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From two-stack sortable permutations to fighting fish.

Corentin Henriet,
joint work with Luca Ferrari and Lapo Cioni.

July 3, 2023

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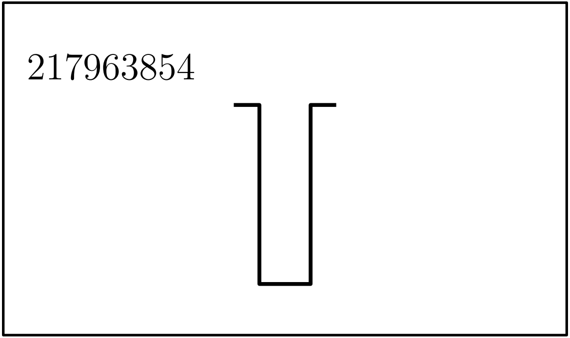
Sorting with a decreasing stack



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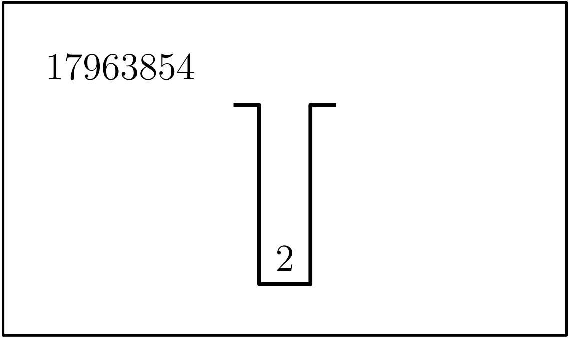

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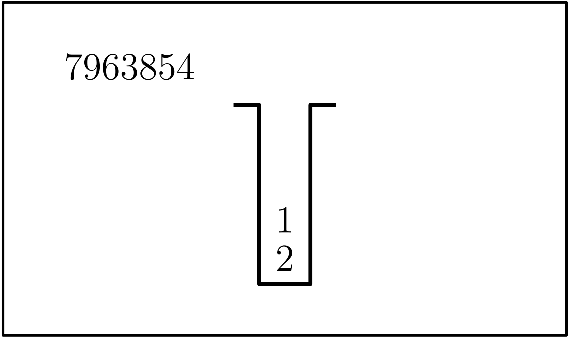

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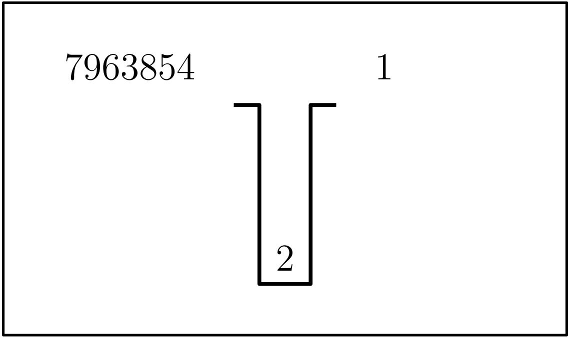

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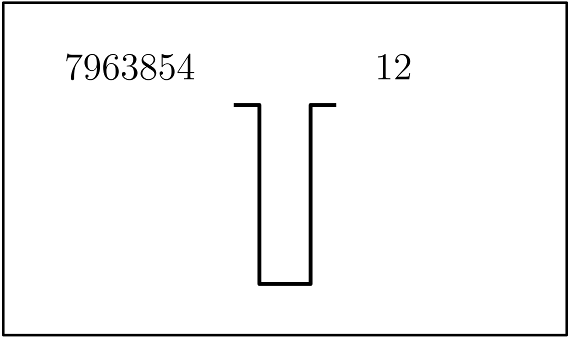

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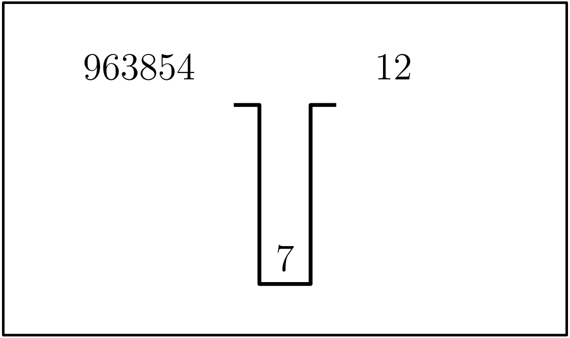

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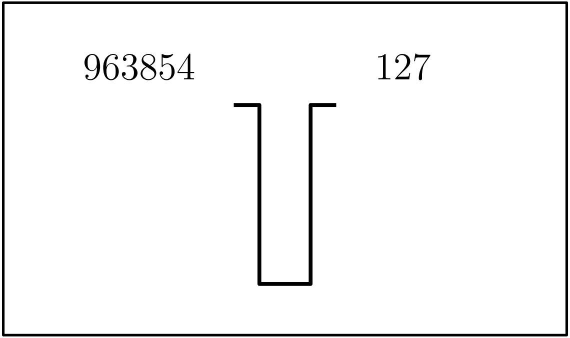

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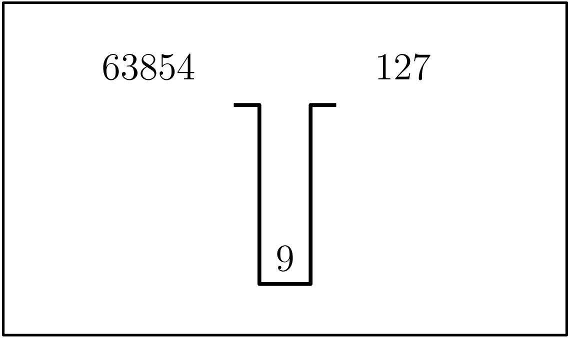

