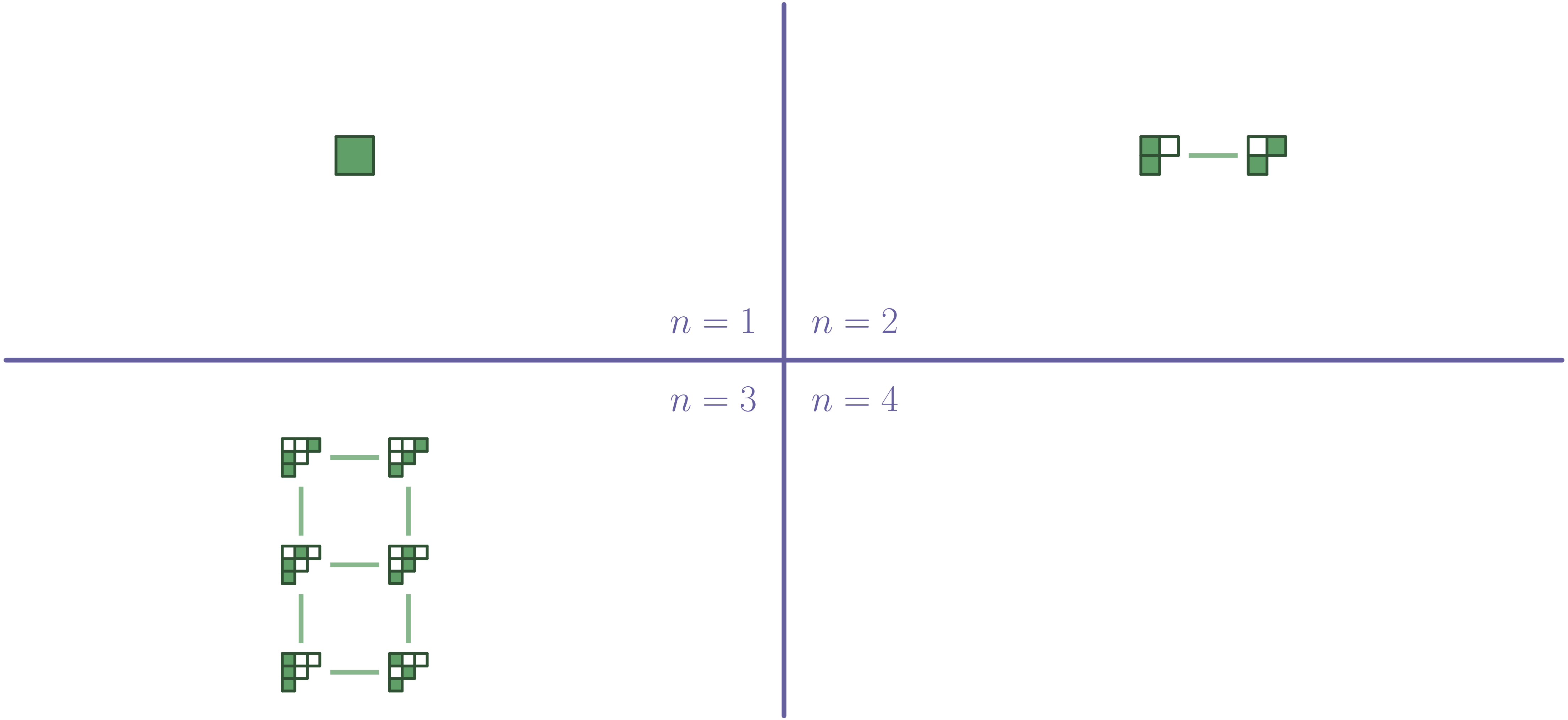
$n = 3$ $n = 4$


$n = 1$
$n = 2$
$n = 3$
$n = 4$

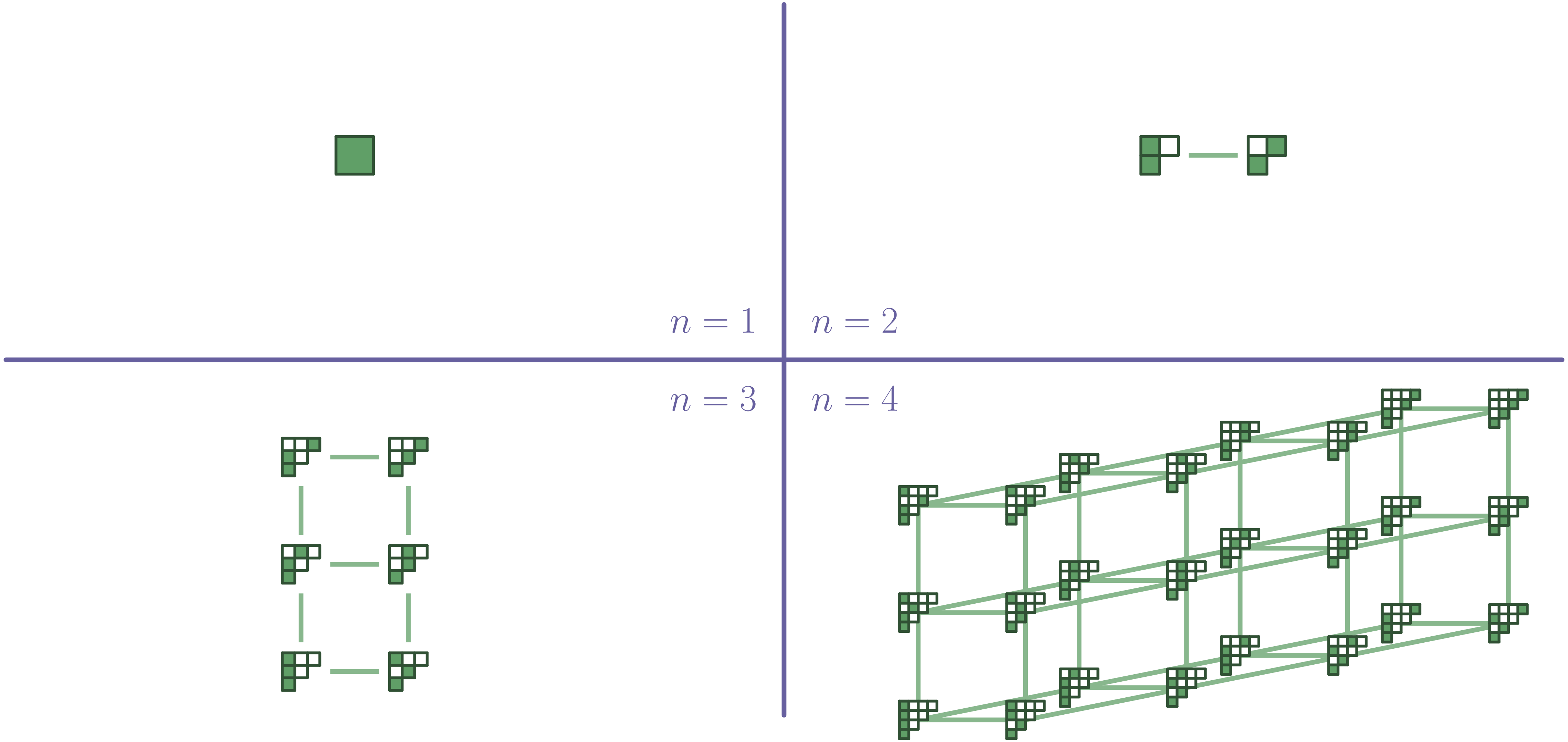
The expanded hypercube



The expanded hypercube



The expanded hypercube



The expanded hypercube

Definition

Write $\mathcal{H}_n$ for the graph on the set of permutations corresponding to exactly one jump to the right on the tableau representation of the permutation. In other words, for any $\sigma, \sigma' \in \mathcal{S}_n$

