Weak ascent sequences, permutations, matrices, and posets

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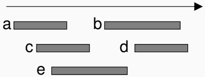
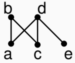
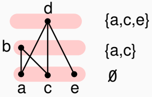
1. Interval orders, ascent sequences and pattern avoiding permutations

A partially ordered set $(P, \leq_P)$ is called $(\mathbf{2}+\mathbf{2})$-free if it contains no induced sub-poset isomorphic to $(\mathbf{2}+\mathbf{2}) =$ 

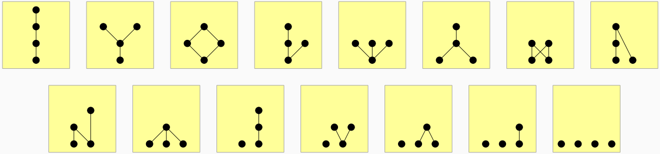
Such posets arise as interval orders:







Let $\mathcal{P}_n$ be the set of unlabeled $(\mathbf{2}+\mathbf{2})$-free posets on n elements.
Hasse diagrams of all members of $\mathcal{P}_4$:



M. Bousquet-Mélou, A. Claesson, M. Dukes and S. Kitaev
(2+2)-free posets, ascent sequences and pattern avoiding permutations
J. Comb. Theory A 2010.

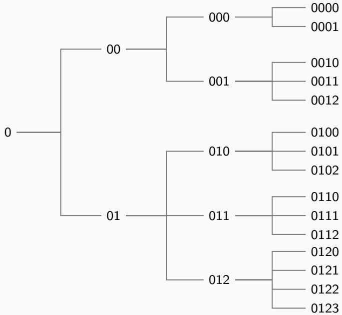
1. Ascent sequences and integer matrices

An integer sequence $(x_1, \ldots, x_n)$ is an ascent sequence if $x_1 = 0$ and

$$0 \leq x_i \leq 1 + \mathrm{asc}(x_1, \ldots, x_{i-1})$$

where $\mathrm{asc}(x_1, \ldots, x_n) = \#\{i : x_i < x_{i+1}\}$.

$$\mathrm{asc}(0, 1, 0, 1, 2, 0, 2, 1, 3) = 5$$



Let Int_n be the collection of all upper triangular matrices containing non-negative integers such that

- the entries sum to n, and
- there are no rows or columns of 0s.

$$\mathrm{Int}_4 = \left\{ \begin{array}{l} \begin{bmatrix}4\end{bmatrix}, \begin{bmatrix}3&0\\0&1\end{bmatrix}, \begin{bmatrix}2&0\\0&2\end{bmatrix}, \begin{bmatrix}1&0\\0&3\end{bmatrix}, \\ \begin{bmatrix}2&1\\0&1\end{bmatrix}, \begin{bmatrix}1&2\\0&1\end{bmatrix}, \begin{bmatrix}1&1\\0&2\end{bmatrix}, \\ \begin{bmatrix}2&0&0\\0&1&0\\0&0&1\end{bmatrix}, \begin{bmatrix}1&0&0\\0&2&0\\0&0&1\end{bmatrix}, \begin{bmatrix}1&0&0\\0&1&0\\0&0&2\end{bmatrix}, \\ \begin{bmatrix}1&1&0\\0&0&1\\0&0&1\end{bmatrix}, \begin{bmatrix}1&1&0\\0&1&0\\0&0&1\end{bmatrix}, \begin{bmatrix}1&0&1\\0&1&0\\0&0&1\end{bmatrix}, \\ \begin{bmatrix}1&0&0\\0&1&1\\0&0&1\end{bmatrix}, \begin{bmatrix}1&0&0&0\\0&1&0&0\\0&0&1&0\\0&0&0&1\end{bmatrix} \end{array} \right\}$$

