A bijective proof of a special case of an equidistribution conjecture

Chetak Hossain

Department of Mathematics, Texas State University

TEXAS STATE MATHEMATICS

Background

The powered Catalan numbers are an integer sequence counting $S_n(1\underline{23}4)$. This appears as OEIS A113227. In [2], Baxter and Shattuck conjecture that $S_n(\underline{23}14)$ is counted by the powered Catalan numbers [2, Conjecture 22]. Beaton et. al. refined this conjecture by the statistic lmin [3, Conjecture 23]. Lin and Fu further refined the conjecture by considering four statistics on permutations as well as four statistics on inversion sequences.

Conjecture

[5, Conjecture 4.2] The quadruple $(rmin, lmin, rmax, asc)$ on $S_n(\underline{23}14)$ has the same distribution as $(zero, max, rmin, rep)$ on $I_n(110)$.

Main Result

Let $S_{n,2}$ be the permutations on $[n]$ that have exactly two ascents. Let $I_{n,2}$ be the inversion sequences of length n that have exactly two repeats.

The quadruple $(rmin, lmin, rmax, asc)$ on $S_{n,2}(\underline{23}14)$ has the same distribution as $(zero, max, rmin, rep)$ on $I_{n,2}(110)$.

Inversion Sequence Statistics

The inversion sequences I_n are the n-tuples $(e_1, \cdots, e_n)$ such that $e_i \le i-1$.

$$\begin{aligned}
\mathrm{Zero}(e) &= \{i \mid e_i = 0\}\\
\mathrm{Max}(e) &= \{i \mid e_i = i-1\}\\
\mathrm{RLmin}(e) &= \{i \mid e_i < e_j \quad \text{for all} \quad j > i\}\\
\mathrm{Row}(e) &= \{i \mid e_i \neq 0,\ e_i \quad \text{does not occur in the suffix} \quad e_{i+1}e_{i+2}\cdots e_n\}\\
\mathrm{Rep}(e) &= \{i \mid e_i \quad \text{does occur in the suffix} \quad e_{i+1}e_{i+2}\cdots e_n\}\\
\mathrm{Emp}(e) &= \{i+1 \mid i \quad \text{does not occur in} \quad e\}
\end{aligned}$$

References

References

[1] Jean-Luc Baril and Vincent Vajnovszki. A permutation code preserving a double Eulerian bistatistic. *Discrete Appl. Math.* 224:9-15, 2017

[2] Andrew M. Baxter and Mark Shattuck. Some Wilf-equivalences for vincular patterns. J. Comb., 6(1-2):19–45, 2015.

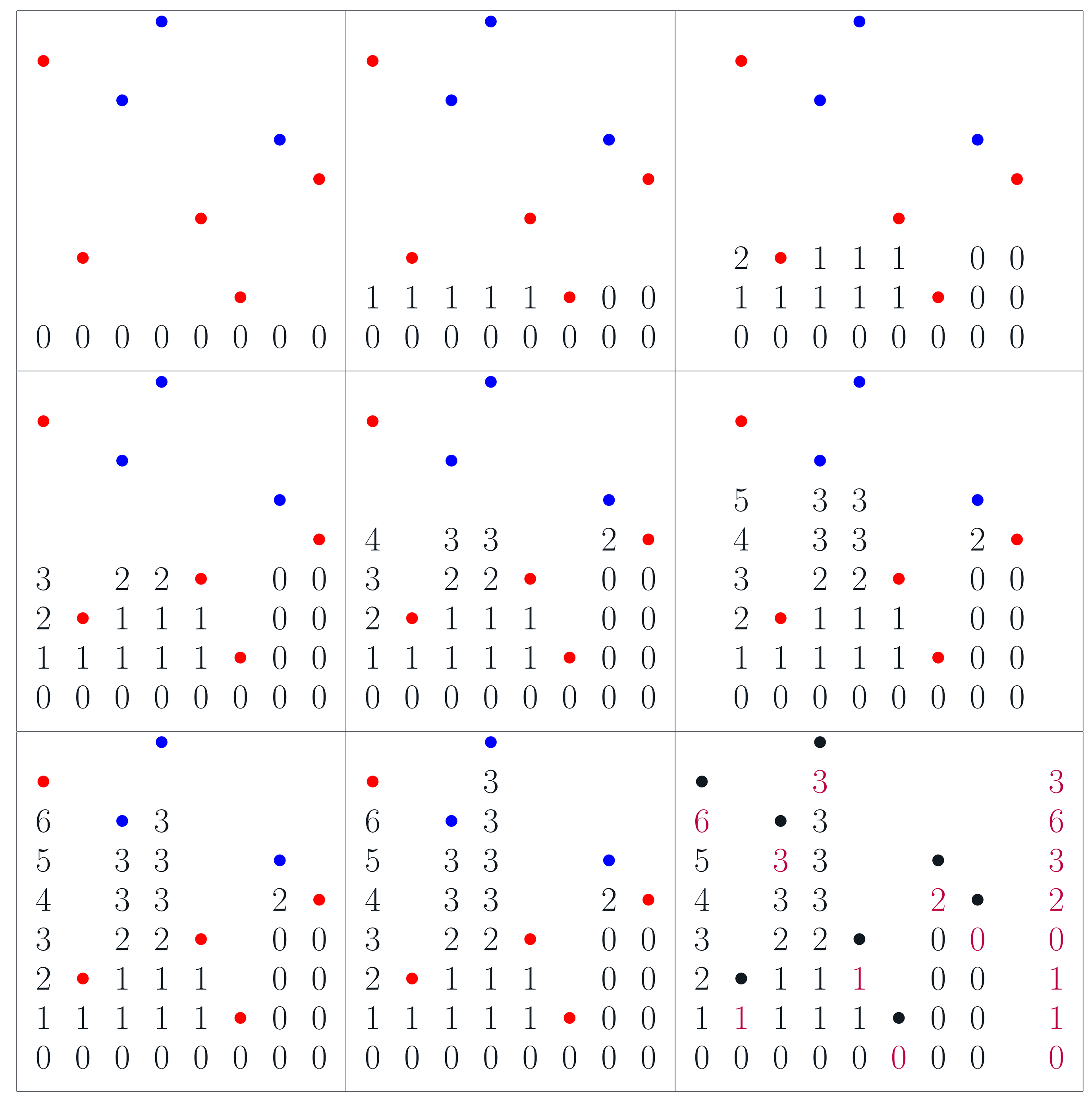
[3] Nicholas R. Beaton, Mathilde Bouvel, Veronica Guerrini, and Simone Rinaldi. Enumerating five families of pattern-avoiding inversion sequences; and introducing the powered Catalan numbers. Theoret. Comput. Sci., 777:69–92, 2019.