$$(\sigma, \sigma') \in \mathcal{H}_n \iff \sum_{j=1}^{n} |\mathrm{Inv}_j(\sigma) - \mathrm{Inv}_j(\sigma')| = 1\,.$$

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Example

$n = 3$

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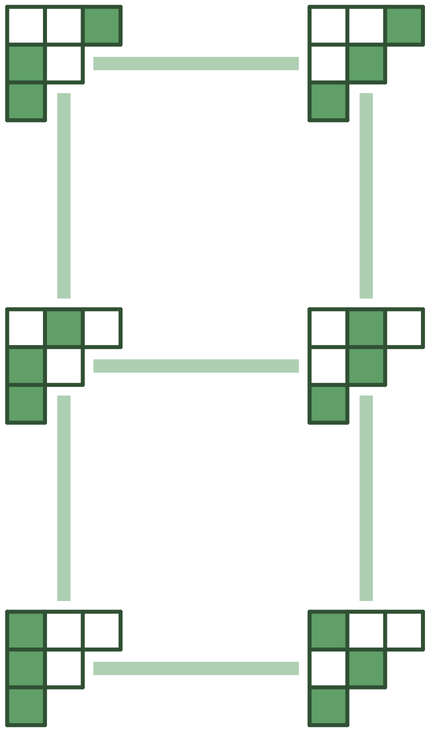
$n = 3$



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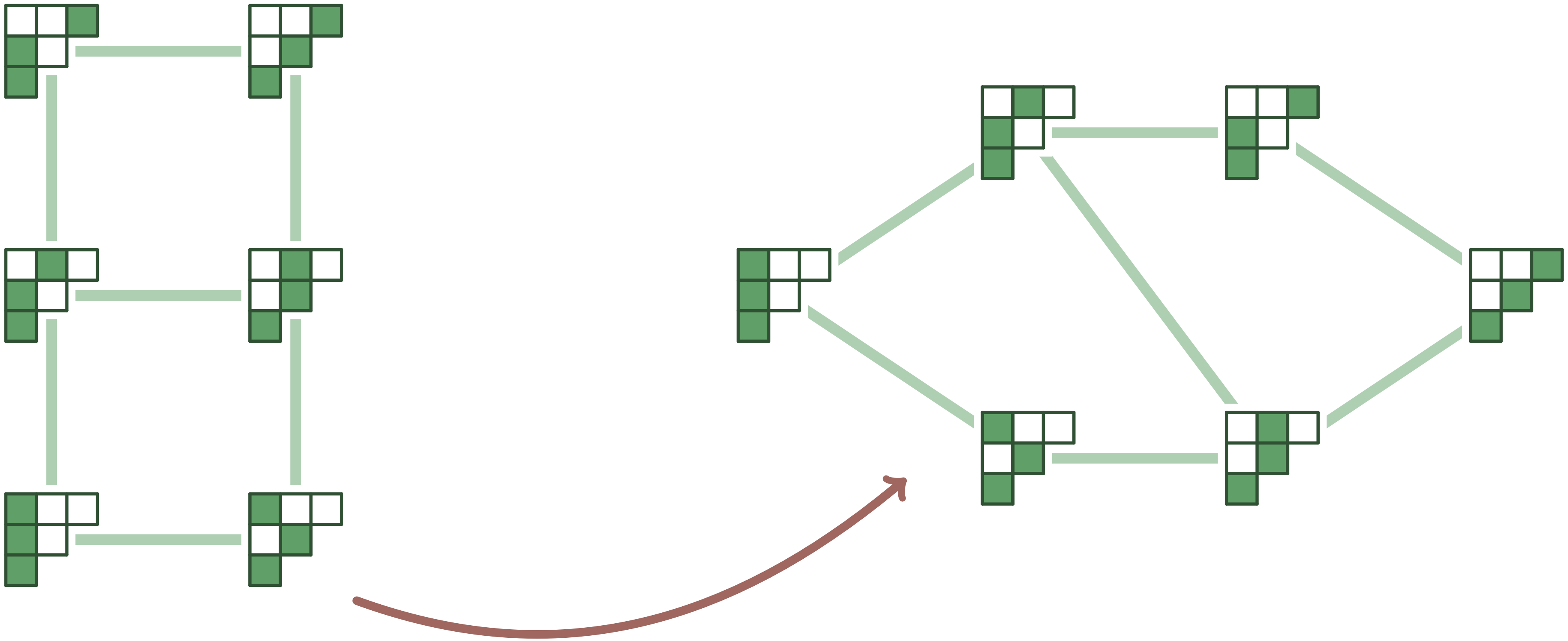