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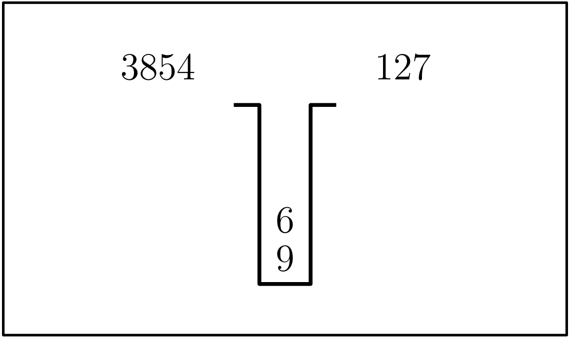

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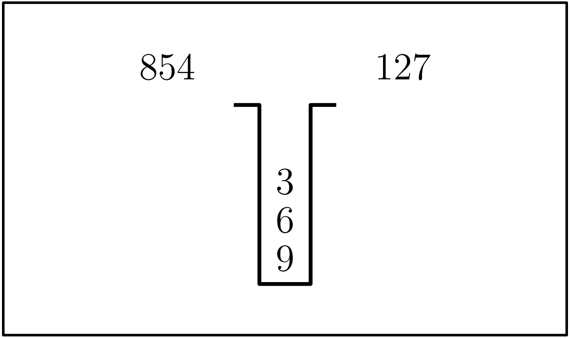

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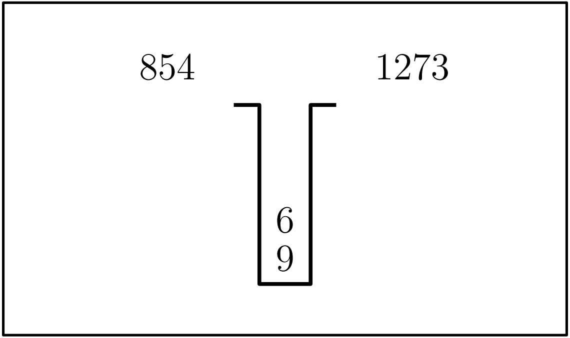

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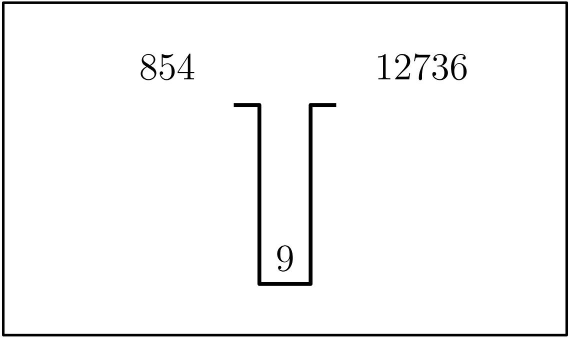

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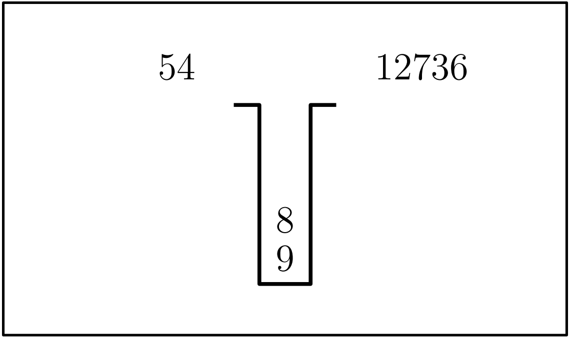

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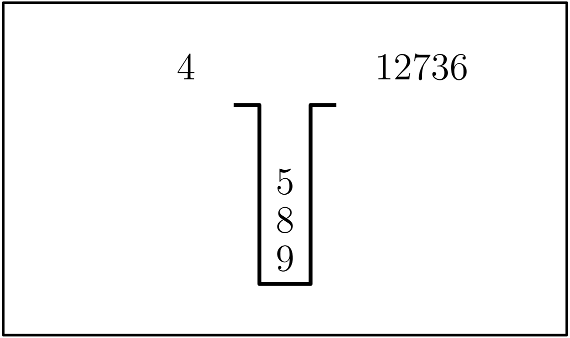

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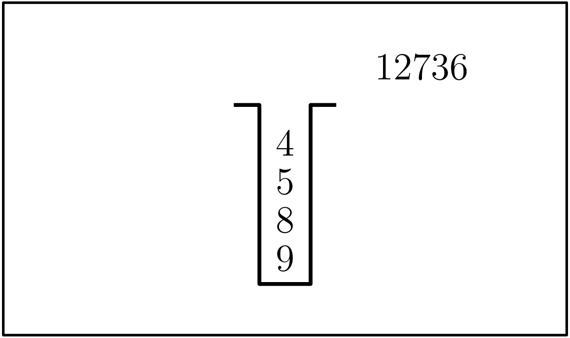

