Continued fractions using a Laguerre digraph interpretation of the Foata–Zeilberger bijection and its variants

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Permutation Patterns 2023
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Cycle classification

For a permutation σ, compare each i with $\sigma(i)$ and $\sigma^{-1}(i)$:

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- cycle valley $\sigma^{-1}(i) > i < \sigma(i)$
- cycle peaks $\sigma^{-1}(i) < i > \sigma(i)$
- cycle double rise $\sigma^{-1}(i) < i < \sigma(i)$
- cycle double fall $\sigma^{-1}(i) > i > \sigma(i)$
- fixed point $i = \sigma(i) = \sigma^{-1}(i)$

Record classification

Consider σ as a word $\sigma(1)\sigma(2)\ldots\sigma(n)$:

- i is record if for every $j < i$ we have $\sigma(j) < \sigma(i)$
  left-to-right-maxima
- i is antirecord if for every $i > j$ we have $\sigma(i) < \sigma(j)$
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Each i is one of the following four types:

- rar - record-antirecord 
- erec - exclusive record 
- earec - exclusive antirecord 
- nrar - neither record-antirecord 

Record-and-cycle classification

Each i is one of the following ten (not 20) types:

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- rar
- nrfix

Continued fractions counting permutation statistics

Consider 10-variable polynomials

$$P_n(x_1, x_2, y_1, y_2, u_1, u_2, v_1, v_2, w, z) = \sum_{\sigma \in \mathfrak{S}_n} x_1^{\mathrm{eareccpeak}(\sigma)} x_2^{\mathrm{eareccdfall}(\sigma)} y_1^{\mathrm{ereccval}(\sigma)} y_2^{\mathrm{ereccdrise}(\sigma)} z^{\mathrm{rar}(\sigma)} \times u_1^{\mathrm{nrcpeak}(\sigma)} u_2^{\mathrm{nrcdfall}(\sigma)} v_1^{\mathrm{nrcval}(\sigma)} v_2^{\mathrm{nrcdrise}(\sigma)} w^{\mathrm{nrfix}(\sigma)}$$

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Theorem (Sokal–Zeng (2022) First J-fraction for permutations)

$$\sum_{n=0}^{\infty} P_n(x_1,x_2,y_1,y_2,u_1,u_2,v_1,v_2,w,z)t^n$$

$$= \cfrac{1}{1-z\cdot t-\cfrac{x_1\,y_1\cdot t^2}{1-(x_2+y_2+w)\cdot t-\cfrac{(x_1+u_1)(y_1+v_1)\cdot t^2}{1-((x_2+u_2)+(y_2+v_2)+w)\cdot t-\cfrac{(x_1+2u_1)(y_1+2v_1)\cdot t^2}{1-\ddots}}}}$$

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Proof uses the Foata–Zeilberger bijection (1990)

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They later found out that Randrianarivony (1998) had a 17 variable continued fraction.

Can we count cycles as well?

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Conjecture (Sokal–Zeng (2022))

$$\sum_{n=0}^{\infty} P_n(x_1, x_2, y_1, y_2, u_1, u_2, y_1, v_2, w, z, \lambda) t^n$$

$$= \cfrac{1}{1 - \lambda z \cdot t - \cfrac{\lambda x_1 y_1 \cdot t^2}{1 - (x_2 + y_2 + \lambda w) \cdot t - \cfrac{(\lambda+1)(x_1+u_1) y_1 \cdot t^2}{1 - ((x_2+v_2) + (y_2+v_2) + \lambda w) \cdot t - \cfrac{(\lambda+2)(x_1+2u_1) y_1 \cdot t^2}{1 - \ddots}}}}$$

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Twist in story:
Can prove their full conjecture using Foata–Zeilberger bijection

We can count cycles in the Foata–Zeilberger bijection

Foata–Zeilberger bijection

A Motzkin path is a path $(0,0) \to (n,0)$ in the non-negative quadrant with steps $(1,1),(1,0),(1,-1)$

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$$\sigma \mapsto (\omega, \xi)$$

where

- $\sigma \in \mathfrak{S}_n$

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  - If step i is $\nearrow$ starting at height h_i, $\xi_i \in [0, h_i]$
  - If step i is $\searrow$ starting at height h_i, $\xi_i \in [0, h_i - 1]$
  - If step i is $\rightarrow$ at height h_i, $\xi_i \in [0, h_i - 1] \cup [0, h_i - 1] \cup \{0\}$

Description of $\sigma \to \omega$

- If i is a cycle valley, step i is $\nearrow$
- If i is a cycle peak, step i is $\searrow$
- If i is a cycle double rise, cycle double fall or fixed, step i is $\color{red}{\rightarrow}$, $\color{blue}{\rightarrow}$ or $\color{green}{\rightarrow}$ respectively.