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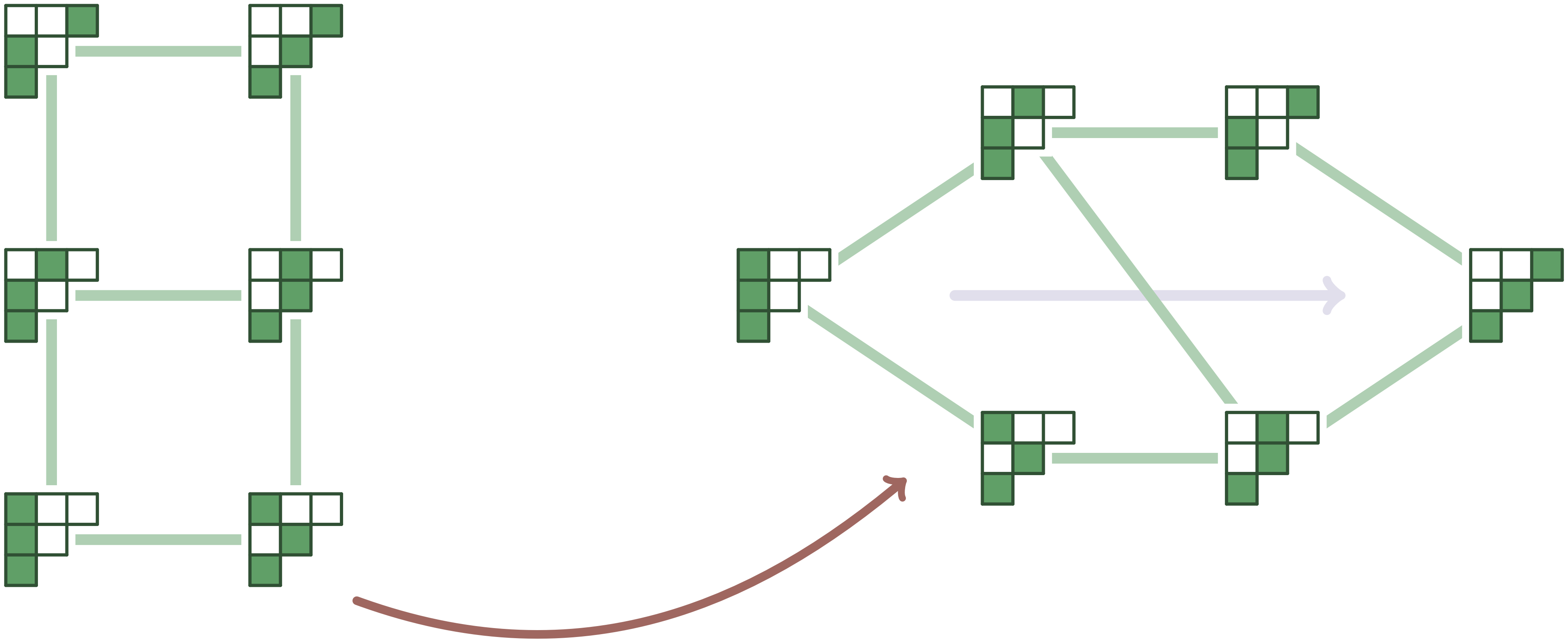
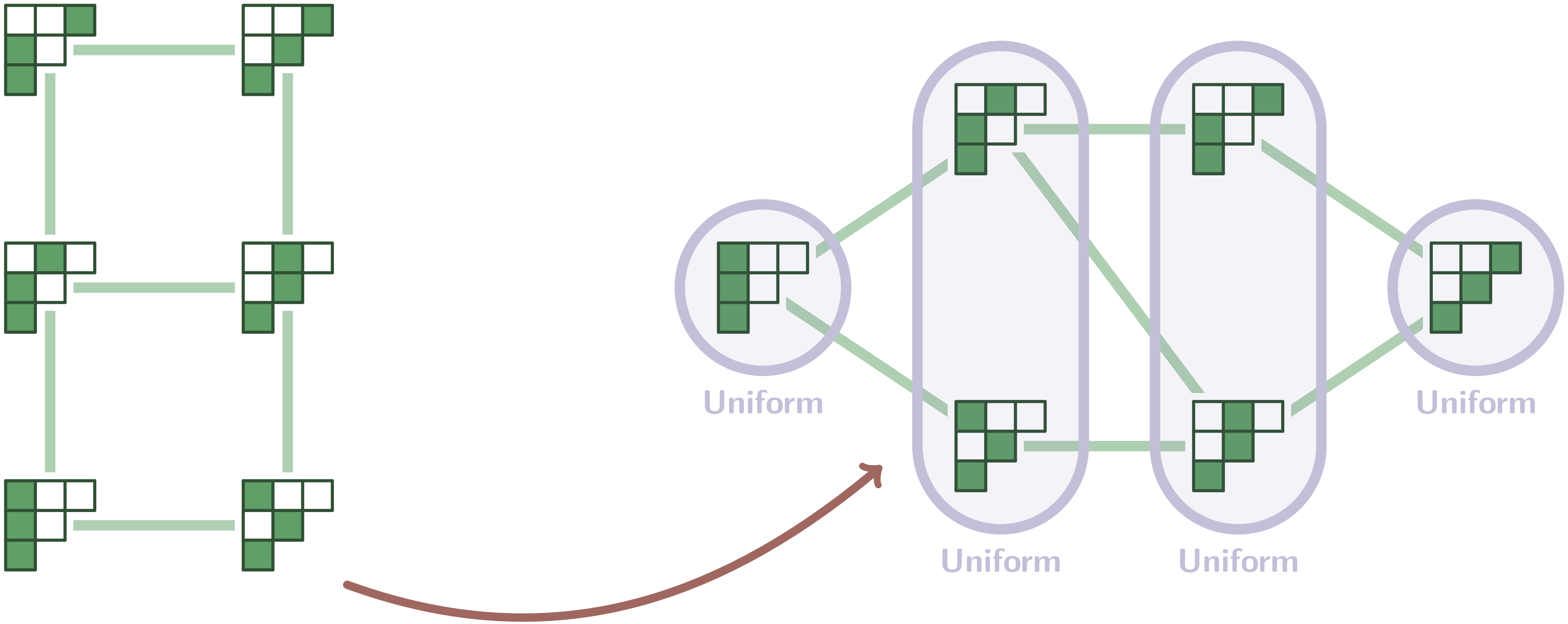
$n = 3$

