On Permutation Classes Defined by Pin Sequences

Ben Jarvis

Based on joint work with Robert Brignall

3rd July 2023

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- Controlling various features of the defining binary sequence (eg., periodic/recurrent, complexity function, Sturmian, etc.) will allow us to control feautures of the resulting permutation class (eg., growth rate, length of longest oscillation, antichains, number of simple permutations, etc.)
- Thus we will end up with a very large example class of permutation classes with 'nice' properties, all of which we are able to enumerate...

Pin Sequences

Definition

A **pin sequence** is a word (finite or infinite) over the language

$$\{1,2,3,4\}(\{l,r\}\{u,d\})^* \cup \{1,2,3,4\}(\{u,d\}\{l,r\})^*$$

Examples:

- 3uruldldl
- 1ldlulurdlulululuId
- 2(drul)* = 2druldruldrul. . .
- 1ulurulururulururulur...

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- $2(\text{drul})^* = 2\text{druldruldrul}\ldots$
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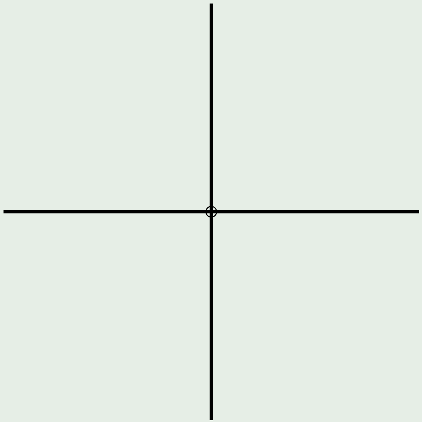
A finite pin sequence can be converted into a permutation by the following procedure:

Pin Permutations

Constructing a permutation from the pin-word 2*lurdld*



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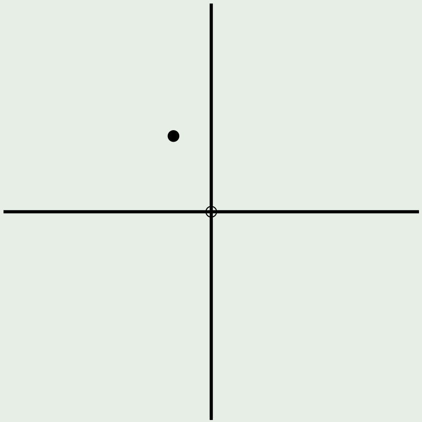

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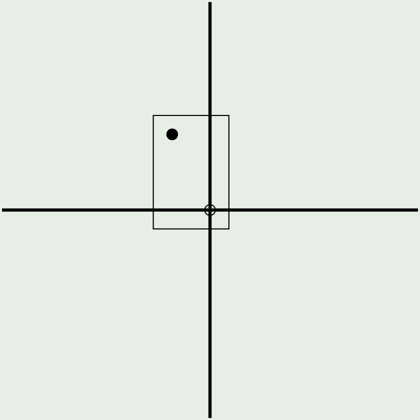

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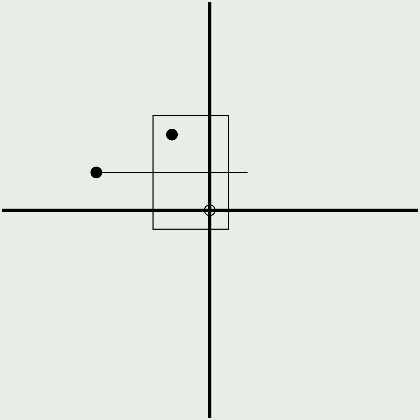

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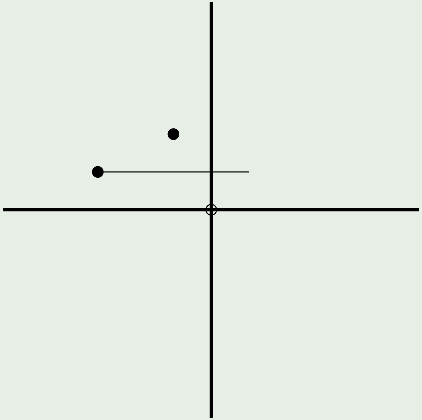

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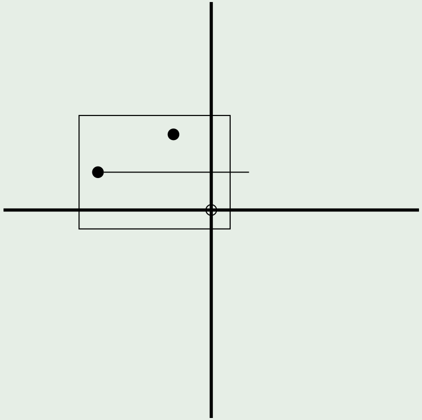

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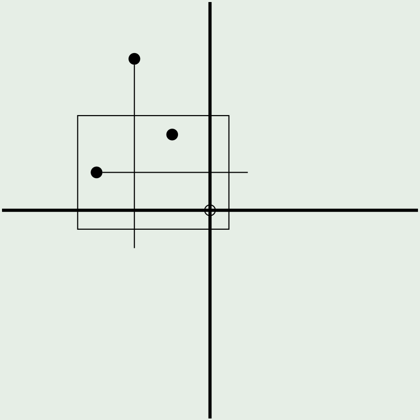

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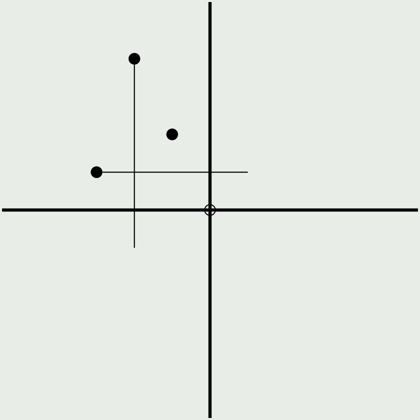

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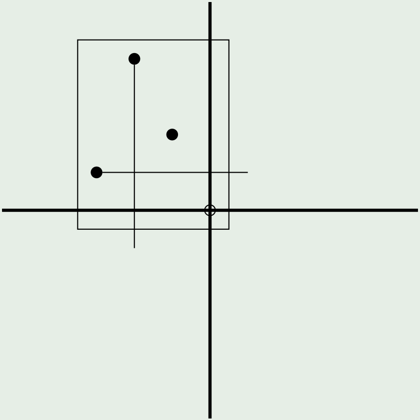

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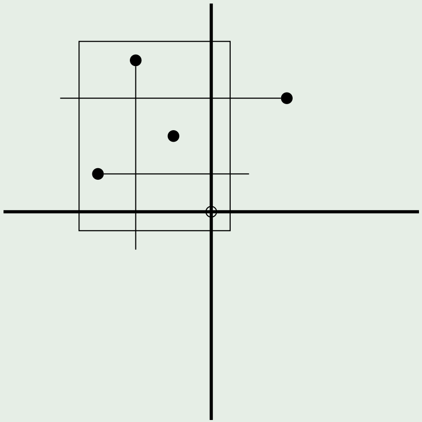