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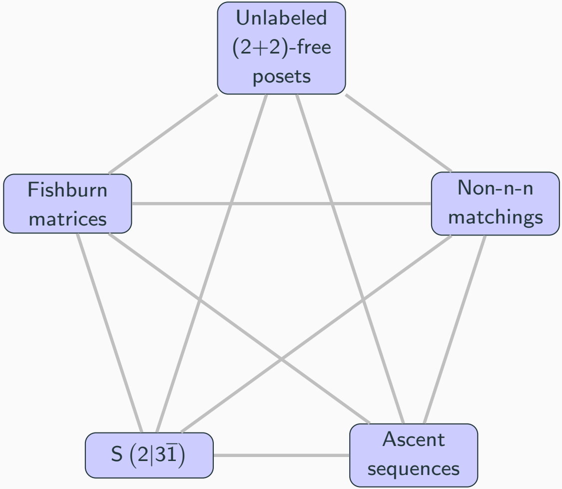
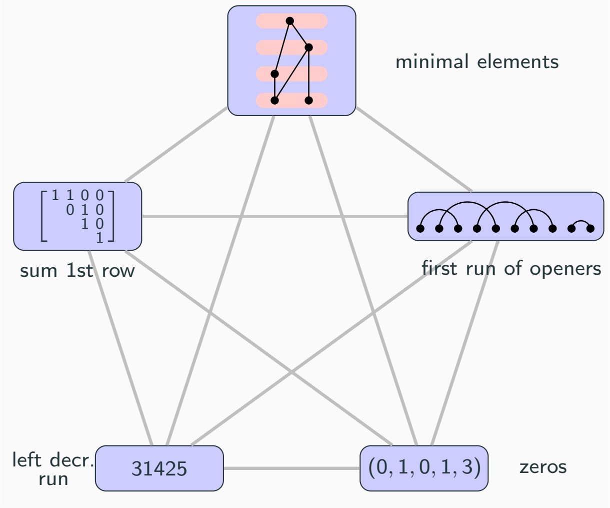
1. A pentagram of correspondences

Theorem 1.1 The grey lines represent bijective correspondences.



Theorem 1.2 Many statistics are translated through these correspondences.



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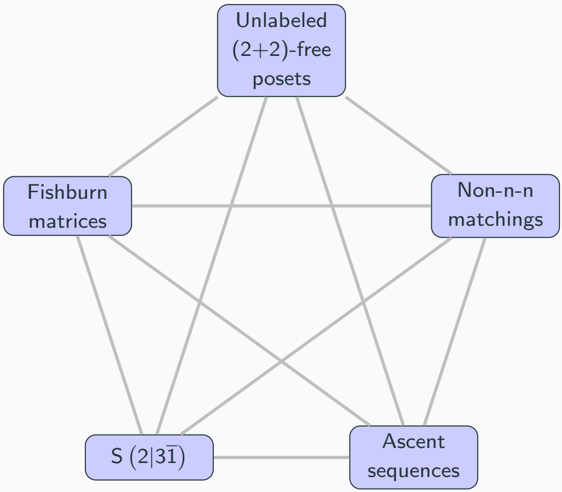

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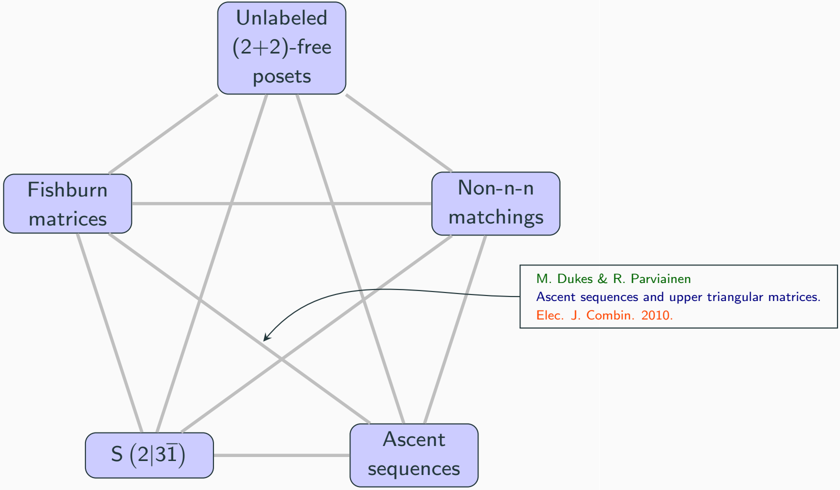