[4] Sylvie Corteel, Megan A. Martinez, Carla D. Savage, and Michael Weselcouch. Patterns in inversion sequences I. Discrete Math. Theor. Comput. Sci., 18(2):Paper No. 2, 21, 2016.

[5] Zhicong Lin and Shishuo Fu. On 120-avoiding inversion and ascent sequences. European J. Combin., 93:Paper No. 103282, 12, 2021.

Example of $\beta =$ bcode $\circ$ inv

$\beta(72683154) = 01102363$



$\mathrm{RLminV}(72683154) = \{1, 4\}$
$\mathrm{LRminV}(72683154) = \{7, 2, 1\}$
$\mathrm{RLmaxV}(72683154) = \{8, 5, 4\}$
$\mathrm{DesTV}(72683154) = \{7, 8, 3, 5\}$
$\mathrm{AscBV}(72683154) = \{2, 6, 1\}$
$\mathrm{AscTV}(72683154) = \{6, 8, 5\}$

$\mathrm{Zero}(01102363) = \{1, 4\}$
$\mathrm{Max}(01102363) = \{1, 2, 7\}$
$\mathrm{RLmin}(01102363) = \{4, 5, 8\}$
$\mathrm{Row}(01102363) = \{3, 5, 7, 8\}$
$\mathrm{Rep}(01102363) = \{1, 2, 6\}$
$\mathrm{Emp}(01102363) = \{5, 6, 8\}$

Involution on Inversion Sequences

Let $\mathrm{Rep}(e) = \{u, w\}$ with $e_u = b$ and $e_w = a$. Let $i, k \in \overline{\mathrm{Row}(e)}$ such that $e_i = a$ and $e_k = b$. Let $\mathrm{Emp}(e) = \{j, z\}$ where $j < z$ with $e_j = c$ and $e_z = h$. If $j > k$, let $e_m = d$ be the unique value appearing in e, but not in the sub-inversion sequence $(e_1, \cdots, e_j)$. If $i < u$, let $e_v = a'$ be the unique value appearing in e, but not in the sub-inversion sequence $(e_1, \cdots, e_u)$.

$$\tau_1(e) = \begin{cases} \begin{pmatrix} k & i & j \end{pmatrix}(e) & u < i,\ j < k \\ (k\ \ i)(m\ \ j)(e) & u < i,\ j > k \\ \begin{pmatrix} k & v & j \end{pmatrix}(e) & u > i,\ j < k \\ (k\ \ v)(m\ \ j)(e) & u > i,\ j > k \end{cases}$$

$$\tau_2(f) = \begin{cases} \begin{pmatrix} j & k & m \end{pmatrix}(f) & u < i,\ f_m < b\ (f_m = f_i = a) \\ (i\ \ k)(j\ \ m)(f) & u < i,\ f_m > b \\ \begin{pmatrix} j & k & m \end{pmatrix}(f) & u > i,\ f_m < b\ (f_m = f_v = a') \\ (v\ \ k)(j\ \ m)(f) & u > i,\ f_m > b \end{cases}$$