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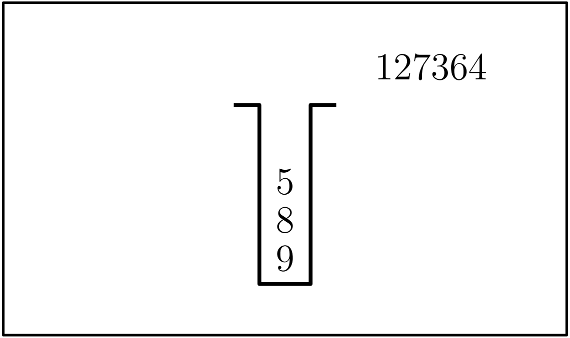

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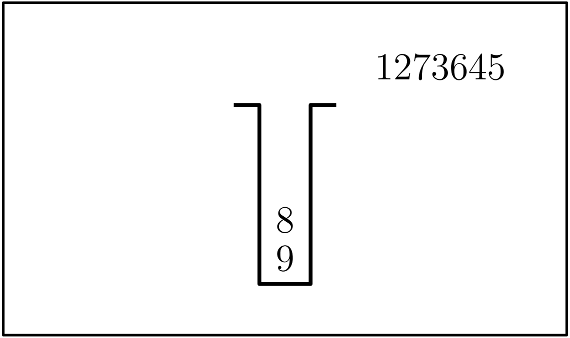

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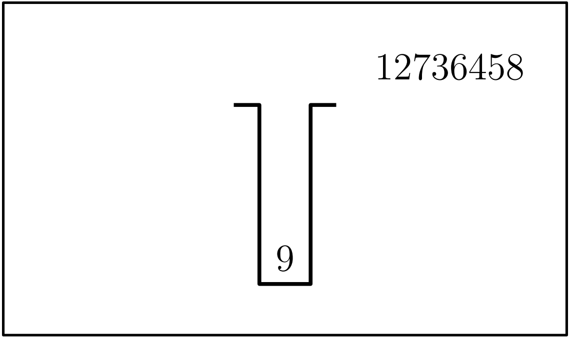

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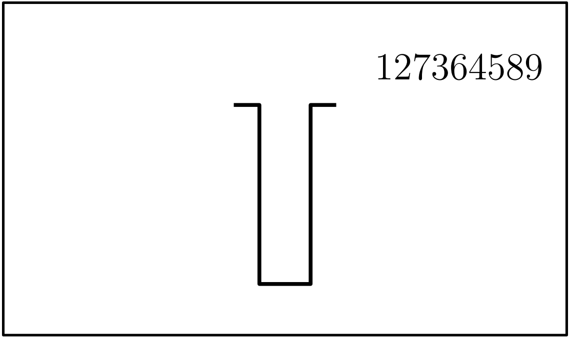


Sorting with a decreasing stack

$$\boxed{\mathbf{S}(217963854) = 127364589}$$

Stack-sorting on permutations

The stack-sorting operator S is also defined recursively:

$$\begin{cases} \mathsf{S}(\varepsilon) = \varepsilon \\ \mathsf{S}(\sigma_1 n \sigma_2) = \mathsf{S}(\sigma_1)\,\mathsf{S}(\sigma_2) n \text{ for } \sigma = \sigma_1 n \sigma_2 \in \mathfrak{S}_n \end{cases}$$

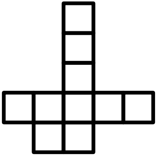
A k-stack sortable permutation is a permutation σ such that $\mathsf{S}^k(\sigma)$ is the identity permutation.

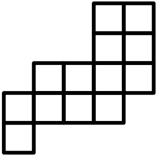
$$|1\mathcal{SS}_n| = \frac{1}{n+1}\binom{2n}{n}$$

$$|2\mathcal{SS}_n| = \frac{2}{(n+1)(2n+1)}\binom{3n}{n}$$

Parallelogram polyominoes

Convex polyomino : Finite connected union of unit squares with convex columns and rows. It is planar.
It is parallelogram if it has a South-West and a North-East cell.

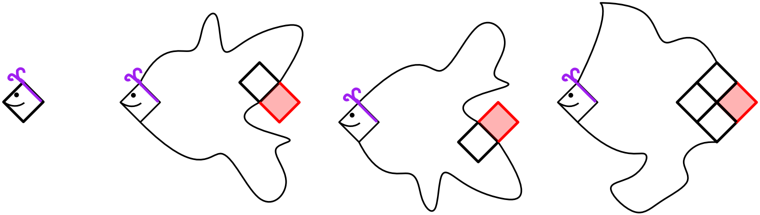




$C_n = \frac{1}{n+1}\binom{2n}{n}$ parallelogram polyominoes of halfperimeter $n-1$.

A generalization of parallelogram polyominoes

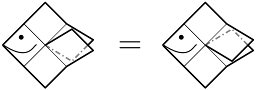
Fighting fish are gluings of cells (= 45° tilted unit squares) that can be obtained from the Head using a finite sequence of operations among the 3 presented below :

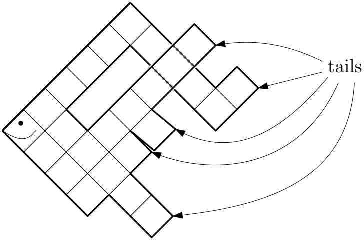


Size = Number of lower free (= not glued) edges minus 1.
Introduced in 2016 by Duchi, Guerrini, Rinaldi and Schaeffer to generalize parallelogram polyominoes.