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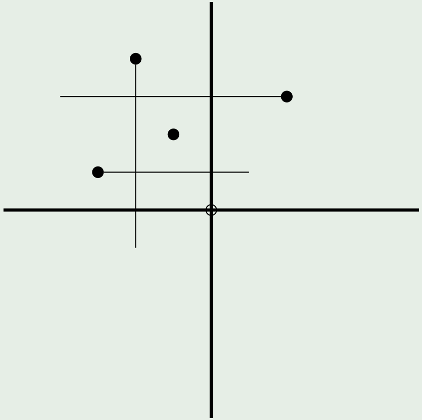

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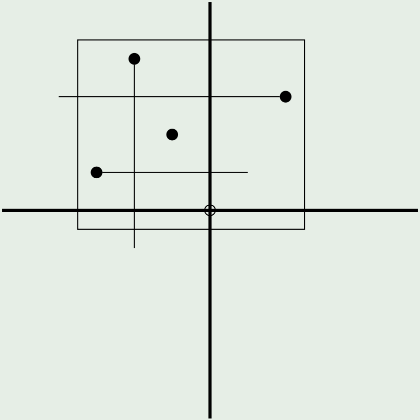

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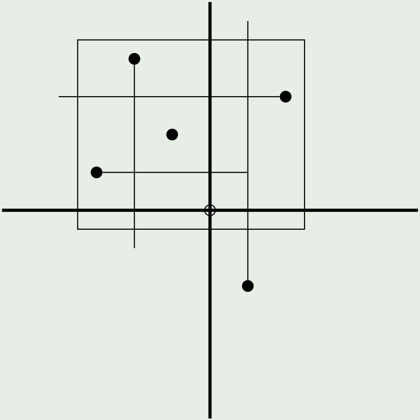

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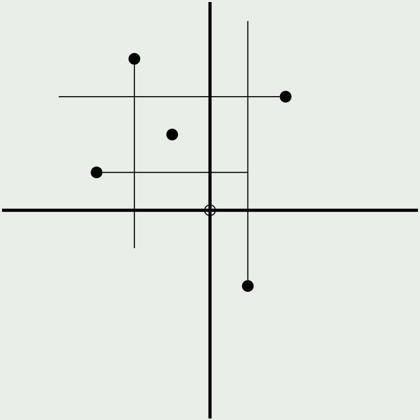

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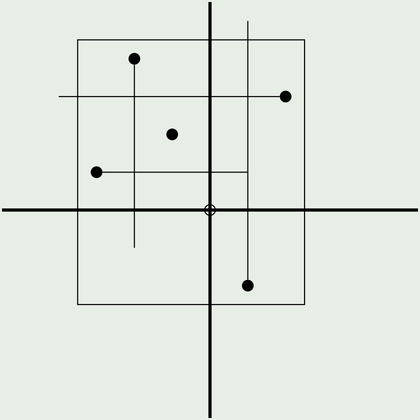

Pin Permutations

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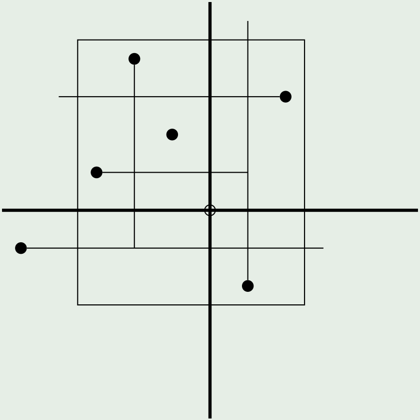

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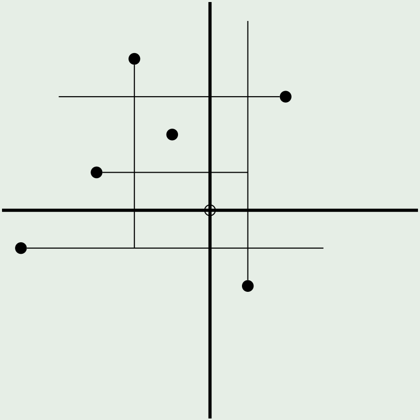

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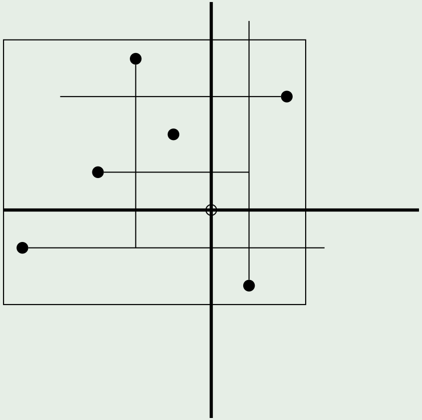

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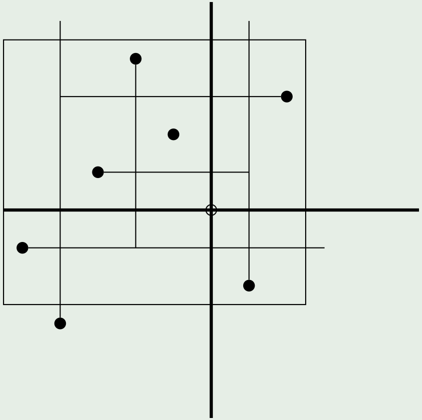

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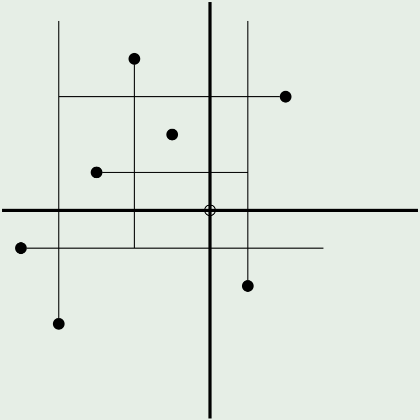

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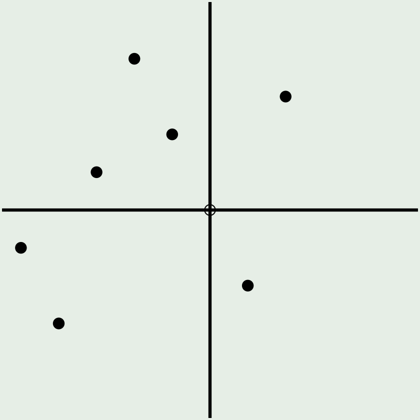

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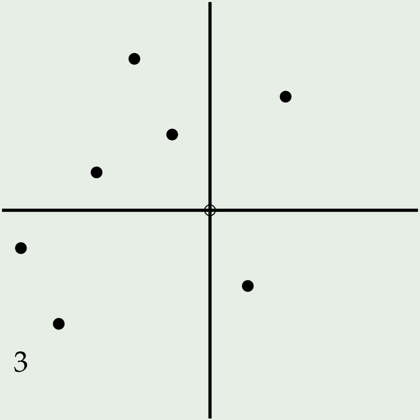

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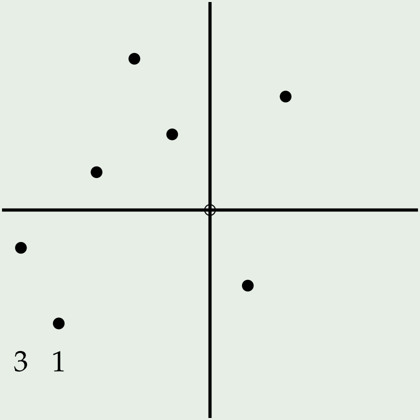

