Combinatorial Stieltjes moment sequences

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STIELTJES MOMENT SEQUENCES

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Example 1: If there is a density function $\mu(x)$, then

$$a_n = \int_0^\infty x^n \mu(x) dx$$

Example 2: If the measure is discrete then there are $c_j, d_j \geq 0$ satisfying

$$a_n = \sum_j c_j d_j^n$$

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$$\begin{bmatrix} a_0 & a_1 & \dots \\ a_1 & a_2 & \dots \\ \vdots & \vdots & \ddots \end{bmatrix} \text{ and } \begin{bmatrix} a_1 & a_2 & \dots \\ a_2 & a_3 & \dots \\ \vdots & \vdots & \ddots \end{bmatrix} \text{ are positive semi-definite.}$$

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Question: Which counting sequences are Stieltjes moment sequences?

PROPERTIES OF STIELTJES MOMENT SEQUENCES

If $(a_n)_{n\geq 0}$ is a Stieltjes moment sequence with generating function $A(t)$ then

- The ratios a_{n+1}/a_n are increasing (i.e., sequence is log-convex).
- The sequence $(a_n a_{n+2} - a_{n+1}^2)_{n\geq 0}$ is also a Stieltjes moment sequence
- All singularities of $A(t)$ lie in $\mathbb{R}_{\geq 0}$.
- We can produce "good" lower bounds on the growth rate

$$\mu = \lim_{n\to\infty} \sqrt[n]{a_n}.$$

Another property: Changing finitely many terms of a Stieltjes moment sequence growing at most exponentially never yields a different Stieltjes moment sequence with the same initial value

Story time: back in 2018...

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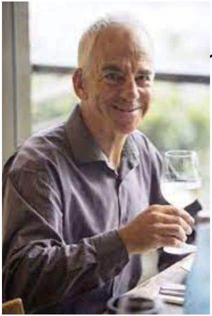

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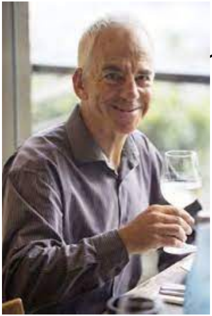

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Conjecture: The sequence $\mathrm{Av}_n(1324)$ is a Stieltjes moment sequence

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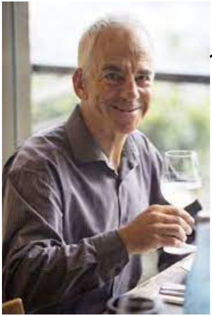

Conjecture: The sequence $\mathrm{Av}_n(1324)$ is a Stieltjes moment sequence
Conjollary: the growth rate μ satisfies $\mu > 10.302$

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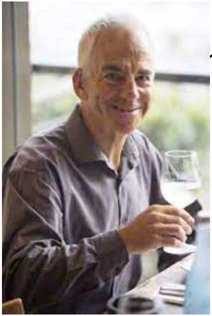

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Conjecture: The sequence $\mathrm{Av}_n(1324)$ is a Stieltjes moment sequence
Conjollary: the growth rate μ satisfies $\mu > 10.302$
Previous best: $\mu \in (10.27, 13.5)$ [Bevan, Brignall, EP, Pantone, 2020]

Story time:

Two other (mostly) independent groups were working on similar ideas:

- [Blitvić, Stiengrímsson,2020]: permutations counted with 14 "natural" parameters form a moment sequence for any non-negative specialisation of the parameters. Includes avoiders of avoiders of patterns of length 3 (classical, consecutive and vincular)
- [Sokal, Zeng, 2022]: Independently same counting sequences and parameters, part of project analysing Hankel total-positivity [Sokal, Zeng, Zhu, Pétréole, E.P., Deb, Gilmore, Chen, . . .]

General belief: For any permutation π, the sequence $|\operatorname{Av}_n(\pi)|$ is a Stieltjes-moment sequence. [Blitvić, Kammoun, Stiengrímsson, Bostan, EP, Guttmann, Maillard, Clisby, Conway, Inoue]

TALK OUTLINE

- **Part 1:** Guessing Stieltjes-ness
  - **Part 1a:** Algorithm
  - **Part 1b:** Stieltjes moment sequences in OEIS
  - **Part 1c:** Examples
- **Part 2:** Proving Stieltjes-ness for exactly solved sequences (Stieltjes inversion formula)
- **Part 3:** Proving Stieltjes-ness for excursions on graphs

Part 1: Guessing Stieltjes-ness

Part 1a: Algorithm for guessing Stieltjes-ness

COMPUTING CONTINUED FRACTION COEFFICIENTS α_j

Definition: Let $a_0, a_1, \ldots$ be a sequence with generating function

$$A(t) = a_0 + a_1 t + a_2 t^2 + \cdots = \cfrac{\alpha_0}{1 - \cfrac{\alpha_1 t}{1 - \cfrac{\alpha_2 t}{1 - \ldots}}}.$$

Recall: Sequence is Stieltjes $\iff$ all $\alpha_j \geq 0$.