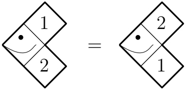
Examples

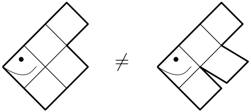
They do not always fit in the plane :





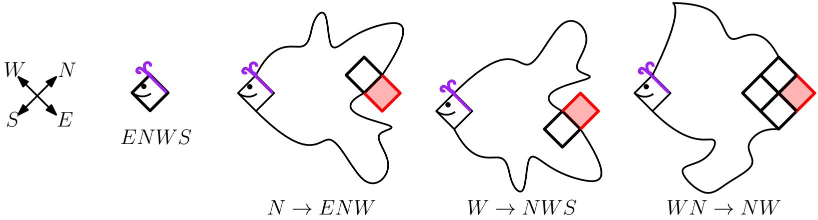
Parallelogram polyominoes are fighting fish with one tail.
Gluing order does not matter but the type of gluing does :





Fish, words and skeletons

Perform a counterclockwise tour of the boundary of the fish



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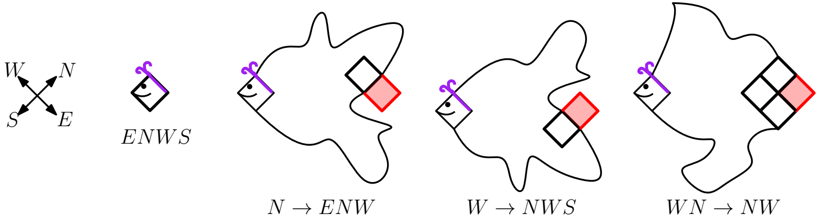
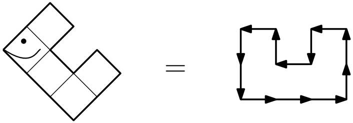


Rotation of 45° → path on the square lattice starting and ending at $(0, 0)$, confined to the quadrant $\{x, y \geq 0\}$.



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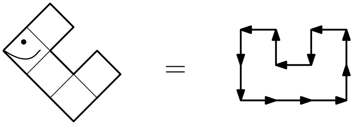
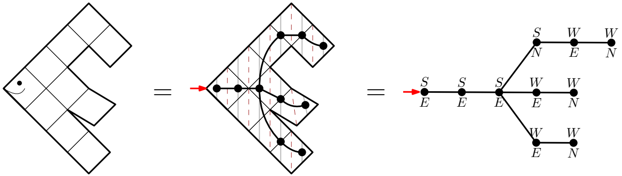
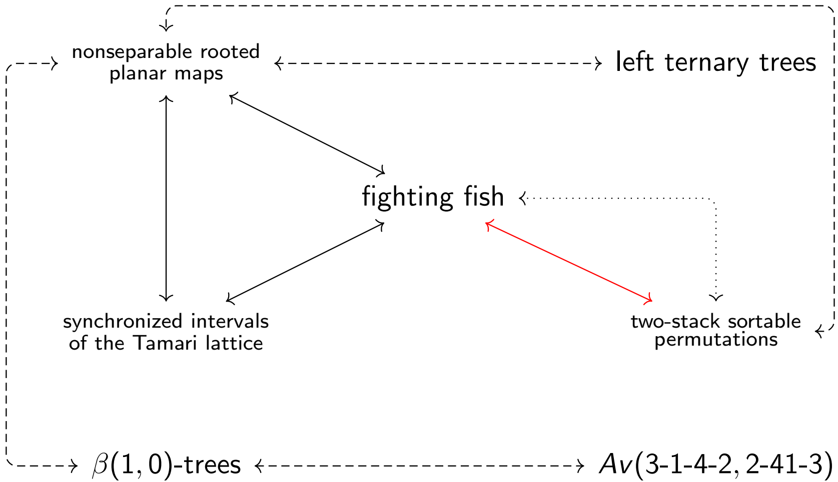


We can alternatively see a fish as its skeleton: a tree where each vertex carries two labels, E or N on one side and W or S on the other side.



Motivation: extend the aquarium of (direct) bijections



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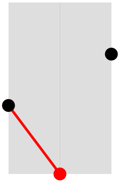
From permutations to sorting trees

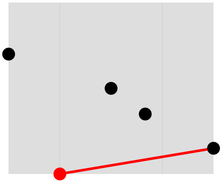
We construct a rooted plane tree using the grid representation of the permutation:

• We add the extra point $(0, n+1)$.

• We proceed from top to bottom by linking each point $(i, \sigma(i))$ to its parent with the rules:



or



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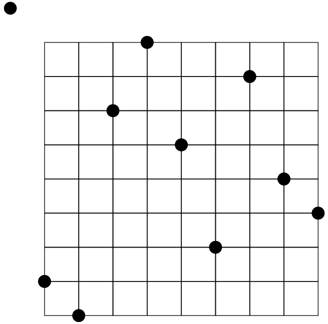

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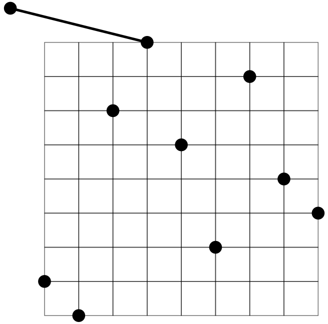

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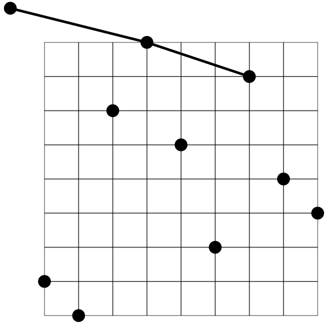