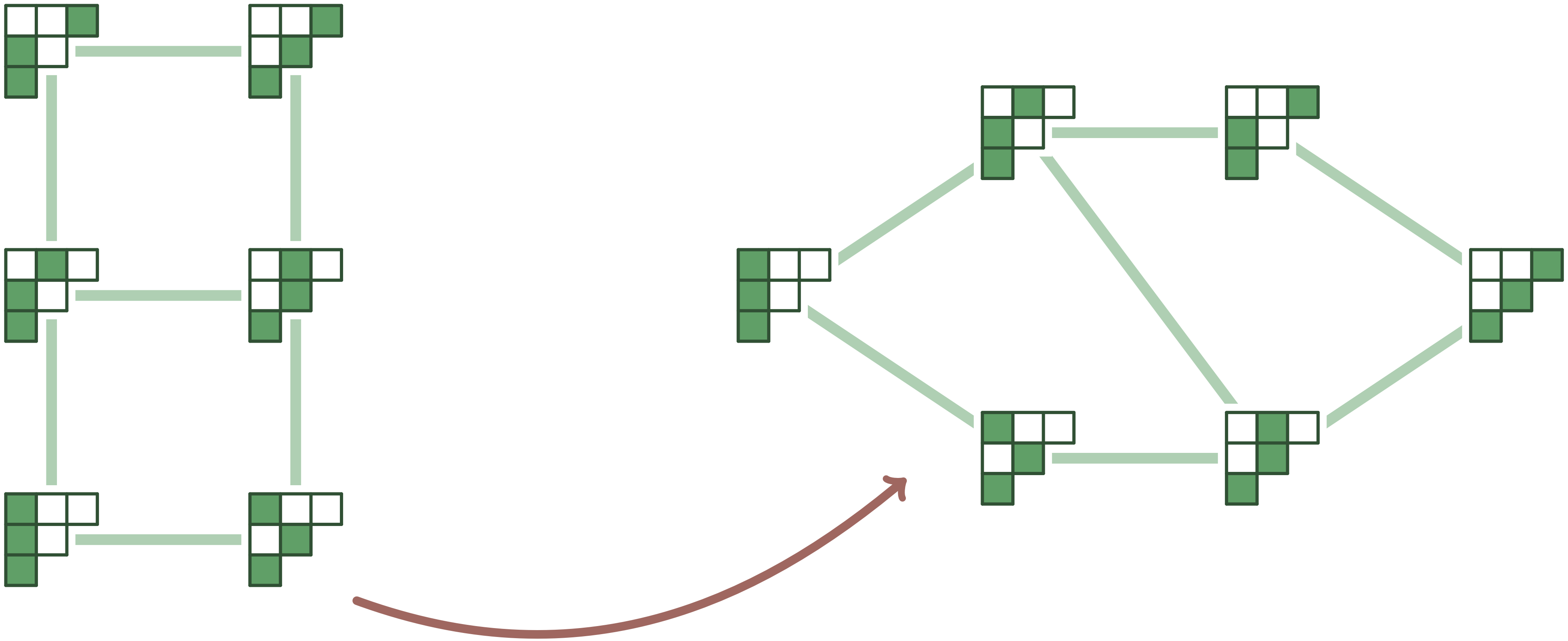




Uniform
Uniform
Uniform
Uniform

Example

$n = 3$



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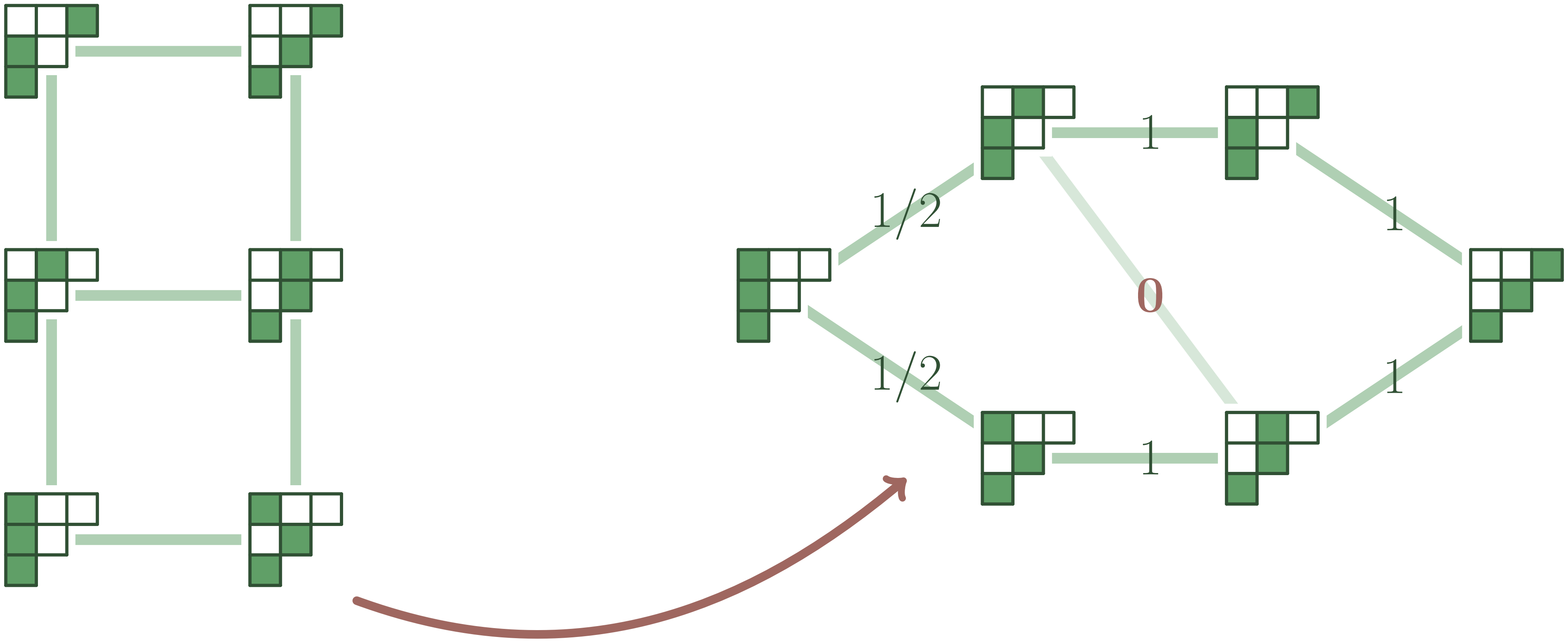