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Recall: Sequence is Stieltjes $\iff$ all $\alpha_j \geq 0$.
Assume: $a_0, a_1, \ldots, a_N$ known exactly.
Define:

$$A_j(t) = \cfrac{\alpha_j}{1 - \cfrac{\alpha_{j+1} t}{1 - \cdots}}.$$

Compute recursively, using

$$A_0(t) = A(t) \quad \text{and} \quad A_j(t) = \frac{\alpha_j}{1 - tA_{j+1}(t)}$$

If $\alpha_0, \ldots, \alpha_N > 0$, we *guess* the sequence is Stieltjes.

Computing continued fraction coefficients α_j

Recall: α_j's and $A_j(t)$ determined by

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Euler-Viskovatov algorithm: [Sokal, 2022]
Recursively define $B_j(t) = A_j(t)B_{j-1}(t)$ and $B_{-1}(t) = 1$. Then

$$\frac{B_j(t)}{B_{j-1}(t)} = \frac{\alpha_j}{1 - t\frac{B_{j+1}(t)}{B_j(t)}}$$

Expanding yields

$$B_j(t) - tB_{j+1}(t) = \alpha_j B_{j-1}(t).$$

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$$B_j(t) - tB_{j+1}(t) = \alpha_j B_{j-1}(t).$$

Algorithm:
Initialise:

$$B_{-1}(t) = 1, \quad B_0(t) = a_0 + a_1 t + \cdots + a_N t^N + O(t^{N+1}).$$

Recursive determine α_j, $B_{j+1}(t) + O(t^{N-j})$ using

$$\alpha_j = B_j(0)/B_{j-1}(0), \qquad B_{j+1}(t) = \frac{1}{t}\left(B_j(t) - \alpha_j B_{j-1}(t)\right).$$

Part 1b: Stieltjes moment sequences in OEIS

Guessing Stieltjes-ness with OEIS

We ran the Euler-Viskovatov algorithm on all 304698 OEIS sequences with at least 15 terms (only considering terms a_n with $n \leq 150$ and $a_n \leq 10^{150}$).

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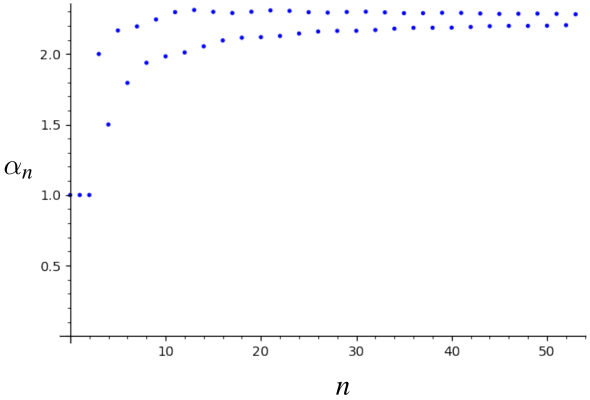
Refined results:

- In 1667 such cases, one of the terms $\alpha_j = 0$, so the generating function $A(t)$ is rational
- In 798 cases (including 328 rational cases), the coefficients α_j are all integers.
- For 7344 sequences the first 15 terms are consistent with being Stieltjes (625 of these not Stieltjes because of later terms)

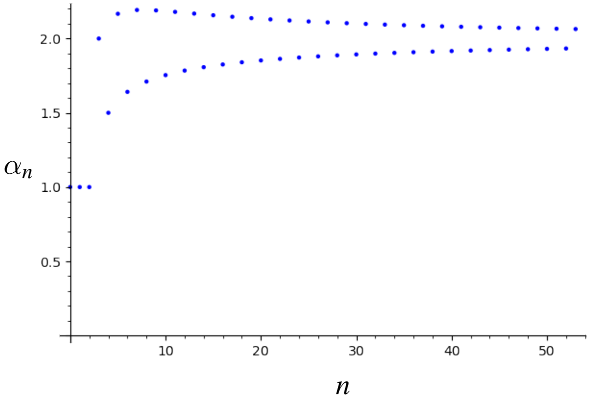
Part 1b: Examples of (possibly) Stieltjes moment sequences in OEIS