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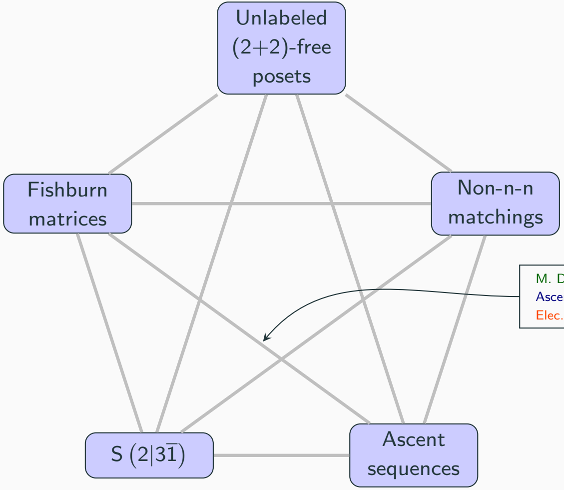
Theorem 1.3 The correspondences can be lifted to higher levels.



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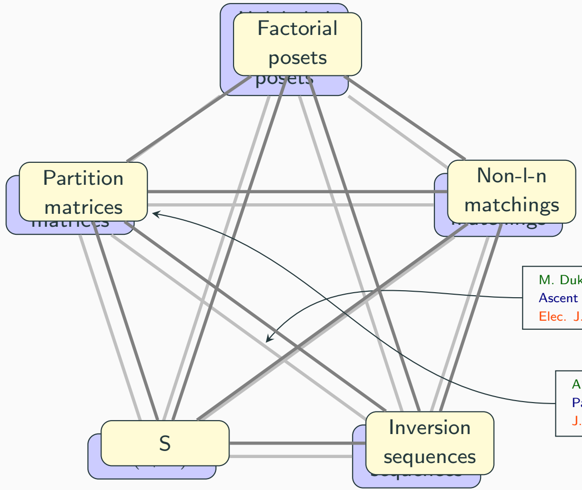

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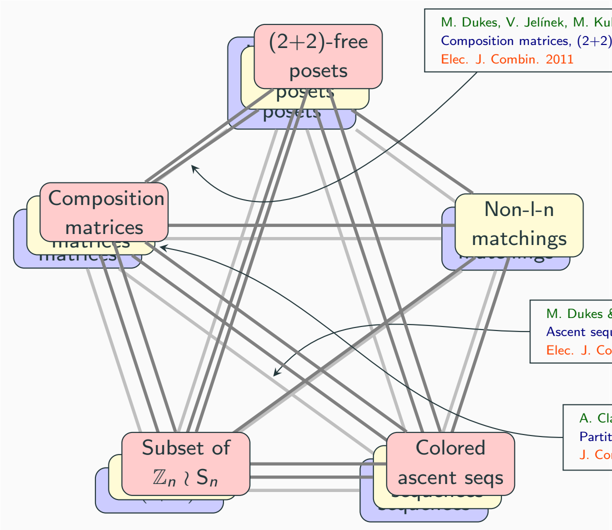