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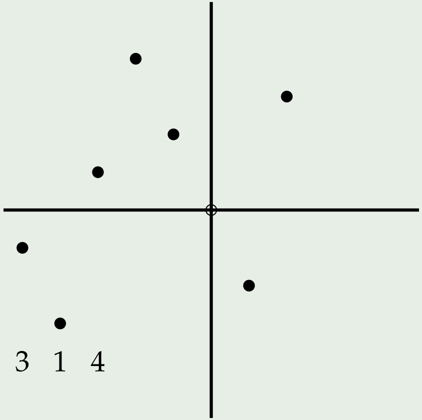

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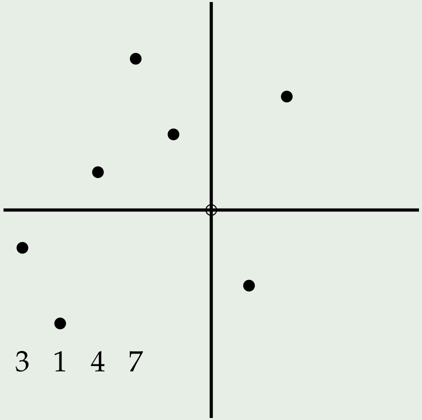

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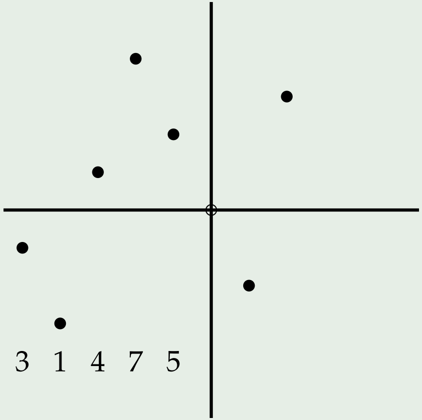

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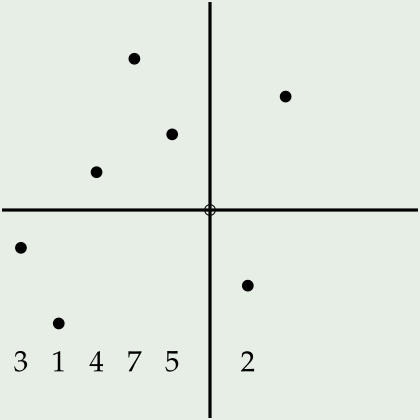

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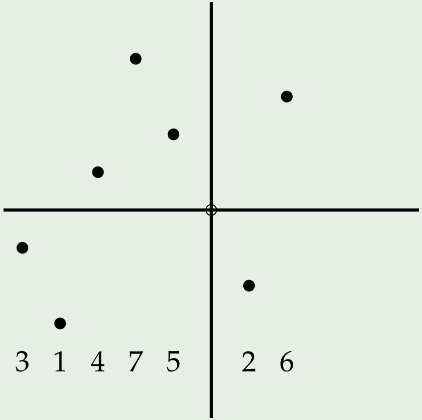

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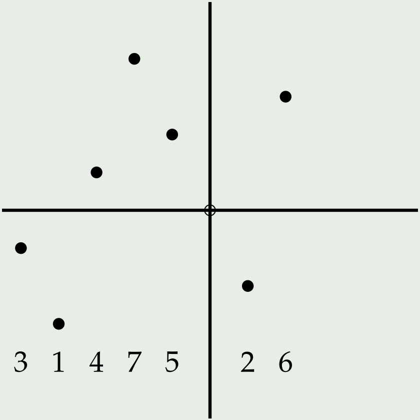

So the pin-word 2*lurdld* constructs the permutation 3147526
(or the *centred* (that is, 2-by-2-gridded) permutation $31475|_3 26$)

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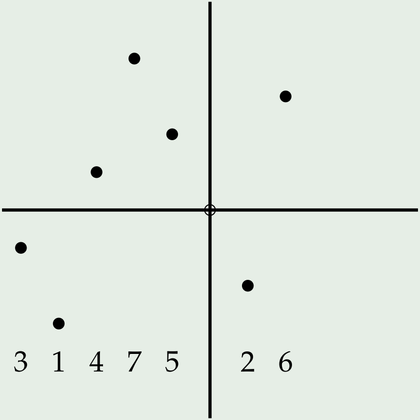
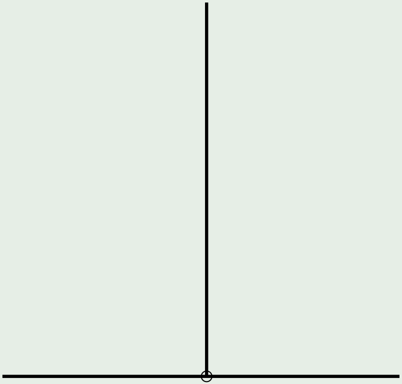


Note: this process is almost guaranteed to generate a simple permutation

The Class $\mathcal{V}$

Constructing a permutation class from the pin sequence $1(ulur)^*$



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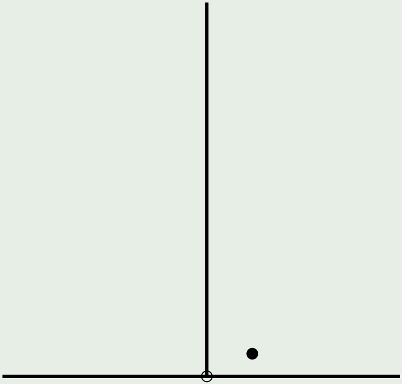

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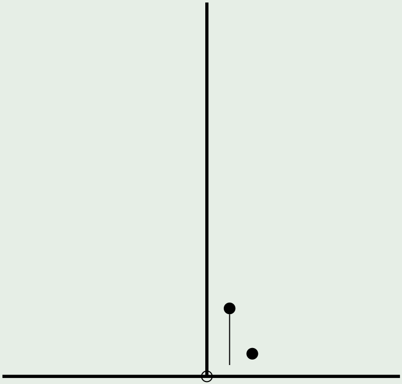

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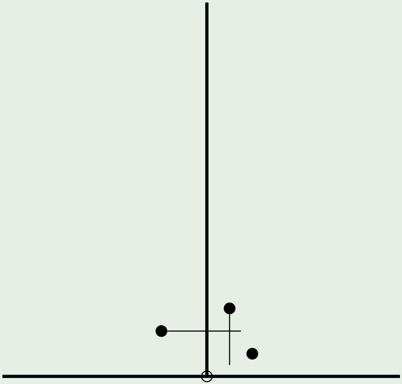

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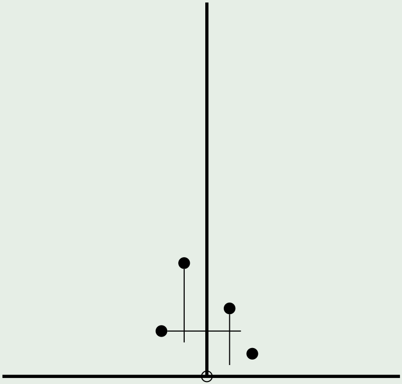

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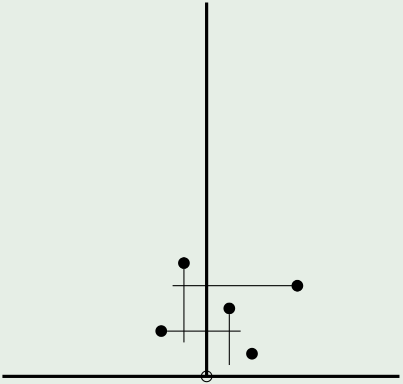