EXAMPLE: 1234-AVOIDERS



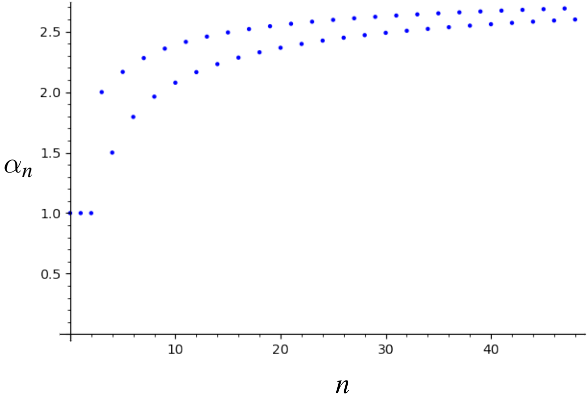
Plot of α_n vs. n for the sequence $a_n = |\operatorname{Av}_n(1234)|$.

EXAMPLE: 1342-AVOIDERS



Plot of α_n vs. n for the sequence $a_n = |\,\mathrm{Av}_n(1342)|$.

EXAMPLE: 1324-AVOIDERS



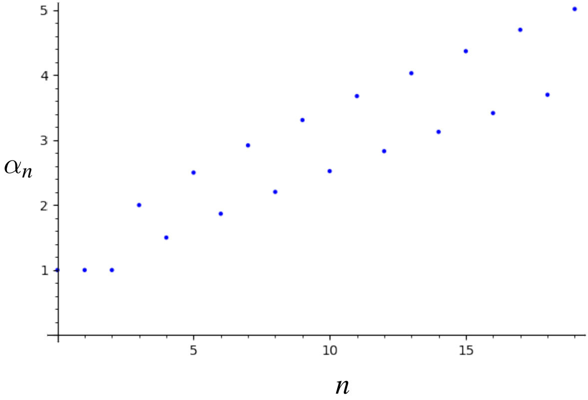
Plot of α_n vs. n.

Other sequences starting 1,1,2,6,23

Of 69 OEIS sequences starting 1,1,2,6,23 there are 16 potential Stieltjes moment sequences

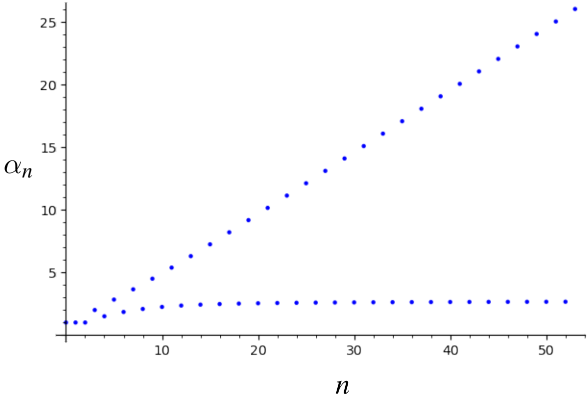
Sequences starting 1,1,2,6,23

A110447: Av(<u>31</u>42)



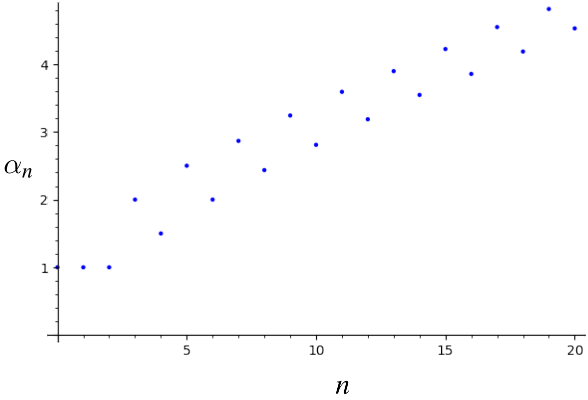
Plot of α_n vs. n.

A113227: Av(1<u>23</u>4)



Plot of α_n vs. n.

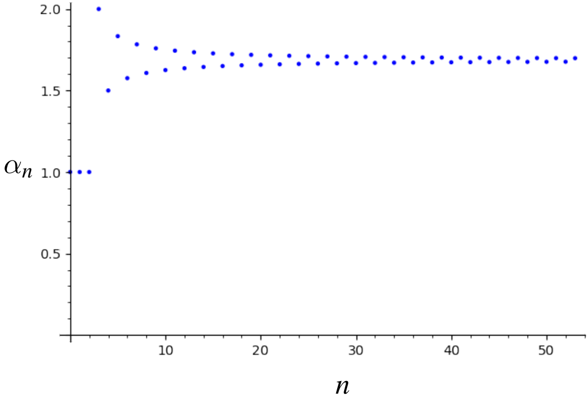
A125273: Av(1<u>42</u>3)



Plot of α_n vs. n.