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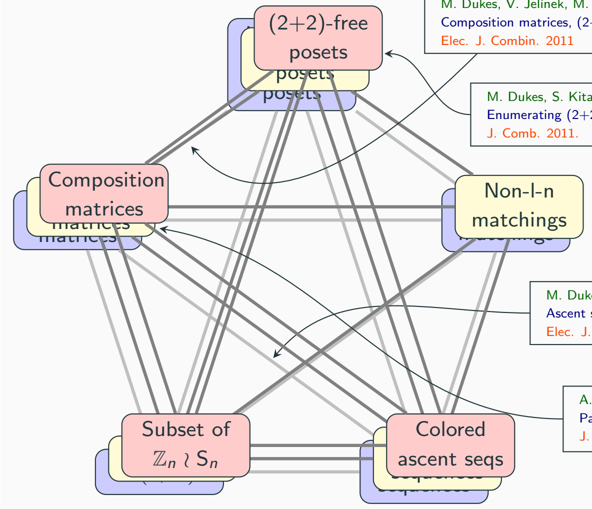

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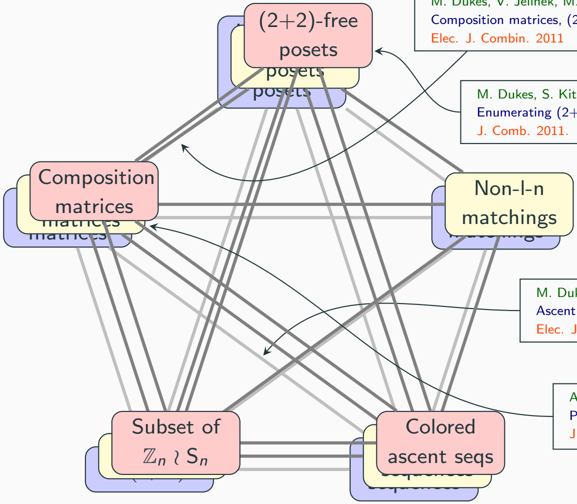

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What happens if we tweak the definition of an ascent sequence to allow adjacent equal entries to represent ascents?

Is there a similarly rich collection of correspondences that we can unearth an prove?

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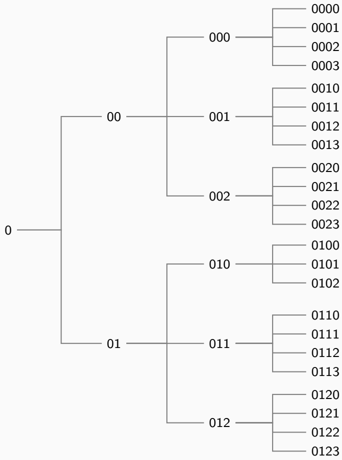
Let WAsc_n be the set of length-n weak ascent sequences.



3. Weak Fishburn permutations = $S_n(3|41\bar{2})$

A permutation $\pi \in S_n$ contains $3|41\bar{2}$ if there exist indices i, j, k, ℓ such that

- $1 \leq i < j < k < \ell \leq n$
- $j = i + 1$
- $\pi_i = \pi_\ell + 1$
- $\pi_k < \pi_\ell < \pi_i < \pi_j$.

In this case we also say that $\pi_i\pi_j\pi_k\pi_\ell$ is an occurrence of $3|41\bar{2}$ in π.

If there are no occurrences of $3|41\bar{2}$ in π, then we say that π avoids $3|41\bar{2}$.

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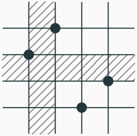

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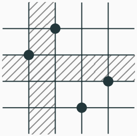

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A nice feature of weak Fishburn permutations is they admit a recursive construction similar to that of $S_n(2|3\overline{1})$.

Every $\pi \in \mathcal{W}_n$ can be written as a pair (π', i) where $\pi' \in \mathcal{W}_{n-1}$ and i is the index of which active site of π' we will place the value n.

For example, $\pi = 62758413 \in \mathcal{W}_8$ can be uniquely written as the pair

$$(\pi', i) = (6275413, 3)$$

where

$$\pi' = \bullet 6\ 2 \bullet 7 \bullet 5 \bullet 4\ 1 \bullet 3 \bullet$$

and $i = 3$ since 8 replaces the (3+1)th bullet from the left.

3. Weak Fishburn permutations cont'd

The positions of these bullets are known as active sites, and for weak Fishburn permutations the rule for placing them is as follows.