Description of $\sigma \to \omega$

Let $\sigma = 715492638 = (1762)(3598)(4) \in \mathfrak{S}_9 x$.

- Cval $= \{1, 3\}$
- Cpeak $= \{7, 9\}$
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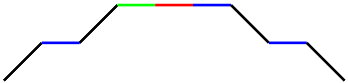

Description of labels $\sigma \to \xi$

For $i \in [n]$

$$\xi_i = \begin{cases} \#\{j\colon j < i \text{ and } \sigma(j) > \sigma(i)\} & \text{if } \sigma(i) > i \quad \text{if } i \in \text{ Cval} \cup \text{Cdrise} \\ \#\{j\colon j > i \text{ and } \sigma(j) < \sigma(i)\} & \text{if } \sigma(i) < i \quad \text{if } i \in \text{Cpeak} \cup \text{Cdfall} \\ 0 & \text{if } \sigma(i) = i \quad \text{if } i \in \text{ Fix} \end{cases}$$

Description of the inverse bijection $(\omega, \xi) \mapsto \sigma$

Define the sets

$$\text{excedance indices } F \quad = \quad \{i \in \sigma : \sigma(i) > i\} \;=\; \mathsf{Cdrise} \cup \mathsf{Cval}$$

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(a) We can clearly read off the sets F, F', G, G', H from the Motzkin path ω.

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(b) Construction of $\sigma|_F : F \to F'$. Let $F = \{i_1 < \ldots < i_k\}$. Let $j_1 j_2 \cdots j_k$ be the permutation of the letters F', such that for any α, the number of letters larger than j_α to the left of j_α is ξ_{i_α}. Then we define $\sigma(i_\alpha) = j_\alpha$ (this is the left-to-right inversion table).

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We similarly define $\sigma|_G : G \to G'$, except now we look at letters smaller to the right (this is the right-to-inversion table).

An example

Let $\sigma = 715492638 = (1762)(3598)(4) \in \mathfrak{S}_9$.

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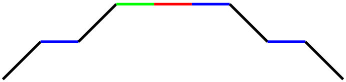

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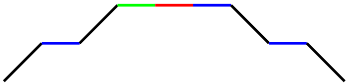

The labels ξ and the sets F, F', G, G' are:

$i \in F$	1	3	5
$\sigma(i) \in F'$	7	5	9
ξ_i	0	1	0

$i \in G$	2	6	7	8	9
$\sigma(i) \in G'$	1	2	6	3	8
ξ_i	0	0	1	0	0

Laguerre digraph

Definition

A Laguerre digraph of size n is a directed graph where each vertex has a distinct label from the label set $\{1, \ldots, n\}$ and has indegree 0 or 1 and outdegree 0 or 1.

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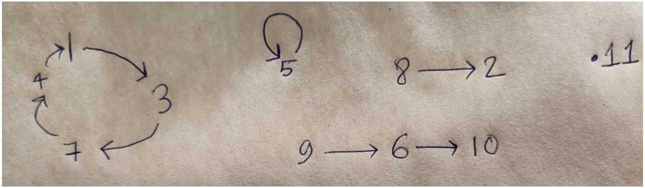
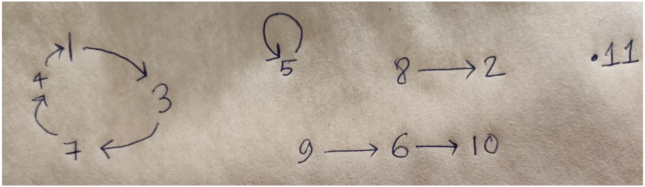


Connected components



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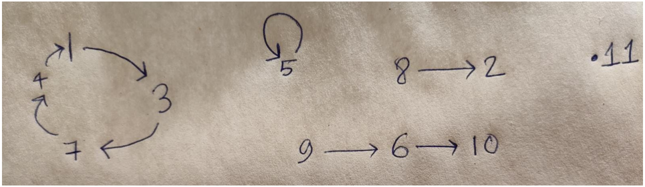

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- Directed paths

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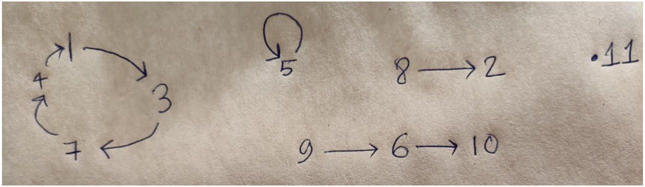