SEQUENCES STARTING 1,1,2,6,23

A098746: $\mathrm{Av}(4231, 42513)$



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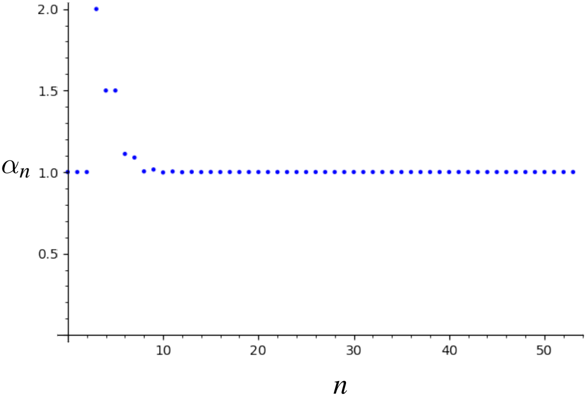
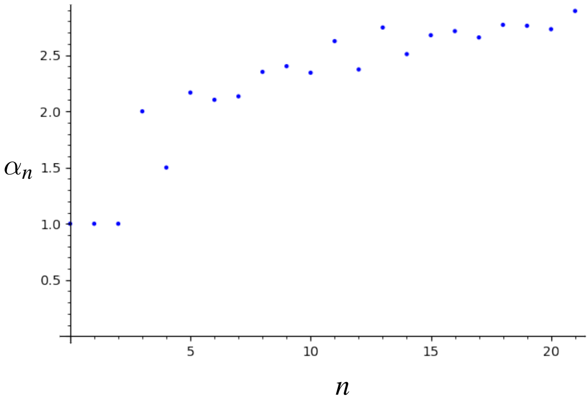
A213090: Av(4231, 35142, 42513, 351624)



Plot of α_n vs. n.

A263778: 120-avoiding inversion sequences



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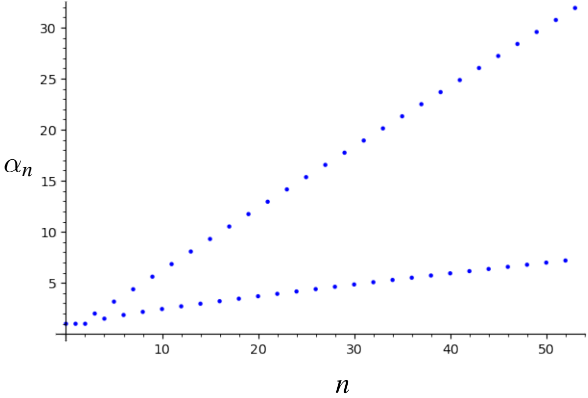
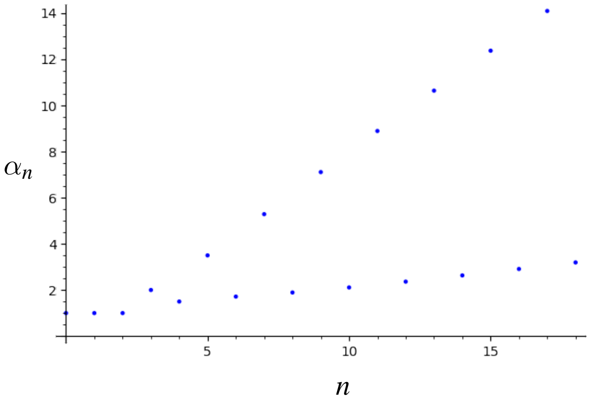
SEQUENCES STARTING 1,1,2,6,23

A187761: Maps $f : \{1,2,\ldots,n\}$ satisfying $f(j) \leq j$ and $f(f(j)) = f(f(f(j)))$.



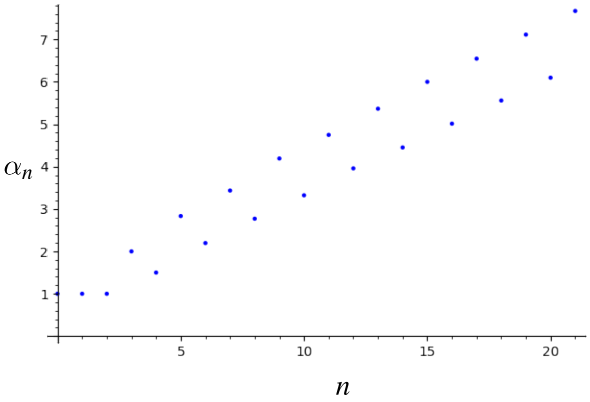
Plot of α_n vs. n.