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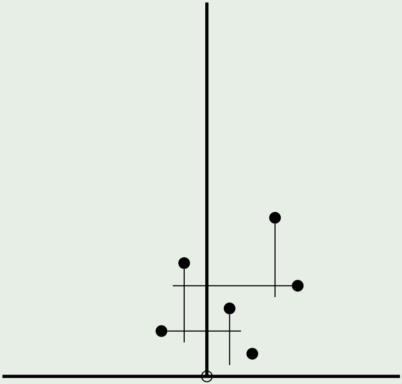

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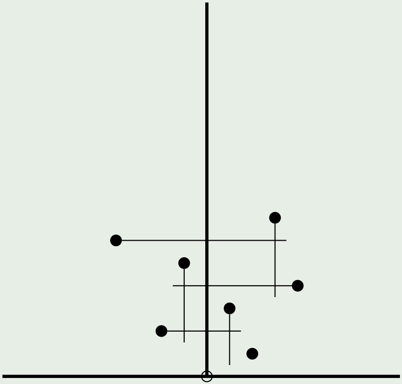

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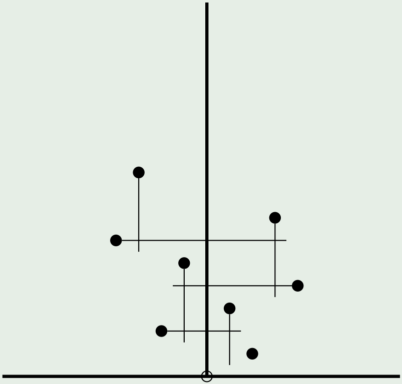

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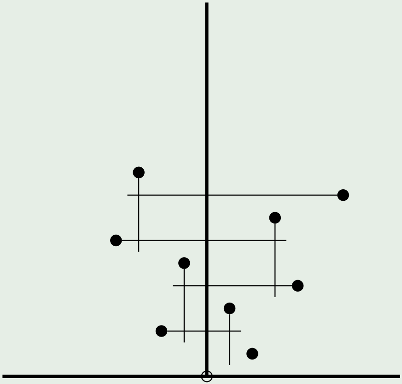

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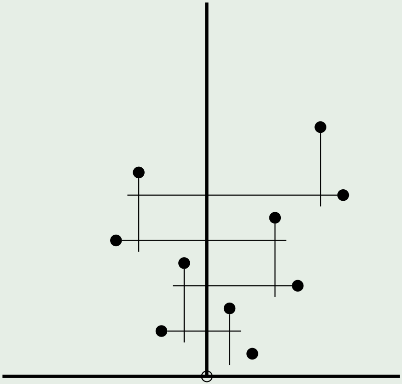

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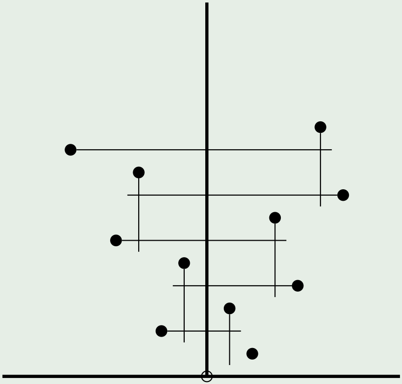

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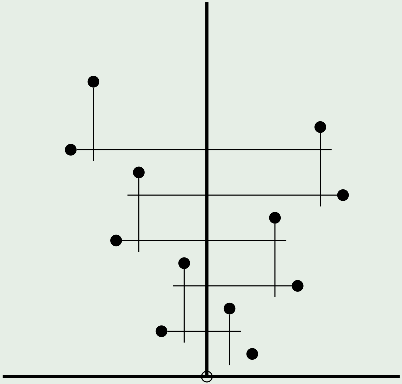

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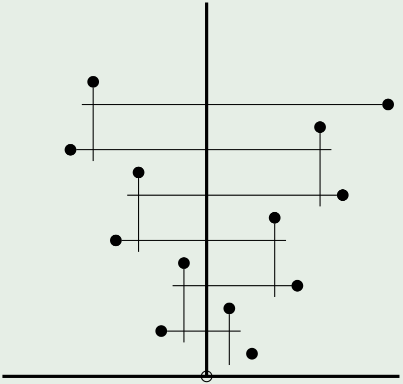

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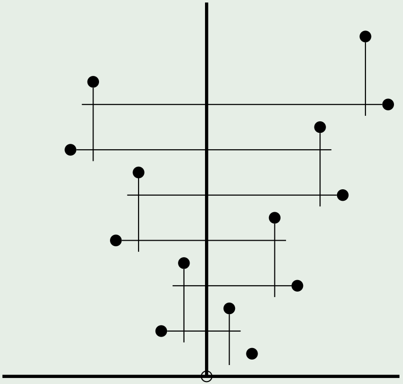

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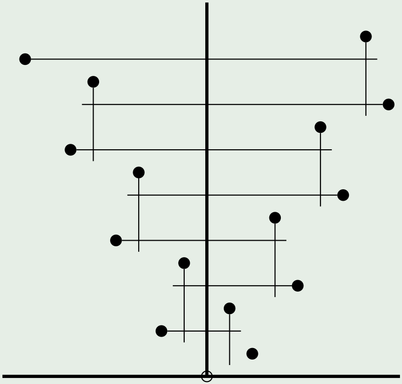

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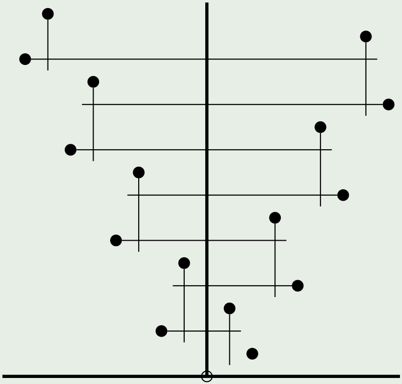

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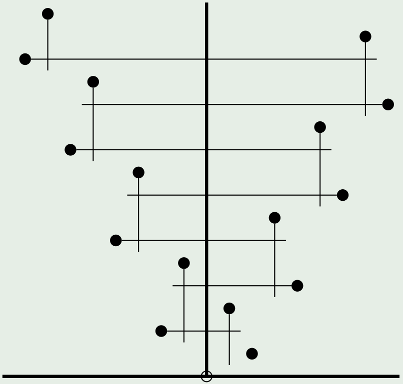

The pin class constructed from this pin sequence (called $\mathcal{V}$) consists of all of the permutations that can be found anywhere inside this (infinite) diagram.