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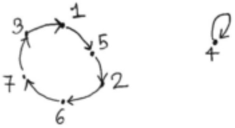
Laguerre digraphs generalise permutations

Laguerre digraphs generalise permutations

1. No paths - Cyclic structure of permutations



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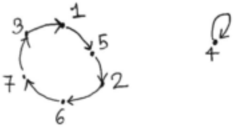

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Laguerre digraphs after Sokal (2022)

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Start with all n vertices and no edges

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Stage (a): $i \in H$ (fixed points) in increasing order

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This order is suggested by the inverse bijection and the inversion tables

"History" of Foata–Zeilberger bijection. Stage (a)

Insert $i \to \sigma(i)$ in order
Stage (a): $i \in H$ **(fixed points) in increasing order**

At the end of this stage, the Laguerre digraph consists of loops at all vertices in H. All other vertices have no adjacent edges.

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At the end of this stage, the resulting Laguerre digraph consists of loops at all vertices in H, and directed paths
cycle peak→ (cycle double fall)* →cycle valley

No edges adjacent to cycle double rises.

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A cycle is closed when i is the final vertex of a path and $\sigma(i)$ is the initial vertex of the same path. We show that in such a situation, i is a cycle valley. Exactly one choice of label ξ_i that makes this possible.

Story continues ....

This resolves the Sokal–Zeng conjecture (2022) for permutations

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Using similar ideas to a slightly different bijection have resolved a 4-variable conjectured continued fraction due to Randrianarivony–Zeng (1996) for objects Genocchi numbers

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Using similar ideas to a slightly different bijection have resolved a 4-variable conjectured continued fraction due to Randrianarivony–Zeng (1996) for objects Genocchi numbers

Similar to Sokal–Zeng, have generalised these continued fractions to families of infinitely many variables

A conjecture of Baril and Kirgizov

Baril and Kirgizov (2021) conjectured the following equidistribution of statistics on $\mathfrak{S}_n$:

Conjecture

The bistatistics $(\mathrm{des}_2, \mathrm{cyc})$ *and* $(\mathrm{pex}, \mathrm{cyc})$ *are equidistributed.*

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Han–Mao–Zeng (2021) showed that this conjecture is equivalent to the following:

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$$\sum_{n=0}^{\infty} y^{\mathrm{des}_2 \sigma} \lambda^{\mathrm{cyc}\sigma} = \cfrac{1}{1 - \lambda z - \cfrac{\lambda y z^2}{1 - (\lambda + 2) z - \cfrac{(\lambda + 1)(y + 1) z^2}{\cdots}}}$$

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How do we resolve this???

Merci pour votre attention

Combinatorial Interpretation of J-fraction

Jacobi-type continued fraction (J-fraction)

$$\cfrac{1}{1-\gamma_0 t-\cfrac{\beta_1 t^2}{1-\gamma_1 t-\cfrac{\beta_2 t^2}{1-\gamma_2 t-\cfrac{\beta_3 t^2}{\ddots}}}}$$

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Associated C-fraction outside of combinatorial literature

Motzkin paths

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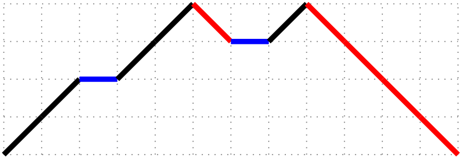

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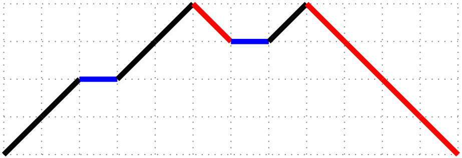

Assign weights:

- $\nearrow$: 1
- $\rightarrow$ from height $i \rightarrow i$: γ_i
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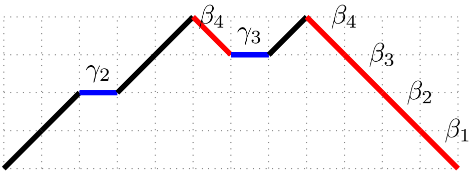

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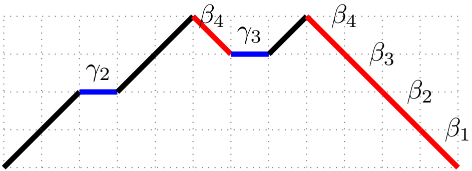

Weight $= \beta_1\beta_2\beta_3\beta_4^2\gamma_2\gamma_3$

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Theorem (Flajolet '80)

The a_n are weighted sum of Motzkin paths with n steps.

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Gateway for proving continued fractions using bijective combinatorics :-D