A007555: Standard paths in composition poset



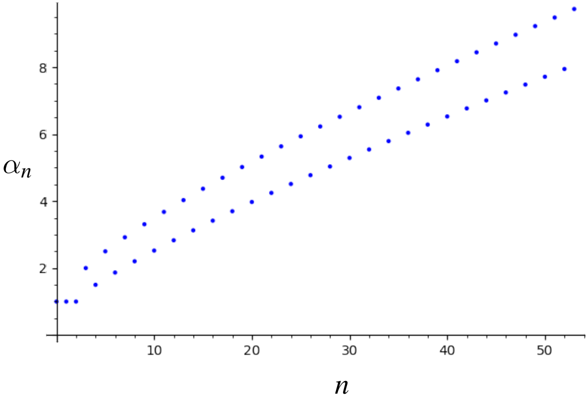
Plot of α_n vs. n.

A352367: Something to do with chordal graphs



Plot of α_n vs. n.

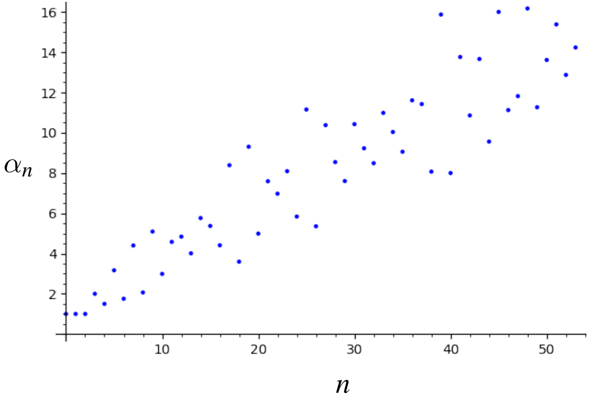
A030266: $A(t) = 1 + tA(t)A(tA(t))$



Plot of α_n vs. n.

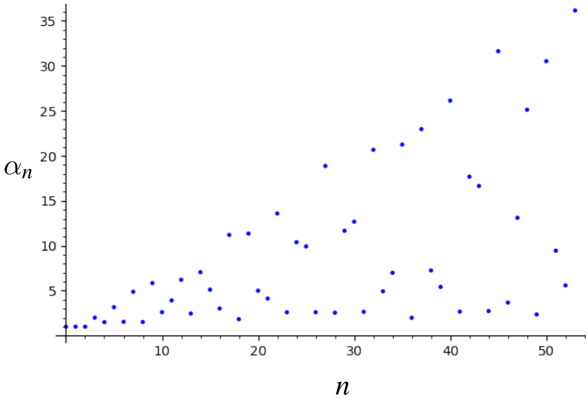
SEQUENCES STARTING 1,1,2,6,23

A125273: $A(t) = 1 + \frac{t}{1-t} A\left(\frac{t}{(1-t)^2}\right)$



Plot of α_n vs. n.

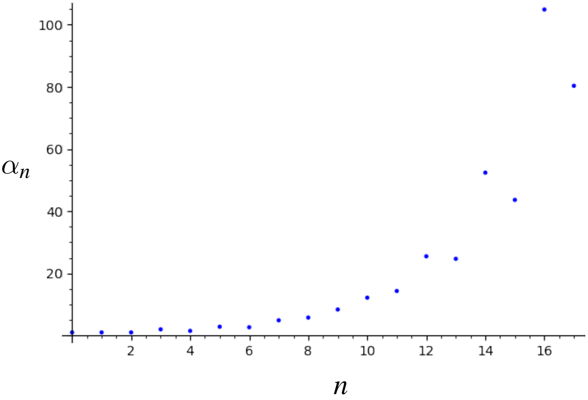
A193321: $A(t) = \sum_{n\geq 0} t^n \left(\prod_{k=1}^{n} \frac{1-kt}{1-2kt} \right)$



Plot of α_n vs. n.

SEQUENCES STARTING 1,1,2,6,23

A192315: $A(t)^2 = \sum_{n \geq 0} t^n A(t)^{2^n}$



Plot of α_n vs. n.

Stieltjes-ness results for permutation classes

Principal classes:

- Solved classes $\mathrm{Av}(1\ldots m)$ and $\mathrm{Av}(1342)$ known to be counted by Stieltjes moment sequence [Rains, 1998], [Bostan,EP,Guttmann,Maillard, 2020]
- The sequence $\mathrm{Av}_n(1324)$ seems to be Stieltjes (using 50 terms [Conway,Guttmann,Zinn-Justin,2018])
- The sequence $\mathrm{Av}_n(12534)$ seems to be Stieltjes (using 38 terms [Biers-Ariel,2019])
- For each (remaining) pattern π of length 5, the sequence $\mathrm{Av}_n(\pi)$ seems to be Stieltjes (using 23 to 27 terms [Clisby,Conway,Guttmann,Inoue,2022])

Principal vincular classes

- Sometimes Stieltjes but not in general
- $\mathrm{Av}(\underline{1234})$ not Stieltjes: in this case $\alpha_6 < 0$
- **Question:** Can one characterise vincular classes that are Stieltjes?