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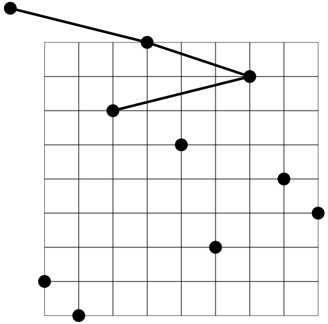

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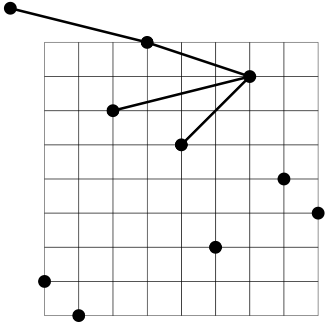

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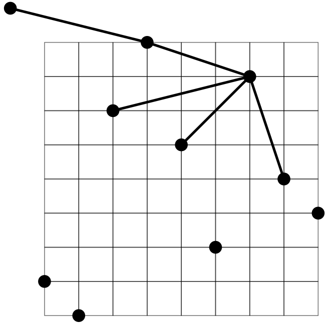

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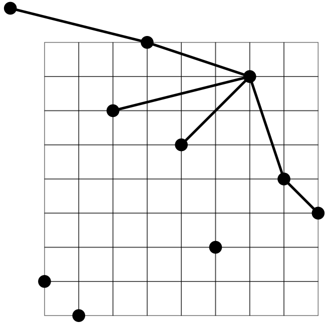

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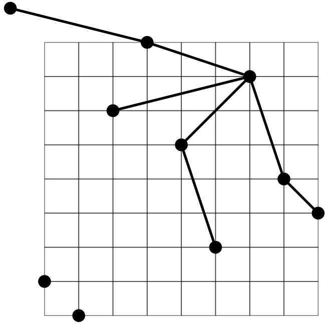

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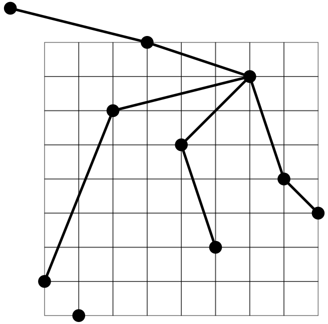

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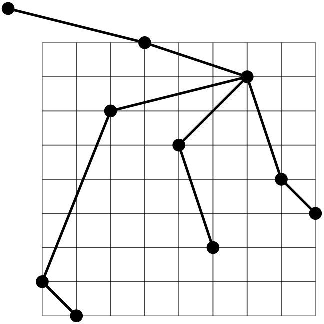

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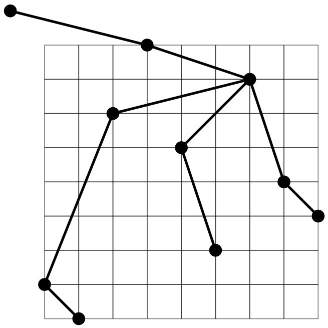
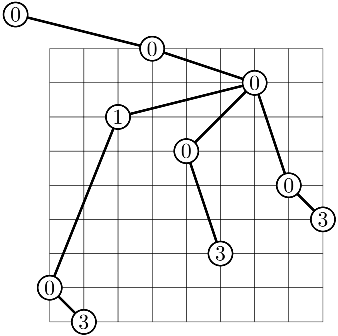


Fact: For $\sigma, \tau \in 2\mathcal{S}\mathcal{S}_n$, $\mathsf{ST}(\sigma) = \mathsf{ST}(\tau) \Leftrightarrow \mathsf{S}(\sigma) = \mathsf{S}(\tau)$.

From permutations to labeled sorting trees

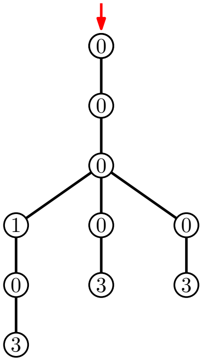
Label each element by:

- 0 if it is followed by a descent,
- $j > 0$ if it is the last point of a descending run of length j.



From permutations to labeled sorting trees

Keep only the labeled rooted plane tree structure:



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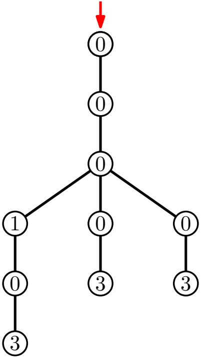
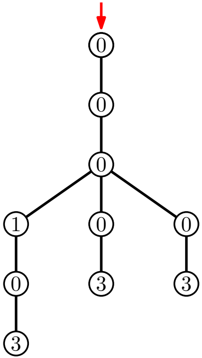


A *labeled sorting tree* is an element of $\mathcal{LST}_n = \mathsf{LST}(\mathfrak{S}_n)$.
Fact: The (restricted) map $\mathsf{LST} : 2\mathcal{SS}_n \to \mathcal{LST}_n$ is bijective.

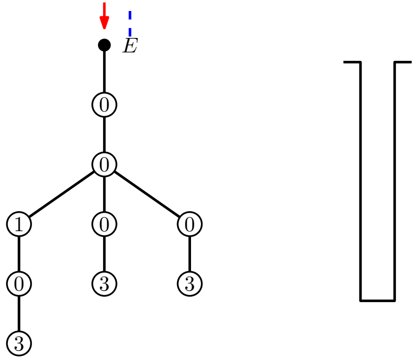
From labeled sorting trees to fighting fish

We perform a clockwise tour of the tree and decorate vertices by a E or a N at the first visit and by a W or a S at the last visit.