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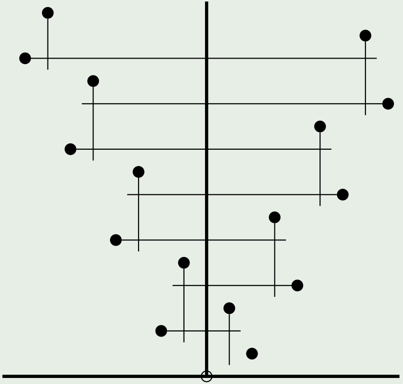
Constructing a permutation class from the pin sequence $1(ulur)^*$



Examples:

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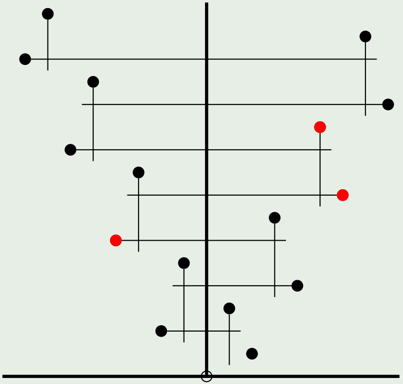

Examples:

- $132 \in \mathcal{V}$

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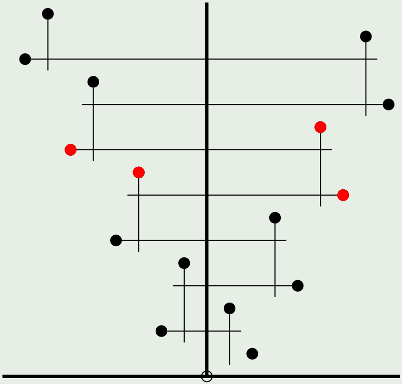

Examples:

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- $3241 \in \mathcal{V}$

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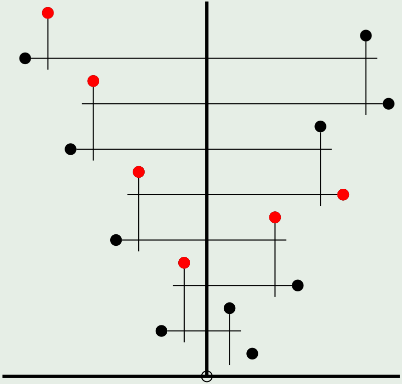

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- $132 \in \mathcal{V}$
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- $654123 \in \mathcal{V}$

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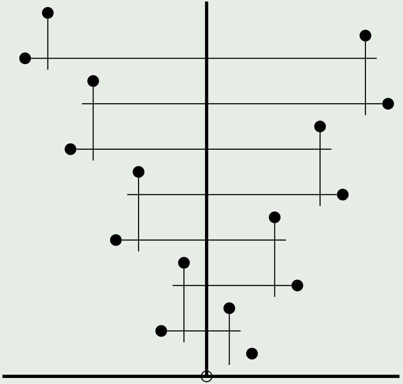

Examples:

- $132 \in \mathcal{V}$
- $3241 \in \mathcal{V}$
- $654123 \in \mathcal{V}$
- $12543 \notin \mathcal{V}$

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Why Are Pin Classes Interesting?

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1.) Structure of simple permutations: ’the primes of permutations’

- This was the inital motivation for their study:
  Brignall, Huczynska and Vatter (2008)

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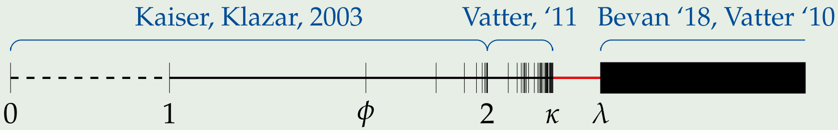
2.) Connection with infinite antichains and well-quasi ordering

- Explains phase-transitions at $\kappa \approx 2.206$ and $\lambda \approx 2.357$ in growth-rate diagram:



Why Are Pin Classes Interesting?

1.) Structure of simple permutations: 'the primes of permutations'

- This was the inital motivation for their study:
  Brignall, Huczynska and Vatter (2008)

2.) Connection with infinite antichains and well-quasi ordering

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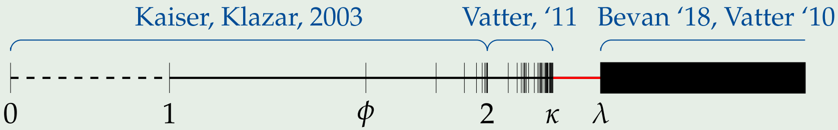

3.) We have a strategy for counting them...

A Structure Theorem for Pin Classes

Most important feature of pin classes is that they are 'easy' to enumerate (at least as long as the defining pin sequence is recurrent...):

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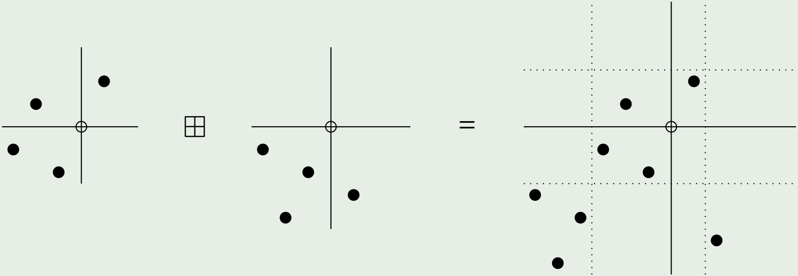
- To describe this structure we will need to consider pin permutations as *centred* (that is, 2-by-2 gridded) permutations (this will affect the enumeration sequence of the permutation class, but crucially does *not* affect the growth rate)
- This is because we need to use the $\boxplus$-sum, a generalisation of the direct sum...

A Structure Theorem for Pin Classes

The ⊞-sum

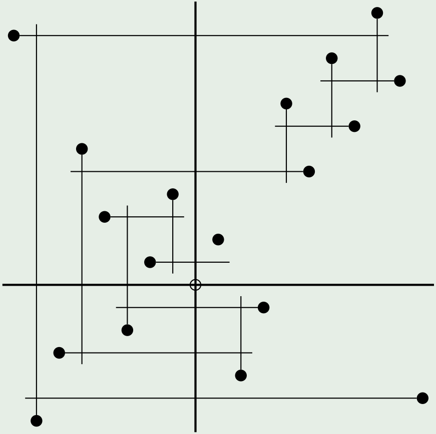
Given two centred (ie., 2-by-2-gridded) permutations π° and σ° their *box sum*, written $\pi^\circ \boxplus \sigma^\circ$, is obtained by replacing (or 'inflating') the ghost point at the centre of σ° with a copy of π°.



The ⊞-sum of the centred permutations $\pi^\circ = (231 \,|_2\, 4)$ and $\sigma^\circ = (413 \,|_4\, 2)$ is $(413675 \,|_6\, 82)$; the ghost point at the centre of σ° is simply replaced ('inflated') by a copy of π°.

A Structure Theorem for Pin Classes

The pin permutation $1luldrdlurururuldr$



- p_n is the *only* point that intersects the bounding rectangle of $p_1, \ldots p_{n-1}$
- Hence when we remove it we create an interval $p_1, \ldots p_{n-1}$
- This decomposes the permutation into a $\boxplus$-sum of two shorter pin permutations

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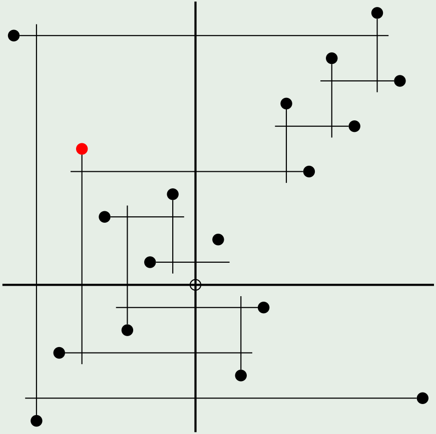