Stieltjes-ness results for permutation classes

Principal vincular classes

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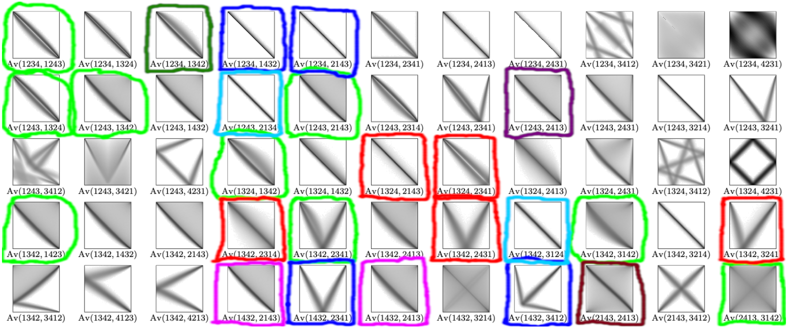
Classical finitely based classes

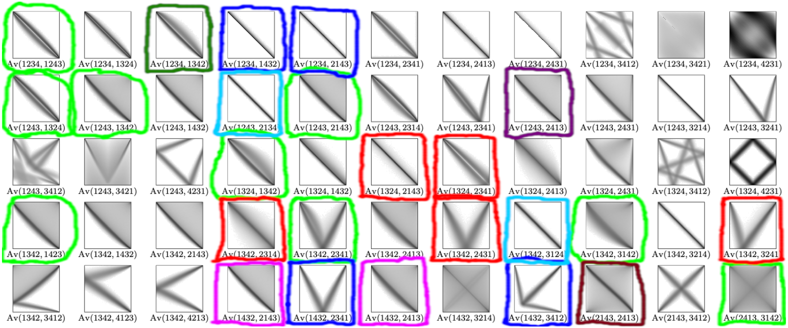
- Not generally Stieltjes !
- For π, τ of lengths 3 and 4, respectively, there are 9 Wilf classes for $\mathrm{Av}(\pi_1, \pi_2)$. Only 1 is Stieltjes: $\mathrm{Av}_n(123, 2143)$
- For π, τ both of lengths 4, respectively, there are 38 Wilf classes for $\mathrm{Av}(\pi_1, \pi_2)$. Only 8 are (possibly) Stieltjes:
  - $\mathrm{Av}(4321, 4123)$ has a rational generating function
  - The other 7 have algebraic generating functions
- **Question:** Can one characterise the permutations classes that are Stieltjes?

Stieltjes 2 by 4 classes



8 (possibly) Stieltjes Wilf classes for 2 by 4 patterns.

STIELTJES 2 BY 4 CLASSES



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Weird property: Every case with non-trivial Wilf equivalence *is* (possibly) Stiletjes.

Part 2: Stieltjes inversion formula

[Bostan,EP,Guttmann,Maillard]

Stieltjes inversion formula

Assume $a_0, a_1, \ldots$ is a Stieltjes moment sequence with

$$a_n = \int_0^\tau x^n \mu(x) dx.$$

When $|z| > \tau$, the generating function $A(t)$ satisfies

$$\frac{1}{z} A\left(\frac{1}{z}\right) = \sum_{n=0}^{\infty} \frac{a_n}{z^{n+1}} = \int_0^\tau \sum_{n=0}^{\infty} \frac{x^n}{z^{n+1}} \mu(x) dx = \int_0^\tau \frac{1}{z-x} \mu(x) dx \qquad .$$

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STIELTJES INVERSION FORMULA

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Stieltjes inversion formula:

$$\mu(x) = -\frac{1}{2\pi i} \lim_{\epsilon \to 0^+} \left(F(x+\epsilon i) - F(x-\epsilon i)\right).$$

Checking Stieltjes-ness with inversion formula

Let $a_0, a_1, \ldots$ be a sequence with exponential growth rate (at most) τ.

To check if sequence is Stieltjes:

- For $|z| > \tau$, define $F(z) = \frac{1}{z}A\left(\frac{1}{z}\right) = \sum_{n=0}^{\infty} \frac{a_n}{z^{n+1}}$.

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- Then $\mu(x)$ is the density. Stieltjes $\iff$ $\mu(x)$ takes only non-negative values.