Let $\tau_1\tau_2\cdots\tau_{n-1} \in \mathcal{W}_{n-1}$.
The site between entries τ_i and τ_{i+1} is active if

- $\tau_i \leq 2$, or
- $\tau_i - 1$ is to the left of τ_i, or
- $\tau_i - 1$ is to the right of τ_i and there is no value $t < \tau_i - 1$ between τ_i and $\tau_i - 1$.

With this notion of active sites let us label the active sites, from left to right, with $\{0, 1, 2, \ldots\}$.

The permutation $\pi = 62754138$ corresponds to the sequence $x = (0, 0, 2, 1, 1, 0, 1, 5)$.

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$$
\begin{aligned}
{}_0 1_1 &\xrightarrow{x_2=0} {}_0 2_1 1_2 \\
&\xrightarrow{x_3=2} {}_0 2_1 1_2 3_3 \\
&\xrightarrow{x_4=1} {}_0 2_1 4\ 1_2 3_3 \\
&\xrightarrow{x_5=1} {}_0 2_1 5_2 4\ 1_3 3_4 \\
&\xrightarrow{x_6=0} {}_0 6\ 2_1 5_2 4\ 1_3 3_4 \\
&\xrightarrow{x_7=1} {}_0 6\ 2_1 7_2 5_3 4\ 1_4 3_5 \\
&\xrightarrow{x_8=5} \ 6\ 2\ 7\ 5\ 4\ 1\ 3\ 8.
\end{aligned}
$$

$\Gamma(6, 2, 7, 5, 4, 1, 3, 8) = (0, 0, 2, 1, 1, 0, 1, 5)$.

Theorem: $\Gamma : \mathcal{W}_n \mapsto \mathrm{WAsc}_n$ is a bijection. If $x = \Gamma(\pi)$ then $\mathrm{wasc}(x) = \mathrm{numact}(\pi)$ and $x_n = \mathrm{lastact}(\pi)$ where numact is the number of active sites in π and $\mathrm{lastact}$ is the label of the site located just before the largest entry of π.

4. Weak Fishburn matrices

Let WMat_n be the set of upper triangular square 0/1-matrices A that satisfy the following three properties:

(a) There are n 1s in A.

(b) There is at least one 1 in every column of A.

(c) For every pair of adjacent columns, the topmost 1 in the left column is weakly above the bottommost 1 in the right column.