From labeled sorting trees to fighting fish

We have a companion stack which starts and ends up empty.
A 0-first visit yields a E, nothing happens on the stack.



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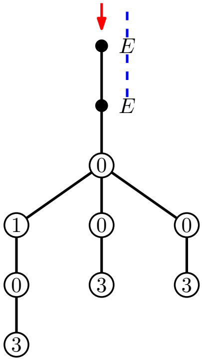

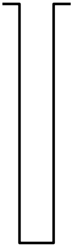

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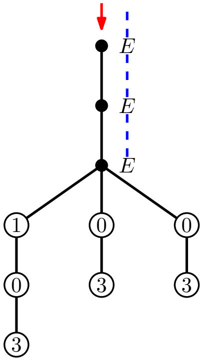

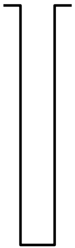

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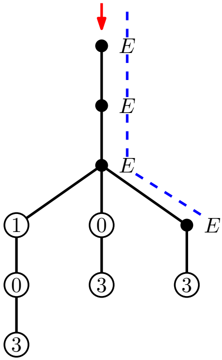
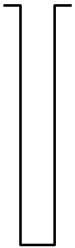
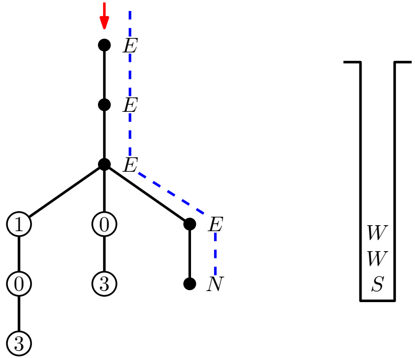




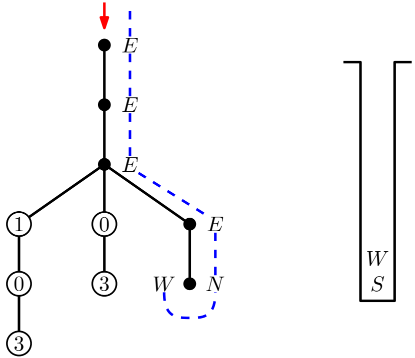
From labeled sorting trees to fighting fish

A j-first visit with $j > 0$ yields a N and we put a S in the stack, followed by $j-1$ W.



From labeled sorting trees to fighting fish

When we visit a vertex for the last time, we pop out one element of the stack that we assign to the vertex.



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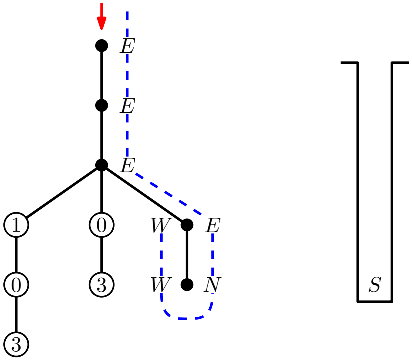

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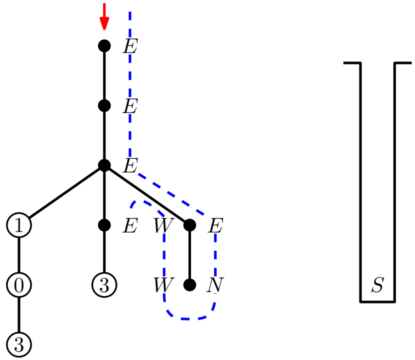

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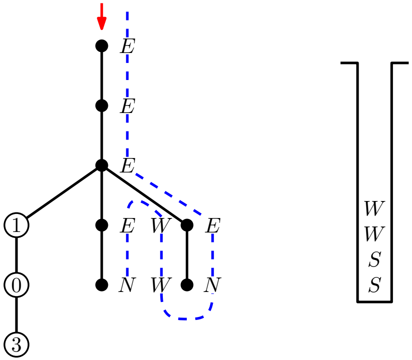

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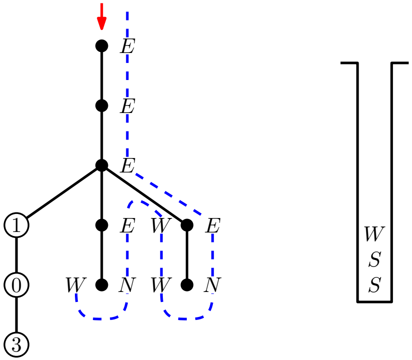

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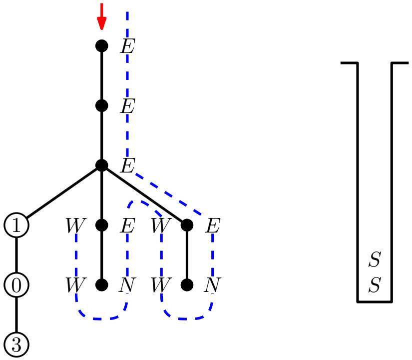

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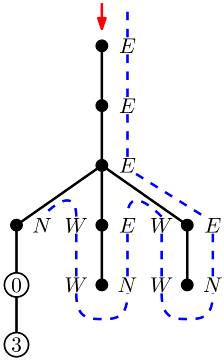
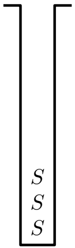
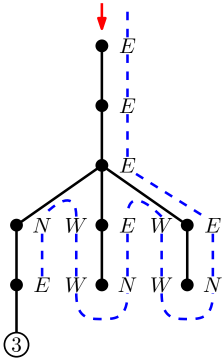
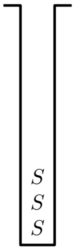