EXAMPLE: CATALAN NUMBERS

Generating function

$$C(t) = 1 + t + 2t^2 + 5t^2 + \cdots = \frac{1 - \sqrt{1-4t}}{2t}.$$

Then

$$F(z) := \frac{1}{z} C\left(\frac{1}{z}\right) = \frac{1}{2}\left(1 - \sqrt{\frac{z-4}{z}}\right).$$

This is analytic on $\mathbb{C} \setminus [0,4]$, so the density function is

$$\mu(x) = -\frac{1}{2\pi i} \lim_{\epsilon \to 0^+} \left(F(x+\epsilon i) - F(x - \epsilon i)\right) = \frac{1}{2\pi}\sqrt{\frac{4-x}{x}}.$$

Positive on $[0,4]$, so sequence is Stieltjes.

EXAMPLE: 1342-AVOIDING PERMUTATIONS

(Bóna, 1997): The generating function

$$A(t) := \sum_{n=0}^{\infty} Av_n(1342)t^n = \frac{1 + 20t - 8t^2 + (1 - 8t)^{3/2}}{2(1 + t)^3}$$

Then

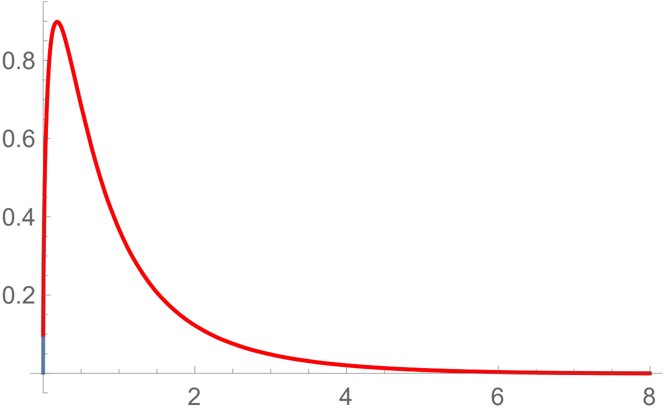
$$F(z) := \frac{1}{z} A\left(\frac{1}{z}\right) = \frac{z^2 + 20z - 8 + \sqrt{z(z - 8)^3}}{2(z + 1)^3}.$$

This is analytic on $\mathbb{C} \setminus [0, 8]$, so the density function is

$$\mu(x) = -\frac{1}{2\pi i} \lim_{\epsilon \to 0^+} \left(F(x + \epsilon i) - F(x - \epsilon i)\right) = \frac{(8 - x)^{3/2}\sqrt{x}}{2\pi(1 + x)^3}.$$

Positive for $x \in [0, 8]$, so sequence is Stieltjes.

Density for 1342-avoiding permutations



Density function for $Av_n(1342)$.

EXAMPLE: 1234-AVOIDING PERMUTATIONS

From known formula for 1234 avoiders, we have (for $|z|$ large)

$$F(z) := \frac{1}{z}A\left(\frac{1}{z}\right) = \frac{z+5}{6} - \frac{(z-1)^{\frac{1}{4}}(z-9)^{\frac{3}{4}}}{6}\,{}_2F_1\left(\left[-\frac{1}{4}, \frac{3}{4}\right], [1], \frac{-64z^3}{(z-1)(z-9)^3}\right).$$

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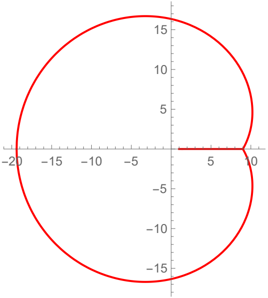
Expression not analytic on $\mathbb{C}\setminus[0,9]$.



Non-analytic points of $F(z)$.

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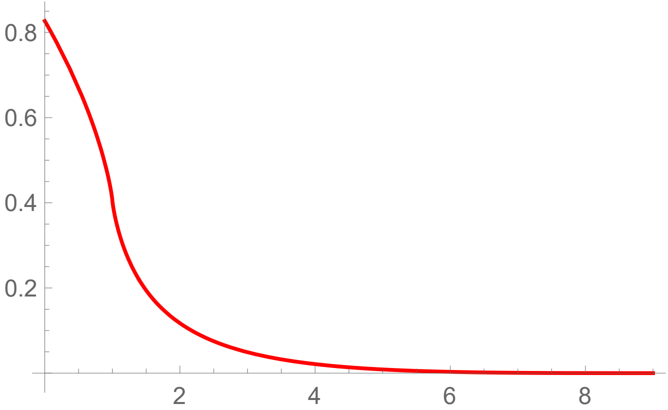
By the Stieltjes inversion formula

$$\mu(x) = -\frac{1}{\pi}\Im(\hat{F}(z)) = -\frac{3}{\pi}\Im(F(z)) \qquad \text{for } x\in[1,9],$$
$$\mu(x) = -\frac{1}{\pi}\Im(\hat{F}(z)) = \frac{3}{\pi}\Re(F(z)) \qquad \text{for } x\in[0,1].$$

These are positive so the sequence is Stieltjes.