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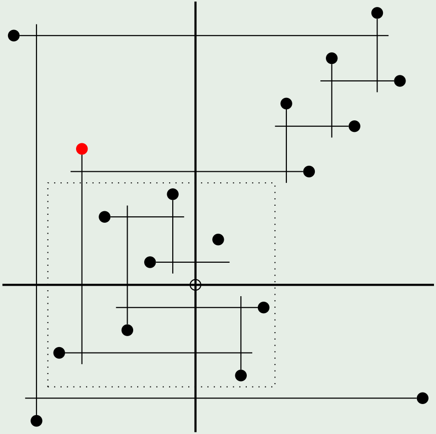

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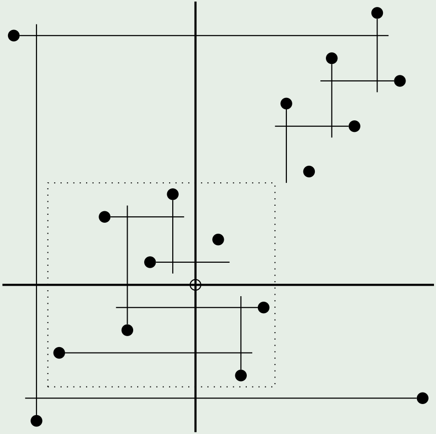

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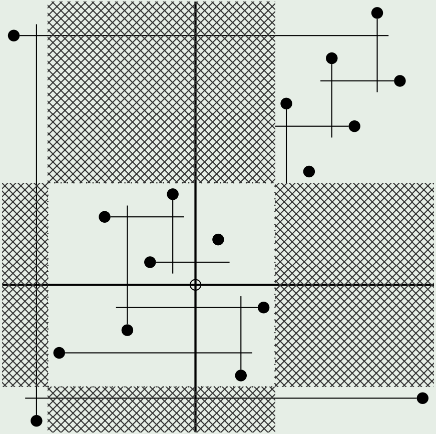

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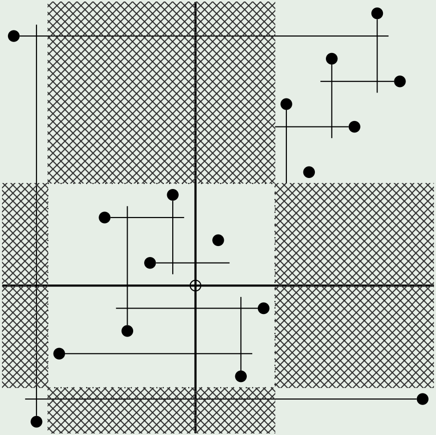

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$1luldrdl\mathbf{u}rururuldr \to 1luldrdl \boxplus 1ururuldr$

A Structure Theorem for Pin Classes

This process equips pin classes with an in-built structure theorem, which we will exploit later on:

Theorem: The Pin Decomposition

Let $\mathcal{C}_w^\circ$ be the pin class generated by pin sequence w. Then:

$$\sigma^\circ \in \mathcal{C}_w^\circ \text{ iff } \sigma^\circ = \pi_{w_1}^\circ \boxplus \pi_{w_2}^\circ \boxplus \ldots \pi_{w_k}^\circ$$

where $w_1, w_2, \ldots w_k$ is a sequence of pin factors of w that occur *in that order, in non-overlapping instances and separated from each other by at least one letter* in w, and $\pi_{w_i}^\circ$ is the (centred) permutation generated from w_i.

A Structure Theorem for Pin Classes

This structure theorem is often awkward to apply due to the conditions on the pin factors w_i; it becomes much easier however, if we assume that w is a **recurrent** pin sequence - that is, every pin factor of w occurs infinitely often. The theorem then becomes:

Corollary: The Recurrent Case

Let $\mathcal{C}_w^\circ$ be the pin class generated by the *recurrent* pin sequence w. Then:

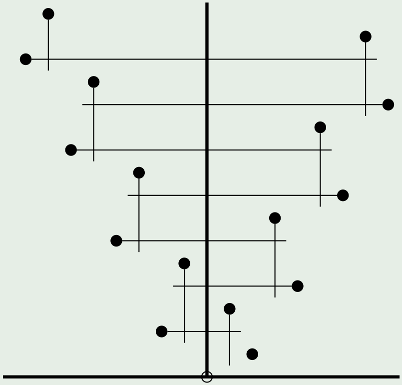
$$\sigma^\circ \in \mathcal{C}_w^\circ \text{ iff } \sigma^\circ = \pi_{w_1}^\circ \boxplus \pi_{w_2}^\circ \boxplus \ldots \pi_{w_k}^\circ$$

where $w_1, w_2, \ldots w_k$ is a sequence of pin factors of w, and $\pi_{w_i}^\circ$ is the (centred) permutation generated from w_i. In particular, C_w° is $\boxplus$-closed.

Counting the pin-class $\mathcal{V}$

The pin class $\mathcal{V}$, defined by the pin sequence $1(ulur)^* = 1ulurulurulur...$

The Class $\mathcal{V}$



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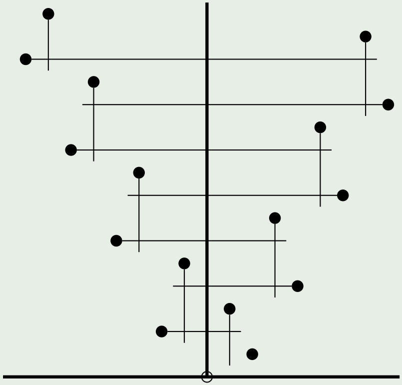

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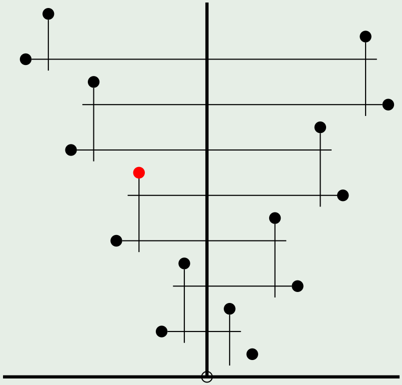

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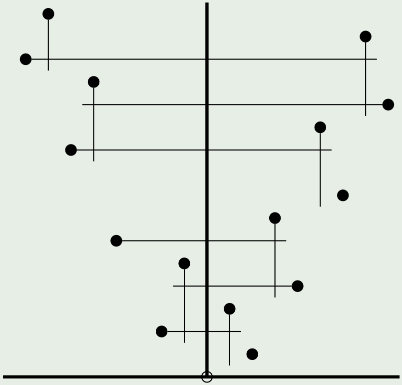

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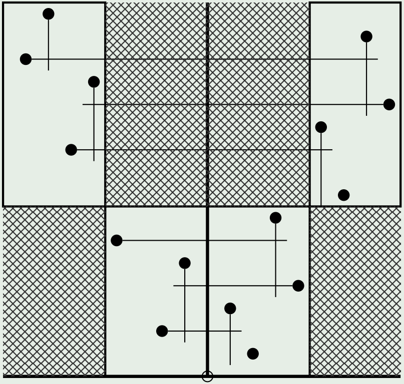

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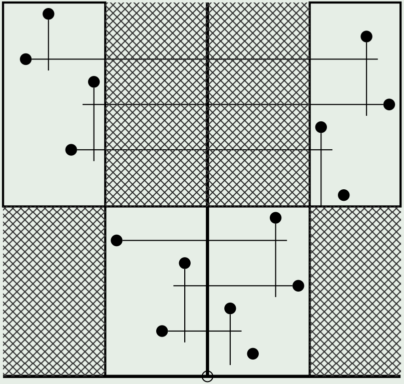