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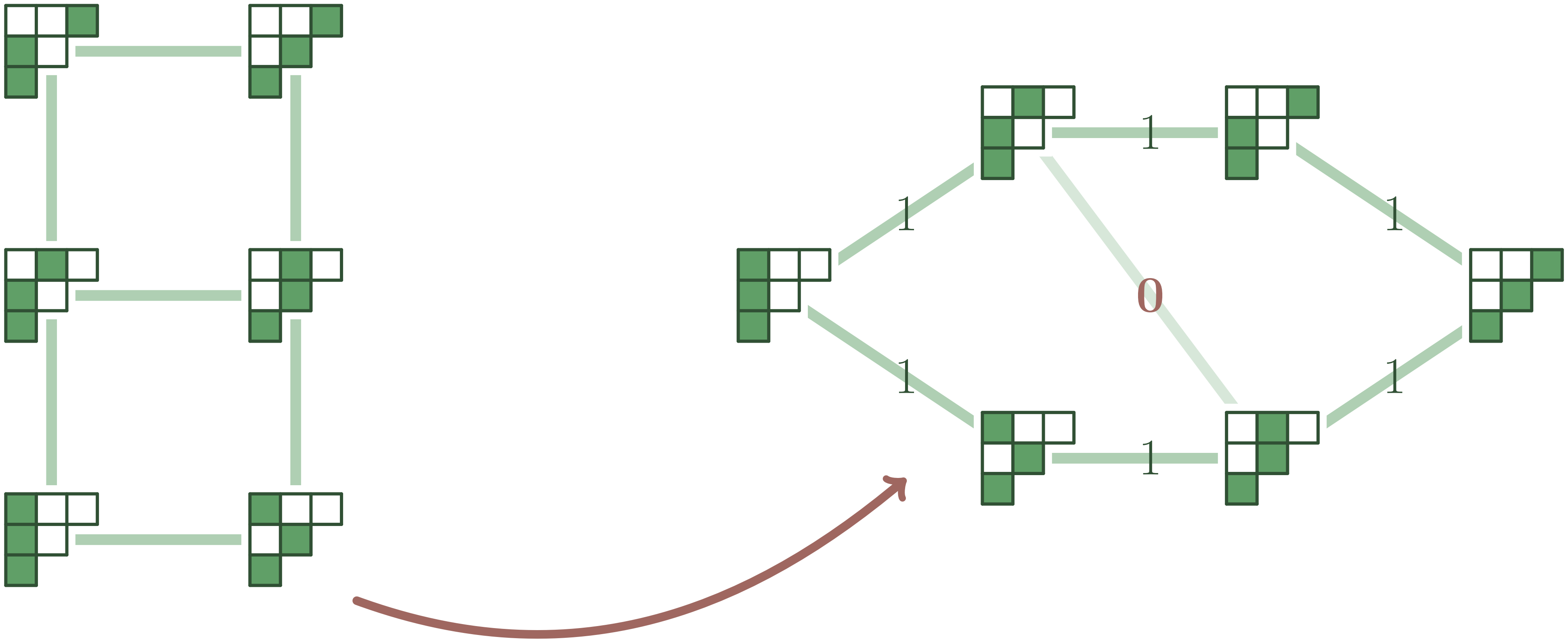
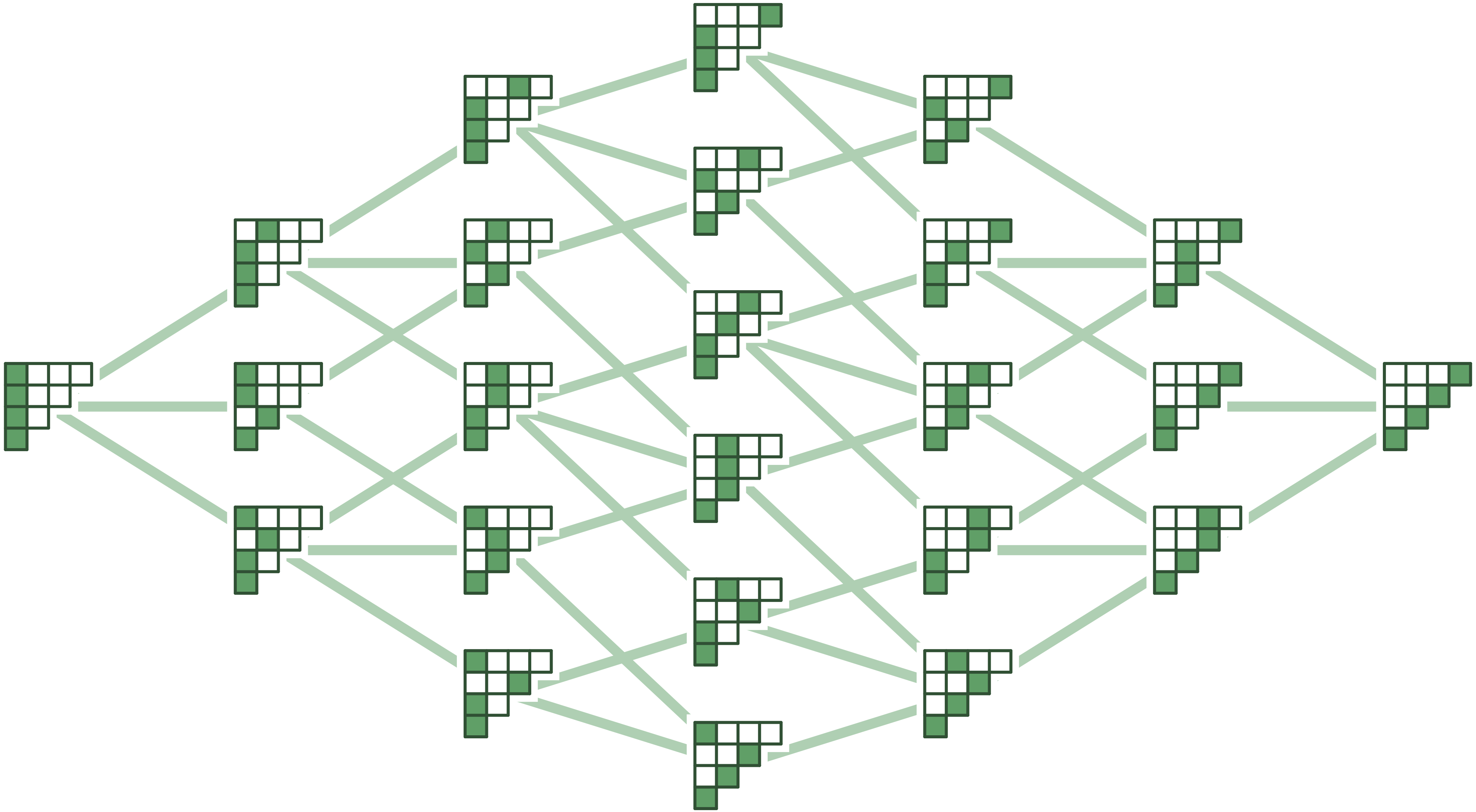


