GRASSMANNIAN PERMUTATIONS AVOIDING IDENTITY

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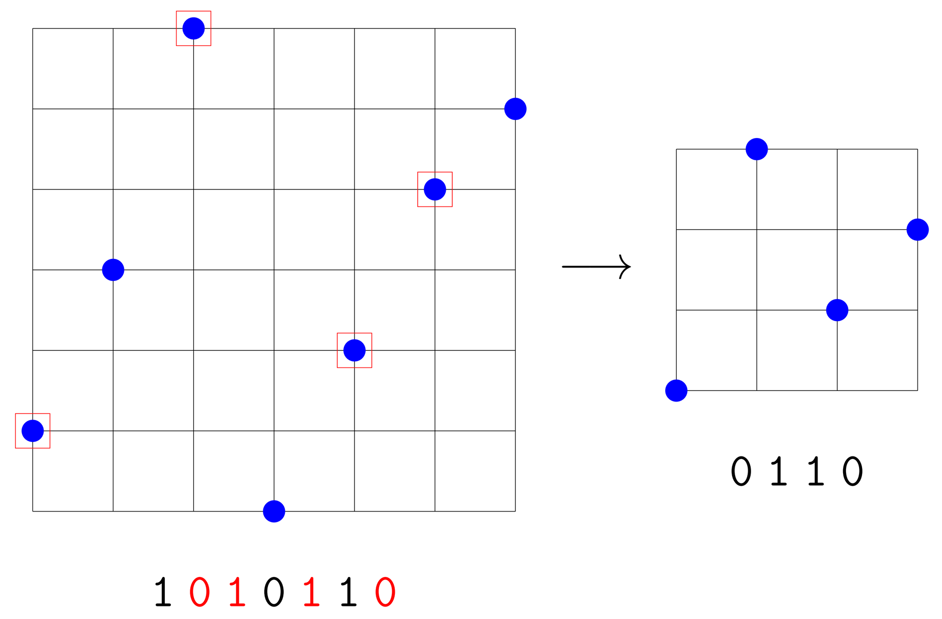
Binary words

Grassmannian permutations of size m:

$$\mathcal{G}_m = \{\pi \in S_m \mid \mathrm{des}(\pi) = 0 \text{ or } 1\}.$$

Associate $G(w)$ to a binary word w as follows:

- Set $A = \{i \in [m] \mid w_i = 0\}$.
- First $\#A$ terms of $G(w)$: A in increasing order.
- Remaining terms: $[m] \setminus A$ in increasing order.



Bijection apart from

$$G(0^i 1^{m-j}) = \mathrm{id}_m \text{ for all } j \in [0, m].$$

Patterns in $G(w)$ are

$$\{G(w') \mid w' \text{ subsequence of } w\}.$$

Focus: Permutations in $\mathcal{G}_m$ avoiding id_k.

$\mathcal{B}(k,m) =$ Set of binary words of length m avoiding $0^j 1^{k-j}$ for all $j \in [0,k]$.

Set $B(k,m) = \#\mathcal{B}(k,m)$.

Count using Dyck Paths

Example. [1, Prop. 3.1] For any $k \geq 2$, we have

$$B(k, 2k-2) = C_{k-1}.$$

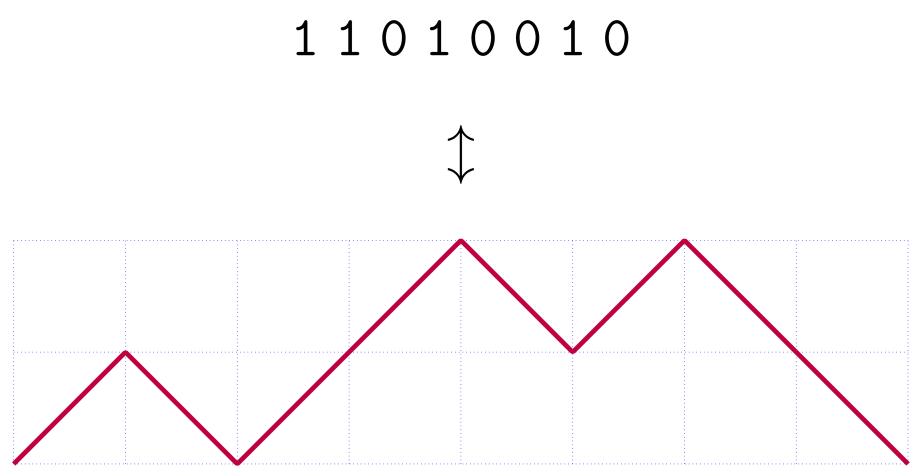
For words in $\mathcal{B}(k, 2k-2)$, $\#0\text{s} = \#1\text{s} = k-1$.