Density for 1234-avoiding permutations



Density function for $Av_n(1234)$.

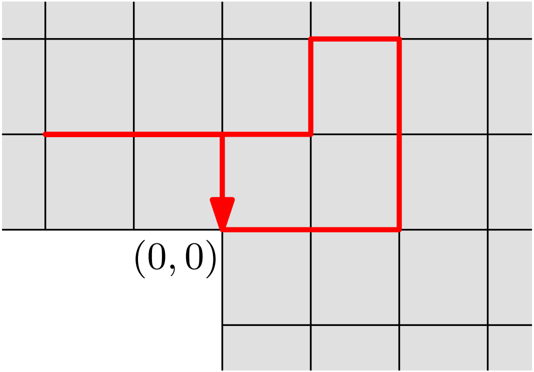
$1234\cdots k$-AVOIDERS AS A MOMENT SEQUENCE

The following is a result of [Rains, 1998]:

- Let U be a Haar random $(k-1)\times(k-1)$ unitary matrix
- Let a_n be the number of $1234\cdots k$-avoiding permutations of length n
- Then $a_n = E(|tr(U)|^2n)$.

So $a_0, a_1, \ldots$ is a Stieltjes moment sequence.

Part 3: Excursions on graphs



PATHS ON GRAPHS

Theorem: [E.P., Guttmann 2019] Let Γ be a graph with vertex set V and edge set E, and let v_0 be a fixed vertex. Let a_n be the number of walks from v_0 to v_0 in Γ of length n. Then $a_0, a_1, \ldots$ is a Hamburger moment sequence.

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Proof of theorem:

- Consider a vector space with basis $\{p_v\}_{v \in V}$.
- Consider a linear operator C on this space defined by $Cp_v = \sum_{(u,v) \in E} p_u$.
- Then C is self-adjoint and $a_n = \langle C^n p_{v_0}, p_{v_0} \rangle$. Hence $a_0, a_1, \ldots$ is a Hamburger moment sequence.

Thank You!

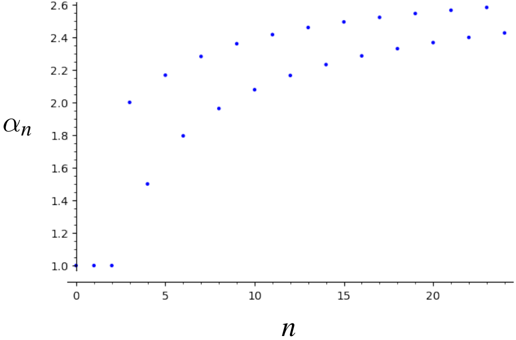
Open problems

Some possible Stieltjes moment sequences from OEIS:

- $(Av_n(\pi))_{n\geq 0}$ for any permutation π
- Fishburn numbers
- A305703: Generalised Fibonacci numbers
- Perfect matchings avoiding certain patterns e.g., A005700 and A220910-A220915
- A319027: Number of preimages of 321-avoiding permutations under West's stack-sorting map

Open problem(s)

For 1324-avoiding permutations the alpha sequence is:



Plot of α_n vs. n for the sequence $a_n = |\operatorname{Av}_n(1324)|$.

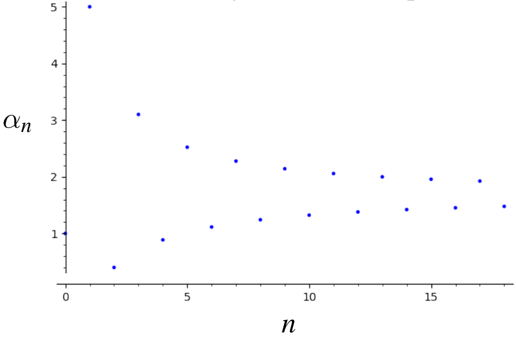
Question: Is this a Stieltjes moment sequence?

GENERALISED FIBONACCI SEQUENCES (A305573)

Definition: A generalised Fibonacci sequence is a sequence $f_0, f_1, f_2, \ldots$ satisfying $f_{j+1} \in \{f_j + f_{j-1}, |f_j - f_{j-1}|\}$.

Definition: Let a_n be the number of generalised Fibonacci sequences with period $3n$.

Question: Is $a_0, a_1, \ldots$ a Stieltjes moment sequence?



Plot of α_n vs. n for the sequence a_n.