Example

Example

$n = 4$



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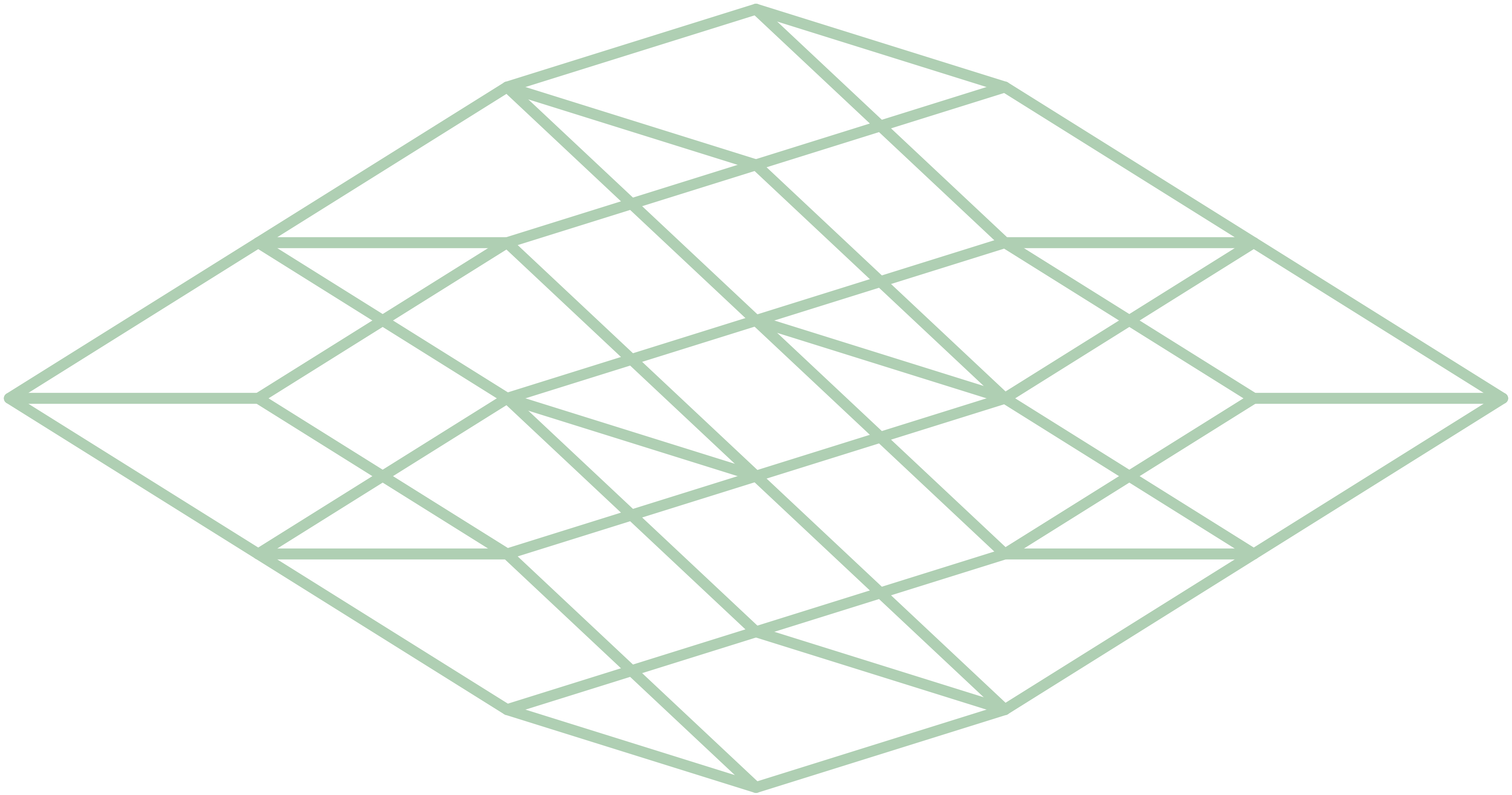
$n = 4$