From labeled sorting trees to fighting fish





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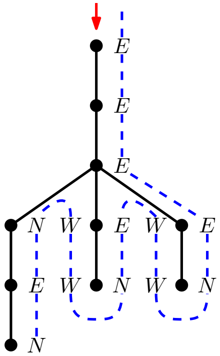

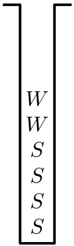

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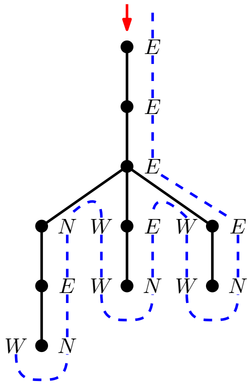

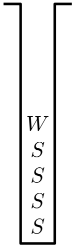

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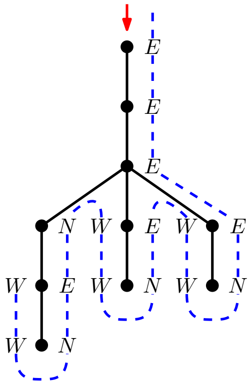

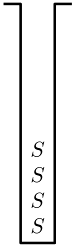

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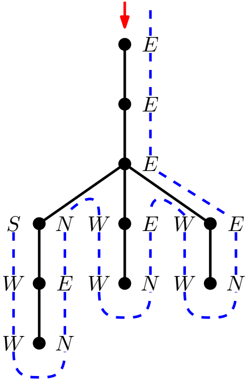

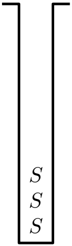

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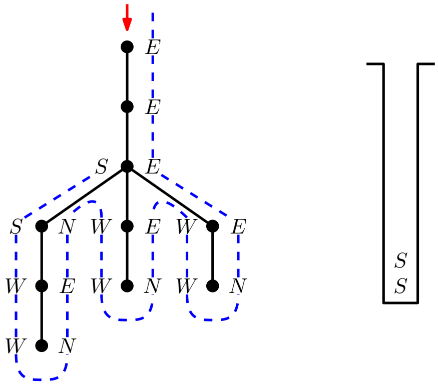

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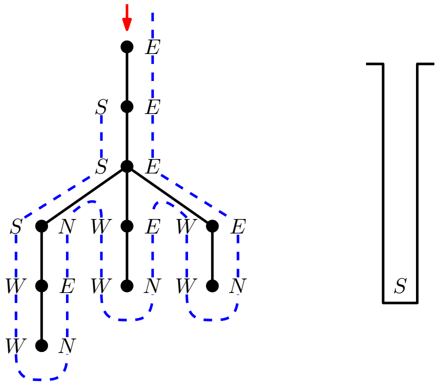

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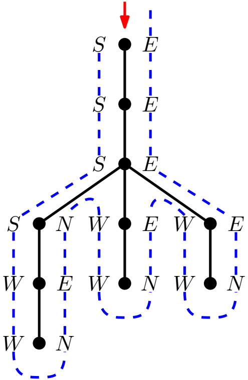
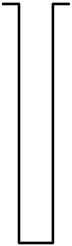
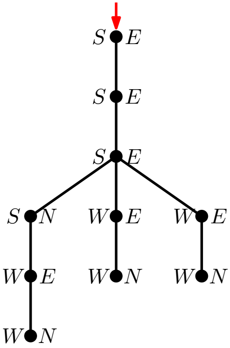




From labeled sorting trees to fighting fish

This procedure yields the skeleton of a fighting fish FW(T) from a tree $T \in \mathcal{LST}$.



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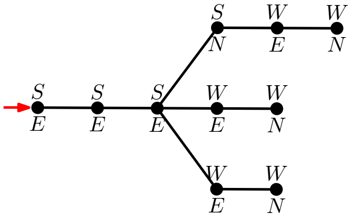

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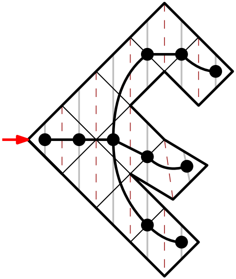
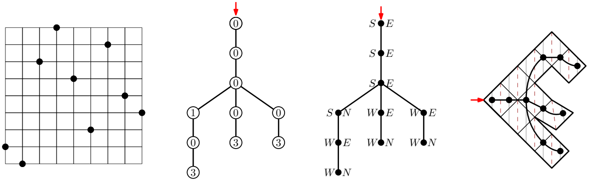


The total bijection



Theorem (Cioni, Ferrari, H. 2023+)

FW $\circ$ LST is a bijection between two-stack sortable permutations and fighting fish. It is the direct version of Fang's recursive bijection (up to symmetry).
Parallelogram polyominoes $\leftrightarrow$ One-stack sortable permutations.
$\#E$ steps $\leftrightarrow$ #descents $+1$