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n	WMat_n
1	$\begin{bmatrix}1\end{bmatrix}$
2	$\begin{bmatrix}1&1\\0&0\end{bmatrix}, \begin{bmatrix}1&0\\0&1\end{bmatrix}$
3	$\begin{bmatrix}1&1\\0&1\end{bmatrix}, \begin{bmatrix}1&1&1\\0&0&0\\0&0&0\end{bmatrix}, \begin{bmatrix}1&1&0\\0&0&1\\0&0&0\end{bmatrix}, \begin{bmatrix}1&1&0\\0&0&0\\0&0&1\end{bmatrix}, \begin{bmatrix}1&0&0\\0&1&1\\0&0&0\end{bmatrix}, \begin{bmatrix}1&0&0\\0&1&0\\0&0&1\end{bmatrix}$
4	$\begin{bmatrix}1&1&1\\0&0&1\\0&0&0\end{bmatrix}, \begin{bmatrix}1&1&1\\0&0&0\\0&0&1\end{bmatrix}, \begin{bmatrix}1&1&0\\0&0&1\\0&0&1\end{bmatrix}, \begin{bmatrix}1&1&1\\0&1&0\\0&0&0\end{bmatrix}, \begin{bmatrix}1&1&0\\0&1&1\\0&0&0\end{bmatrix}, \begin{bmatrix}1&1&0\\0&1&0\\0&0&1\end{bmatrix}, \begin{bmatrix}1&0&1\\0&1&1\\0&0&0\end{bmatrix}, \begin{bmatrix}1&0&1\\0&1&0\\0&0&1\end{bmatrix}, \begin{bmatrix}1&0&0\\0&1&1\\0&0&1\end{bmatrix},$ $\begin{bmatrix}1&1&1&1\\0&0&0&0\\0&0&0&0\\0&0&0&0\end{bmatrix}, \begin{bmatrix}1&1&1&0\\0&0&0&1\\0&0&0&0\\0&0&0&0\end{bmatrix}, \begin{bmatrix}1&1&1&0\\0&0&0&0\\0&0&0&1\\0&0&0&0\end{bmatrix}, \begin{bmatrix}1&1&1&0\\0&0&0&0\\0&0&0&0\\0&0&0&1\end{bmatrix}, \begin{bmatrix}1&1&0&0\\0&0&1&1\\0&0&0&0\\0&0&0&0\end{bmatrix}, \begin{bmatrix}1&1&0&0\\0&0&1&0\\0&0&0&1\\0&0&0&0\end{bmatrix}, \begin{bmatrix}1&1&0&0\\0&0&1&0\\0&0&0&0\\0&0&0&1\end{bmatrix},$ $\begin{bmatrix}1&1&0&0\\0&0&0&0\\0&0&1&1\\0&0&0&0\end{bmatrix}, \begin{bmatrix}1&1&0&0\\0&0&0&0\\0&0&1&0\\0&0&0&1\end{bmatrix}, \begin{bmatrix}1&0&0&0\\0&1&1&1\\0&0&0&0\\0&0&0&0\end{bmatrix}, \begin{bmatrix}1&0&0&0\\0&1&1&0\\0&0&0&1\\0&0&0&0\end{bmatrix}, \begin{bmatrix}1&0&0&0\\0&1&1&0\\0&0&0&0\\0&0&0&1\end{bmatrix}, \begin{bmatrix}1&0&0&0\\0&1&0&0\\0&0&1&1\\0&0&0&0\end{bmatrix}, \begin{bmatrix}1&0&0&0\\0&1&0&0\\0&0&1&0\\0&0&0&1\end{bmatrix}$

4. Mapping weak ascent sequences to weak Fishburn matrices

Let us construct the weak Fishburn matrix A that corresponds to the weak ascent sequence $x = (0, 0, 2, 1, 1, 0, 1, 5) \in \mathrm{WAsc}_8$.

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$$\begin{bmatrix} 1 \end{bmatrix} \xrightarrow{x_2=0} \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix} \xrightarrow{x_3=2} \begin{bmatrix} 1 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \xrightarrow{x_4=1} \begin{bmatrix} 1 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 1 \end{bmatrix} \xrightarrow{x_5=1} \begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\xrightarrow{x_6=0} \begin{bmatrix} 1 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \xrightarrow{x_7=1} \begin{bmatrix} 1 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\xrightarrow{x_8=5} \begin{bmatrix} 1 & 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} = A$$

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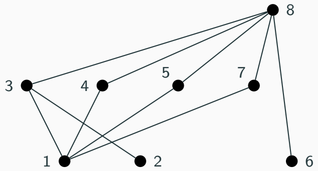
We write $\Omega(A) = x$.

Theorem: $\Omega : \mathrm{WMat}_n \to \mathrm{WAsc}_n$ is a bijection. Let $\Omega(M) = w$. Then

- \# occurrences of j in w = the sum of the entries of the $(j+1)$th row in M,
- \# weak ascents in w = the dimension of M reduced by 1,
- length of final decreasing run = sum of the entries in rightmost column of M,
- w_n, the last entry of w, is equal to $\mathrm{topone}(M) - 1$.