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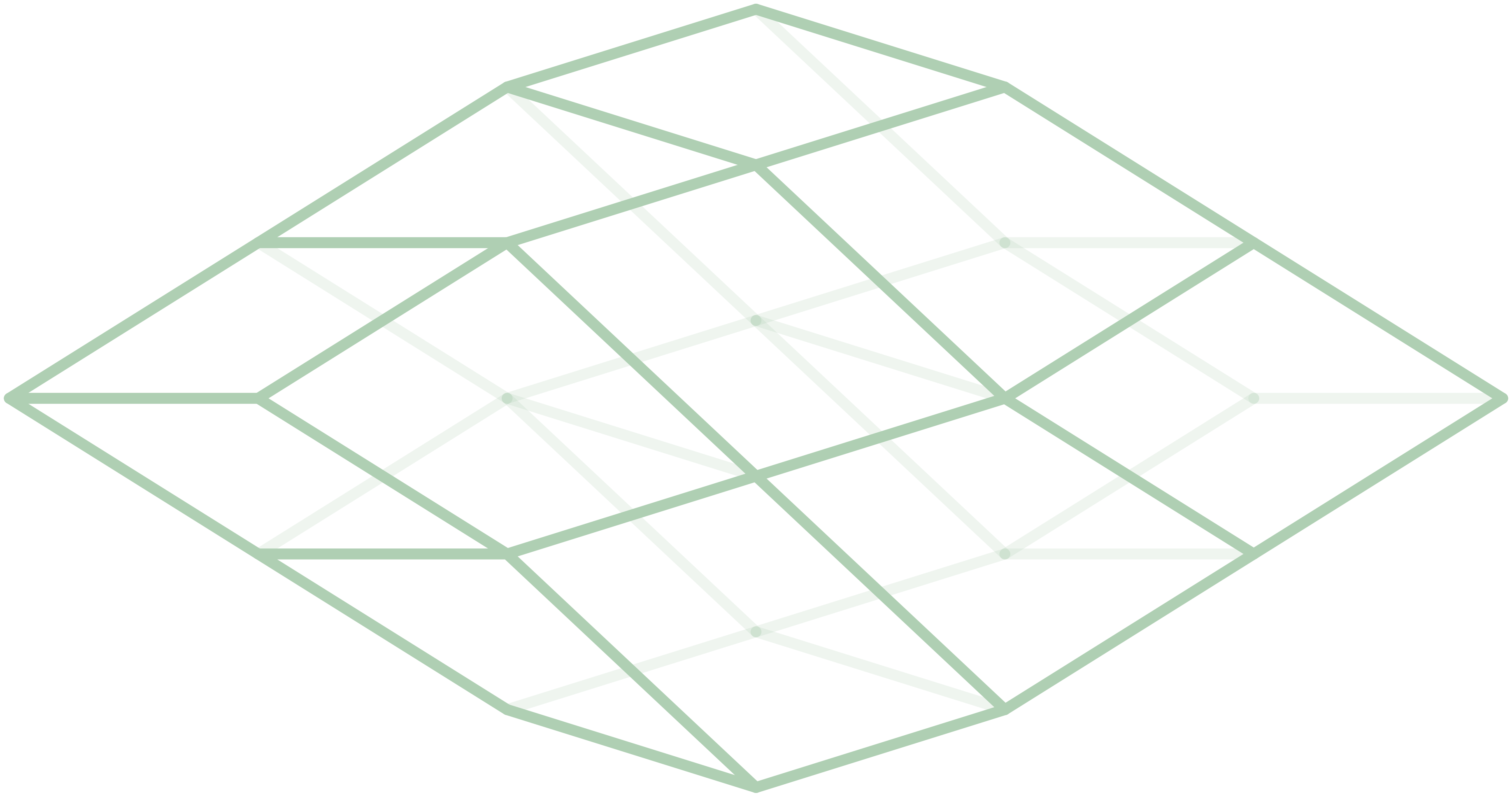

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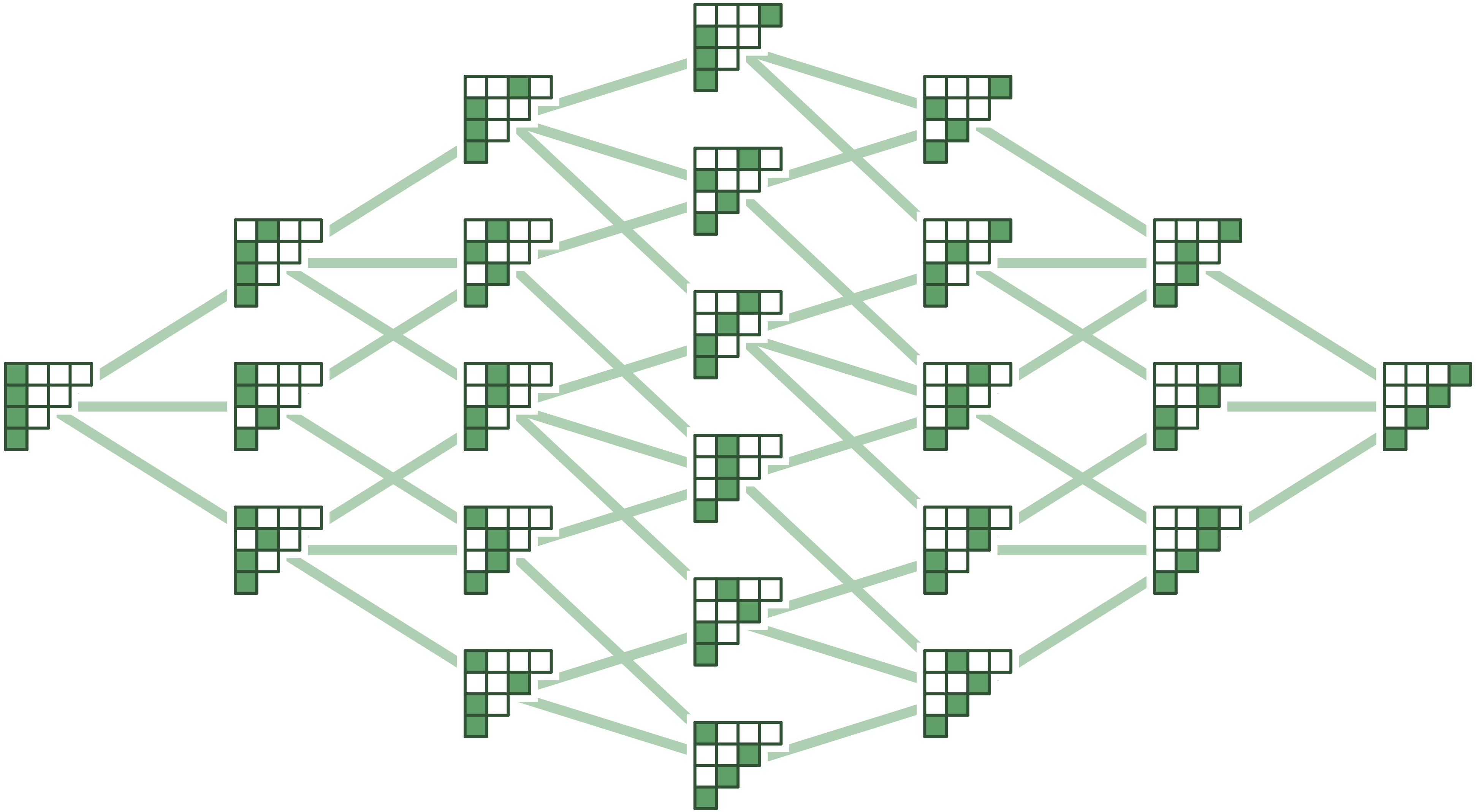