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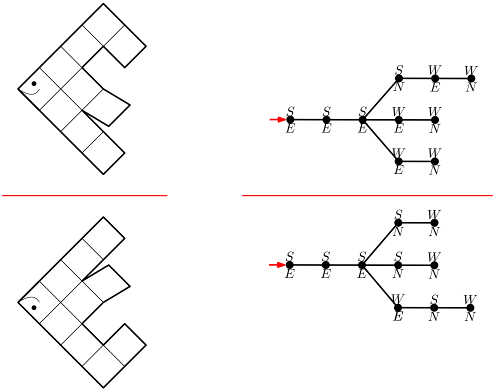
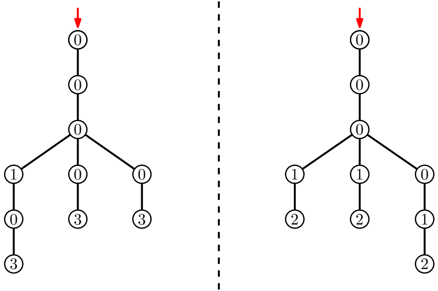
Mirroring fish

Conjugation of fish is the mirror involution wrt the x-axis. On skeletons, the tree is mirrored and letters are exchanged $E \leftrightarrow S$, $N \leftrightarrow W$.



Mirroring fish

How to describe the corresponding involutions on $\mathcal{LST}$? On $2\mathcal{SS}$?



A surprising symmetry on labeled sorting trees

Proposition

A labeled rooted plane tree T with n non-root vertices belongs to $\mathcal{LST}_n$ iff:

$$\sum_{v \in T} \lambda(v) = n + 1$$

$$\forall v \in T, \quad \sum_{w \in \mathrm{anc}(v)} \big(2 - \deg(w)\big) - 1 \geq \lambda(v)$$

$$\forall v \in T \setminus \{r\}, \quad \sum_{w \in \mathrm{sub}(v)} \big(\lambda(w) - 1\big) \geq 1$$

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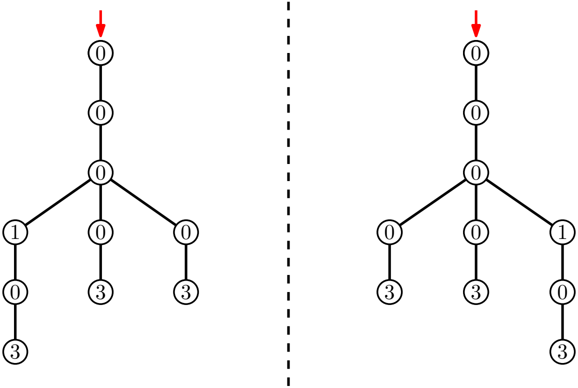
These three conditions **do not** depend on the order of the subtrees rooted at the children of v for any vertex v.

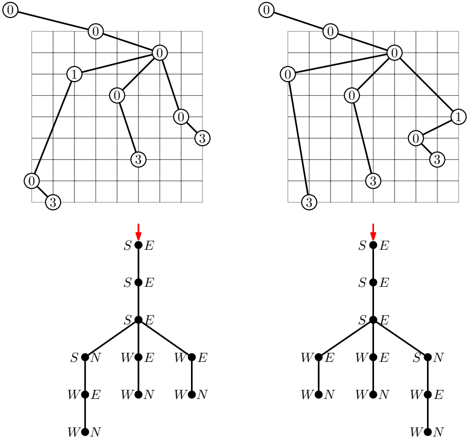
A surprising symmetry on labeled sorting trees

Hence $\mathcal{LST}_n$ is stable by the mirror symmetry:



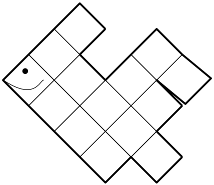
A surprising symmetry on labeled sorting trees

It is very surprising and it would be nice to have a description of the induced involutions on $2\mathcal{SS}_n$ and $\mathcal{FF}_n$.



The area statistic on fighting fish

The (shifted) area of a fighting fish is the number of cells in it (minus its size).



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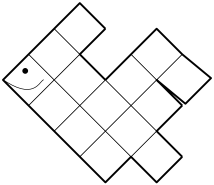
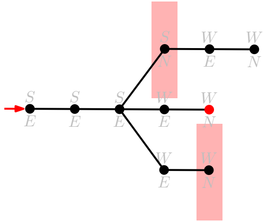


What is the corresponding statistic on $2\mathcal{SS}$? On $\mathcal{LST}$?

The area statistic on fighting fish

We also conjecture the dinv statistic on fighting fish to be equidistributed with the shifted area.



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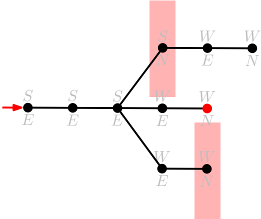

Even more interesting, the joint symmetry seems to hold, i.e. $G_n(q,t) = G_n(t,q)$, where $G_n(q,t) = \sum_{F \in \mathcal{FF}_n} q^{\mathrm{area}(F)-n} t^{\mathrm{dinv}(F)}$.

Extension to $\mathfrak{S}_n$

The maps ST and LST are defined for any permutation in $\mathfrak{S}_n$.

- For a given tree $T \in (\mathcal{L})\mathcal{ST}_n$, can we describe the set $\{\sigma \in \mathfrak{S}_n | \, (\mathrm{L})\mathrm{ST}(\sigma) = T\}$? Enumerate it ?
- The sequence (1,2,5,16,64,308,...) counting permutations giving rise to fighting fish of area 0 appears to count some plane labeled increasing binary trees avoiding some pattern (OEIS A131178). Is there more structure hidden ?

Thank you for your attention !