5. Weakly factorial posets

A poset P on the elements $\{1, \ldots, n\}$ is naturally labeled if $i <_P j$ implies $i < j$:



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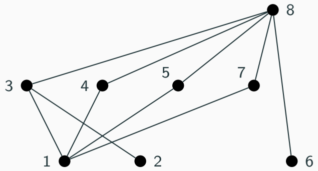


Definition

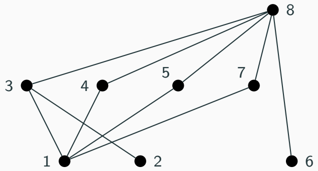
A naturally labeled poset P on $[1, n]$ such that, for all $i, j, k \in [1, n]$, we have

$$i < j \text{ and } j <_P k \implies i <_P k$$

is called a factorial poset.

5. Weakly factorial posets

A poset P on the elements $\{1, \ldots, n\}$ is naturally labeled if $i <_P j$ implies $i < j$:



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Factorial posets are (2+2)-free.

(This and further properties of factorial posets can be found in Anders Claesson, Svante Linusson. n! matchings, n! posets. Proc. AMS, 2011.)

Definition (The mapping Ψ)

Let $A \in \mathrm{WMat}_n$. Form a matrix B as follows. Make a copy of A. Beginning with the leftmost column, and within each column one goes from bottom to top, replace every 1 that appears with the elements $1, 2, \ldots, n$. Further, define a partial order $(P, <)$ on $[1, n]$ as follows: $i <_P j$ if the index of the column that contains i is strictly less than the index of the row that contains j. Let

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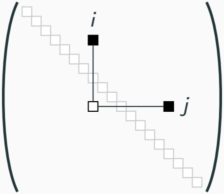
Diagrammatically the relation in Definition 2 is equivalent to i being north-west of j in the matrix and the "lower hook" of i and j being strictly beneath the diagonal:



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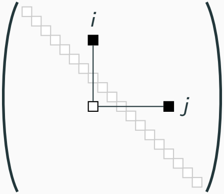

Note that the set of entries contained in the first s columns for an s is the complete set $\{1, 2, \ldots, s_k\}$ for some s_k.

Consider the matrix A from the previous example. Form B by relabeling the 1s in the matrix according to the rule:

$$A = \begin{bmatrix} 1 & 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} \quad \mapsto \quad B = \begin{bmatrix} 1 & 2 & 0 & 6 & 0 & 0 \\ 0 & 0 & 4 & 5 & 7 & 0 \\ 0 & 0 & 3 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 8 \end{bmatrix}$$

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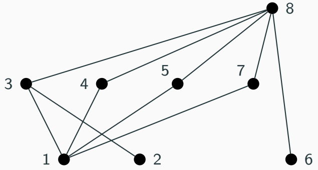
The Hasse diagram of this poset is:



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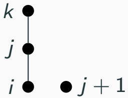

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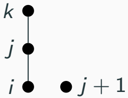

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Let WPoset_n be the set of weakly $(3+1)$-free factorial posets on $[1, n]$.

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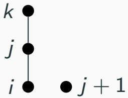

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Theorem

The mapping $\Psi : \mathrm{WMat}_n \to \mathrm{WPoset}_n$ *is a bijection. Moreover, if* $P = \Psi(M)$ *then*

- *the sum of the top row of* M *is the number of minimal elements in* P*,*
- *the number of non-zero rows in* M *equals the number of levels in the poset* P*.*

6. Some comments on enumeration

On looking at the first few values of this sequence

$$1, 1, 2, 6, 23, 106, 567, 3440, 23286, 173704, 1414102.$$

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This expression and the bijection imply the following:

Let $A_n = |\mathrm{WAsc}_n|$. Then $A_n = \sum_{k=0}^{n} a_{n,k}$, where $a_{n,k}$ is given by the following formula. The initial values $a_{0,0} = 1$, $a_{n,0} = a_{0,k} = 0$ and

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Theorem: The number of weak ascent sequences of length n having k weak ascents is $a_{n,k+1}$.

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Open question: Is it possible to given a closed form for the generating function $A(z)$?

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Thank you!

A. Claesson, B. Benyi and M. Dukes.
Weak ascent sequences and related combinatorial structures.
Eur. J. Comb. **108** 2023, 103633.