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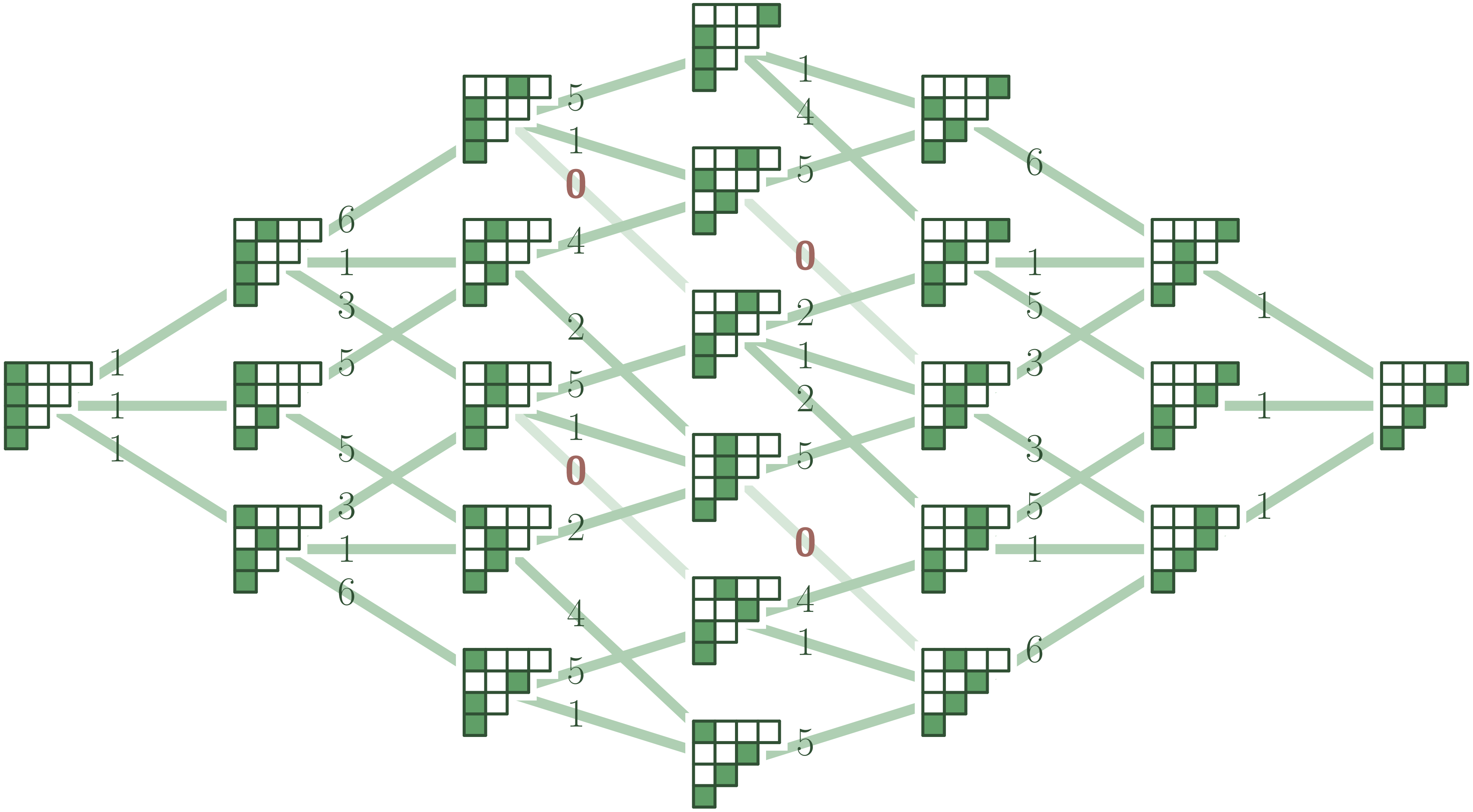

Open problem

Questions:

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Can we prove the existence of such weights on $\mathcal{H}_n$ for any $n \geq 1$?

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Can we prove the existence of such weights on $\mathcal{H}_n$ for any $n \geq 1$?

Can we characterize the values of these weights?

Open problem

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Can we prove the existence of such weights on $\mathcal{H}_n$ for any $n \geq 1$?

Can we characterize the values of these weights?

This would prove that doubly Markovian smooth and strictly increasing Mallows processes exist.

Thank you!

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