- Every $\pi \in \mathcal{V}$ is contained in this (infinite) diagram
- As soon as we remove an interior point of a pin permutation it decomposes into the $\boxplus$-sum of two consecutive pin permutations
- Hence every $\pi \in \mathcal{V}$ ca be expressed (uniquely) in the form $\pi = \sigma_1 \boxplus \sigma_2 \boxplus \cdots \boxplus \sigma_k$, where the σ_i are $\boxplus$-indecomposables

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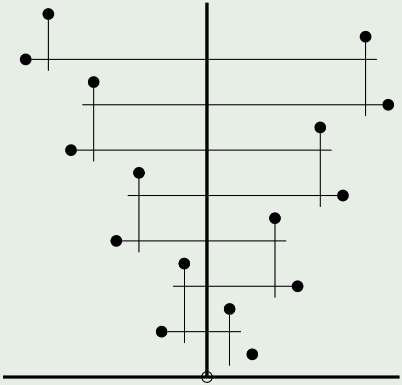

Strategy: first we count the $\boxplus$-indecomposables in $\mathcal{V}$:

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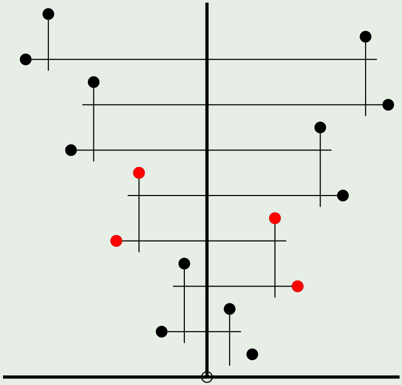

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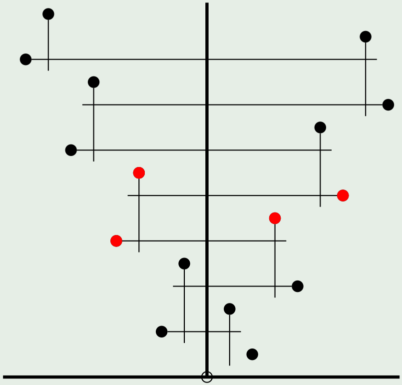

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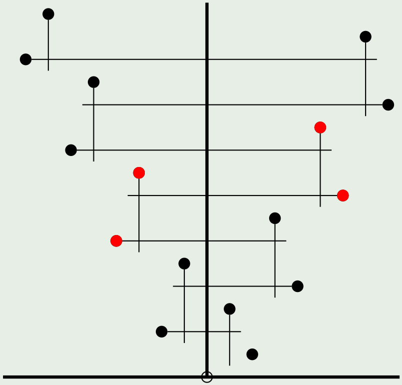

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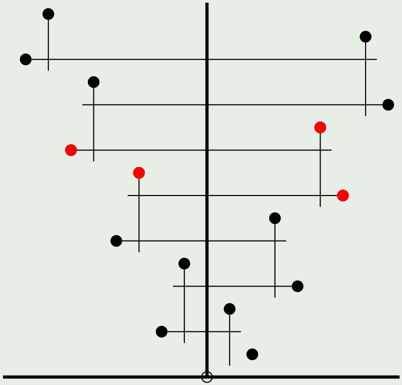

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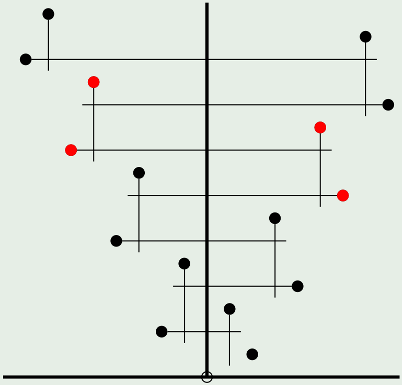

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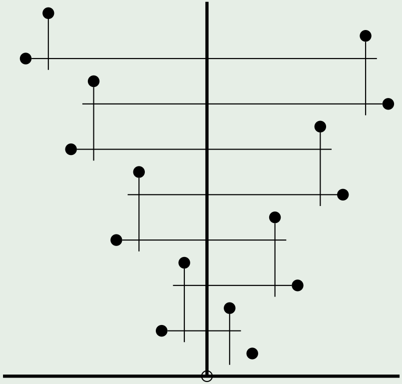

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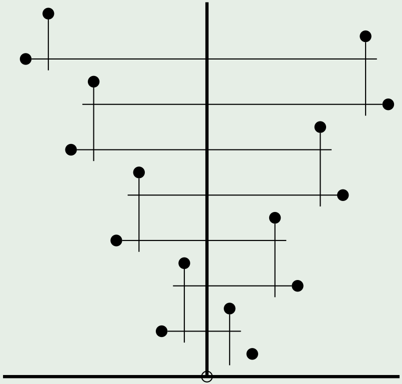
We can store this information as a generating function:

$$
\begin{aligned}
g(z) &= 2z + 2z^2 + 2z^3 + 4z^4 + 4z^5 + 4z^6 + \ldots \\
&= 2z + 2z^2 + 2z^3 + 4z^4(1 + z + z^2 + z^3 + \ldots)
\end{aligned}
$$

Counting the pin-class $\mathcal{V}$

The Class $\mathcal{V}$



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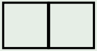
$$g(z) = 2z + 2z^2 + 2z^3 + \frac{4z^4}{1-z} = \frac{2z(1+z^3)}{1-z}$$

Counting the pin-class $\mathcal{V}$



- One ⊞-indecomposable: g.f. $g(z) = L_1 z + L_2 z^2 + L_3 z^3 + \ldots$

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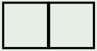

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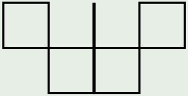

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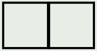

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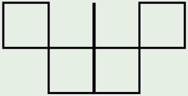

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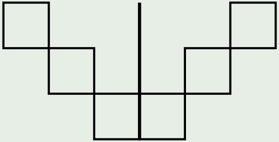

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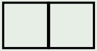

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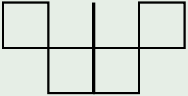

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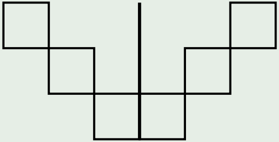

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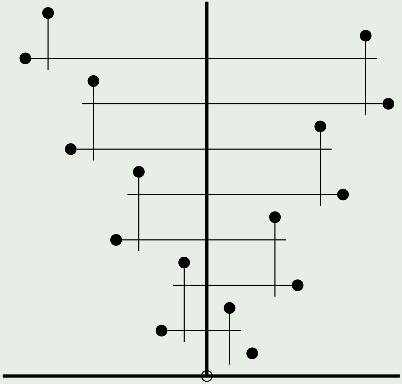
So the generating function for the entire class $\mathcal{V}$ is given by:

$$\begin{aligned} f(z) &= 1 + g(z) + g(z)^2 + g(z)^3 + g(z)^4 + \ldots \\ &= \frac{1}{1 - g(z)} = \frac{1}{1 - \frac{2z(1+z^3)}{1-z}} = \frac{1 - z}{1 - 3z - 2z^4} \end{aligned}$$