Bijection

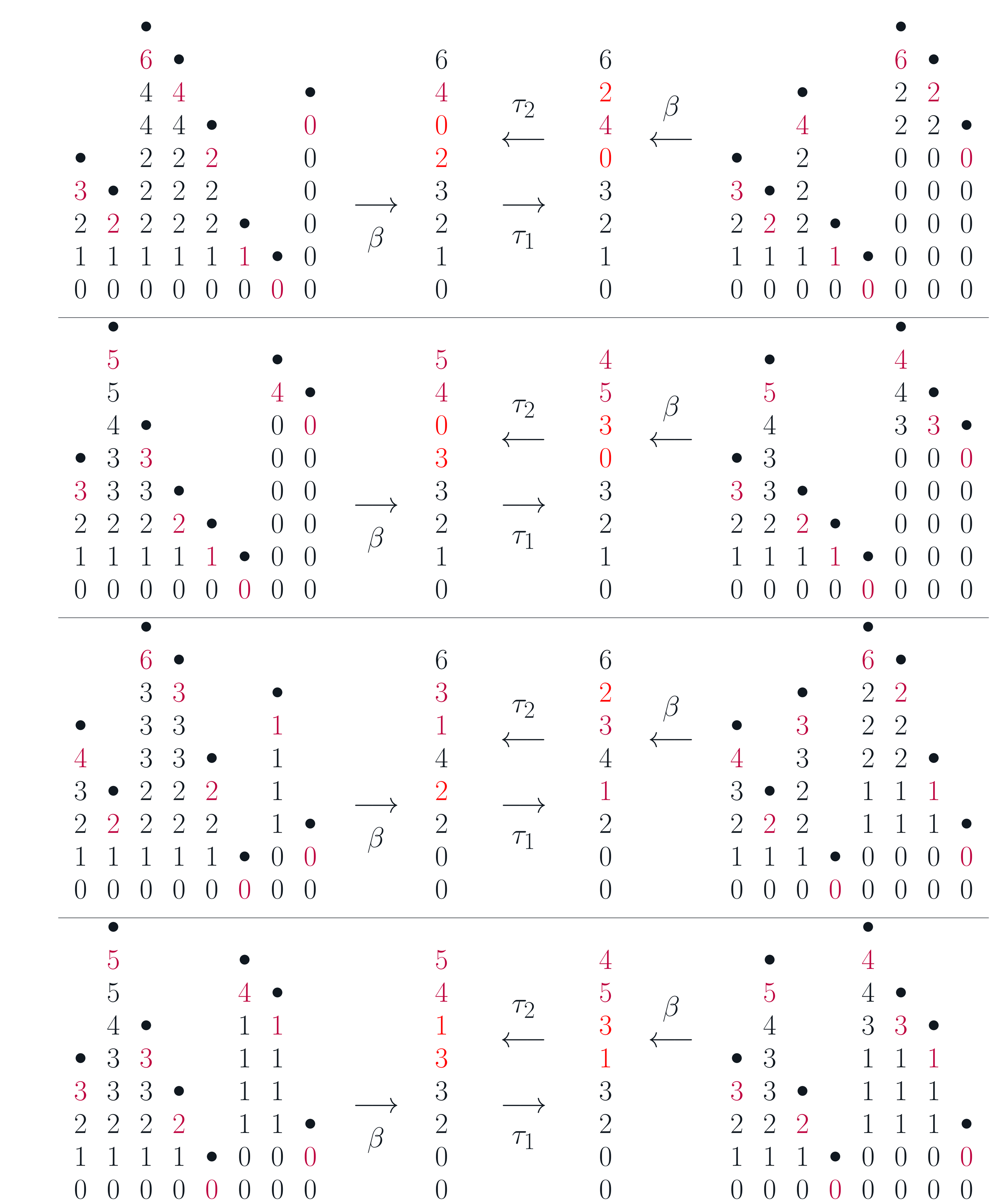
Consider the following map $\varphi : S_{n,2} \to I_{n,2}$.

$$\varphi(x) = \begin{cases} \beta(x) & x \in S_n(\underline{23}14),\ \beta(x) \in I_n(110) \\ \beta(x) & x \notin S_n(\underline{23}14),\ \beta(x) \notin I_n(110) \\ (\tau_1 \circ \beta)(x) & x \in S_n(\underline{23}14),\ \beta(x) \notin I_n(110) \\ (\tau_2 \circ \beta)(x) & x \notin S_n(\underline{23}14),\ \beta(x) \in I_n(110) \end{cases}$$

Bijection Examples

$\varphi(43875216) = 01230426$ $\varphi(48532176) = 01230354$

$\varphi(53874162) = 00214326$ $\varphi(48531762) = 00231354$



$(rmin, lmin, rmax, asc)(43875216) = (2, 4, 3, 2) = (zero, max, rmin, rep)(01230426)$
$(rmin, lmin, rmax, asc)(48532176) = (2, 4, 3, 2) = (zero, max, rmin, rep)(01230354)$
$(rmin, lmin, rmax, asc)(53874162) = (2, 3, 4, 2) = (zero, max, rmin, rep)(00214326)$
$(rmin, lmin, rmax, asc)(48531762) = (2, 3, 4, 2) = (zero, max, rmin, rep)(00231354)$