Counting the pin-class $\mathcal{V}$

Growth Rate of $\mathcal{V}$



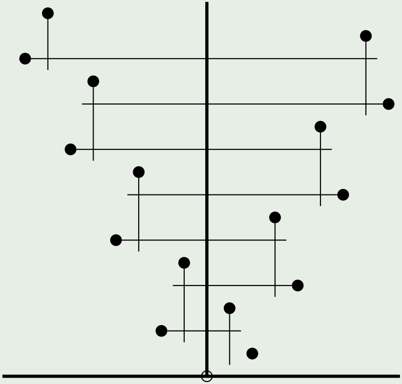
$\mathcal{V}$ has generating function

$$f(z) = \frac{1 - z}{1 - 3z - 2z^4}$$

Now we can use the generating function of $\mathcal{V}$ to calculate its growth rate using Pringsheim's Theorem

Counting the pin-class $\mathcal{V}$

Growth Rate of $\mathcal{V}$



$\mathcal{V}$ has generating function

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Growth rate $\nu \approx 3.069$

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Moral of this example:

The above process works for (recurrent) pin sequences more generally. Reduces the problem of enumerating a pin class to a strategy:

Enumerating (Recurrent) Pin Classes

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Enumerating (Recurrent) Pin Classes

1. Pin construction gives structure theorem: basically the same as above example.

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The above process works for (recurrent) pin sequences more generally. Reduces the problem of enumerating a pin class to a strategy:

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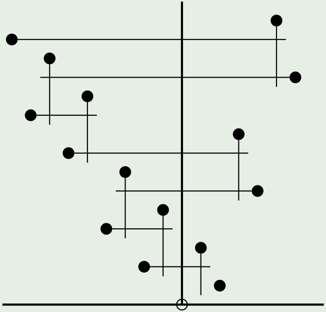
4. Generating function theory: deduce g.f. for whole class from g.f. of box-indecomposables and investigate asymptotics through analysis

Applications of Pin Classes in Two Quadrants

Application 1: A Wealth of Classes



The pin class generated by the pin sequence $w = 1(ululur)^*$. This has growth rate ≈ 3.25

- We now have a natural correspondence between binary sequences and pin classes in two quadrants (eg., 100100100 . . .)
- This gives us a huge class of permutation classes which we can enumerate by determining the complexity of the sequence
- $\rightarrow$ see Robert's talk (uncountably many permutation classes with distinct enumerations)

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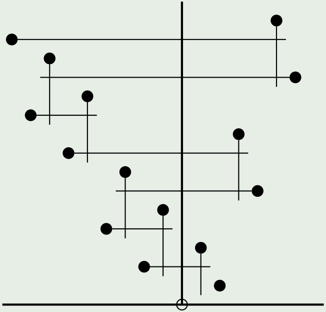

The pin class generated by the pin sequence $w = 1(ululur)^*$.
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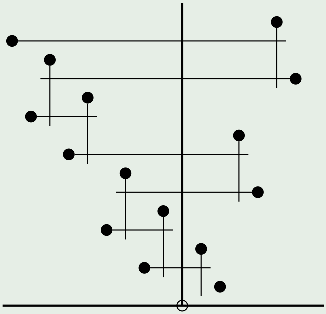
Ongoing work:

- Classify growth rates of periodic pin classes in two quadrants
- See how far this extends to recurrent classes more generally
- Non-recurrent pin classes...

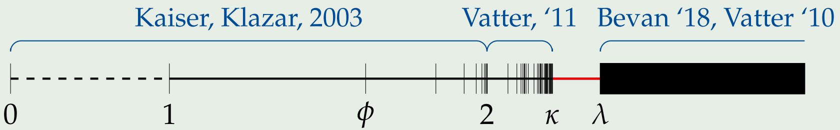
Applications of Pin Classes in Two Quadrants

Application 2: Classes with Bounded Oscillations



- Very easy to control the maximum length of an oscillation in periodic pin classes
- Thus has applications to establishing growth rates of permutation classes with bounded oscillations

Possible Growth Rates of Permutation Classes



Applications of Pin Classes in Two Quadrants

Application 3: Well-Quasi-Ordering and Antichains

- Pin sequences are a good way of producing antichains
- Thus pin classes have potential applications of well-quasi-ordering and classifying antichains
- *Conjecture:* $\mathcal{V}^{+2}$ contains the ‘second-smallest’ antichain?

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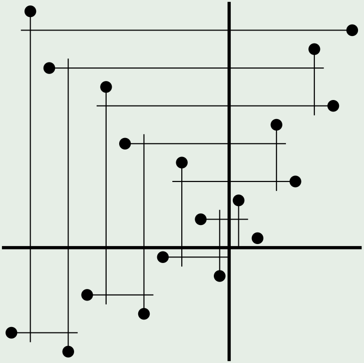
Further Directions

Pin Classes in Three and Four Quadrants

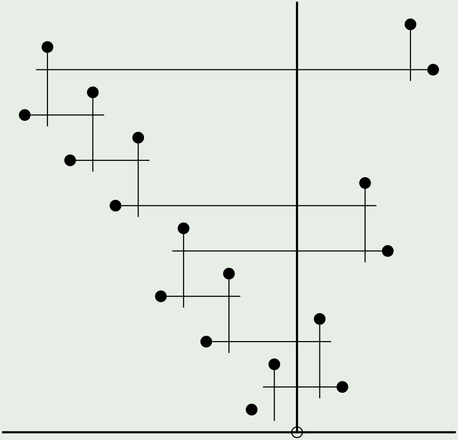


The pin class $\mathcal{Y}$ generated by $w = 1(uldlur)^*$.

- Once we move beyond two quadrants things get more difficult: the $\boxplus$-decomposition is no longer unique and the correspondence between contiguous pin factors and $\boxplus$-indecomposables breaks down
- Fortunately, these problems are somewhat pathological, and have now been fully classified
- This allows the process to be amended, though some control over the resulting pin class is lost

Non-Recurrent Pin Classes

The Liouville V, $\mathcal{V}_{\mathcal{L}}$



- A non-recurrent pin class that we *can* enumerate: its growth rate is ≈ 3.283
- Idea is to bound below by the *box interior*, $\mathcal{V}_{\mathcal{L}}^{\boxplus}$, the largest $\boxplus$-closed class contained in $\mathcal{V}_{\mathcal{L}}$
- This is in fact 'enough' of the class to dominate its growth rate

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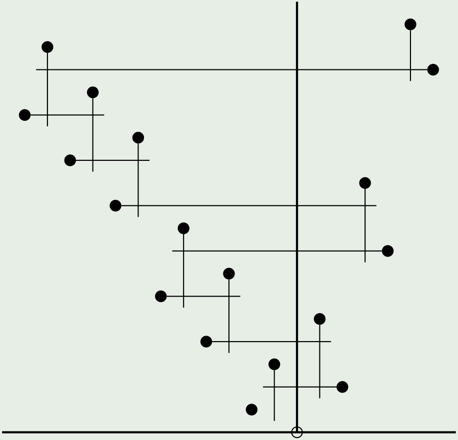

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Open Problem: Is the growth rate of a non-recurrent pin class always equal to that of its $\boxplus$-interior?

Open Problems & Further Directions for Work

- Classification of growth rates of (periodic, recurrent) pin permutation classes in two quadrants
- Is the antichain at $\mathcal{V}$ the 'next' one after the antichain of oscillations?
- Applications to growth rates of permutation classes with bounded oscillations
- Explore pin classes in three and four quadrants
- Is the growth rate of a non-recurrent pin class always equal to that of its $\boxplus$-interior?