$$1^{a_{k-1}} 0 1^{a_{k-2}} 0 \cdots 1^{a_2} 0 1^{a_1} 0$$
$$\updownarrow$$
$$U D^{a_1} U D^{a_2} \cdots U D^{a_{k-1}}$$

where $\sum_{r=1}^{i} a_r \leq i$ for all $i \in [k]$.

Theorem. For any $k, m \geq 1$, we have

$$B(k,m) = \sum_{a=1}^{2k-m-1} \left[\binom{m}{k-a} - \binom{m}{k} \right].$$



$\mathcal{B}(k,m) \leftrightarrow$ Dyck paths of semilength $(k+1)$ with sum of heights of first and last peak $(2k-m)$.

Words in $\mathcal{B}(k,m)$ with $\#0\text{s} = j \in [0, k-1]$:

$$1^{a_j} 0 1^{a_{j-1}} 0 \cdots 1^{a_2} 0 1^{a_1} 0 1^{a_0}$$
$$\updownarrow$$
$$U^{k-j} D^{a_0+1} U D^{a_1} \cdots U D^{a_{j-1}} U D^{a_j} U D^{k+j-m}$$

where $\sum_{r=0}^{i} a_r < (k-j) + i$ for all $i \in [0, j]$.

Lemma. The number of Dyck paths of semilength $n+1$ with first peak of height a and last peak of height b is

$$\binom{2n-a-b}{n-a} - \binom{2n-a-b}{n}.$$

Count using Recurrences

Theorem. [1, Conjecture by Michael Weiner] For any $k, m \geq 1$, we have

$$B(k,m) = \sum_{j=1}^{2k-m} (-1)^{j-1} j \cdot \binom{2k-m-j}{j} \cdot C_{k-j}.$$

Both sides satisfy same initial conditions and recurrence

$$B(k,m) = B(k-1, m-1) + B(k, m-1) - T(k, m-k)$$

$T(a,b) = \#$ Dyck paths of semilength a with last peak of height $a-b$.

Suppose $w = w_1 w_2 \cdots w_m$ and $w' = w_1 w_2 \cdots w_{m-1}$.

- If $w_m = 1$, $w \in \mathcal{B}(k,m) \Leftrightarrow w' \in \mathcal{B}(k-1, m-1)$.
- If $w_m = 0$, $w \in \mathcal{B}(k,m) \Leftrightarrow w' \in \mathcal{B}(k, m-1)$ and $\#0\text{s} \neq k-1$.

Words in $\mathcal{B}(k, m-1)$ with $\#0\text{s} = k-1$:

$$1^{a_{k-1}} 0 1^{a_{k-2}} 0 \cdots 1^{a_1} 0$$
$$\updownarrow$$
$$U D^{a_1} U D^{a_2} \cdots D^{a_{k-1}} U D^{2k-m}$$

Corollary. The number of Dyck paths of semilength n with sum of heights of first and last peaks equal to s is

$$\sum_{j=1}^{\lfloor s/2 \rfloor} (-1)^{j-1} j \binom{s-j}{j} C_{n-1-j}.$$

Direct proof?

Question

For any $r \geq 1$, count the permutations in $\mathcal{G}_m$ with exactly r occurrences of id_k.

Parity Restrictions

If $w = 1^{a_k} 0 1^{a_{k-1}} 0 \cdots 1^{a_1} 0 1^{a_0}$, then $\mathrm{inv}(G(w)) = \sum_{i=1}^{k} i \cdot a_i$.

Hence, $G(w)$ is odd if and only if number of odd terms in $(a_1, a_3, a_5, \ldots)$ is odd.

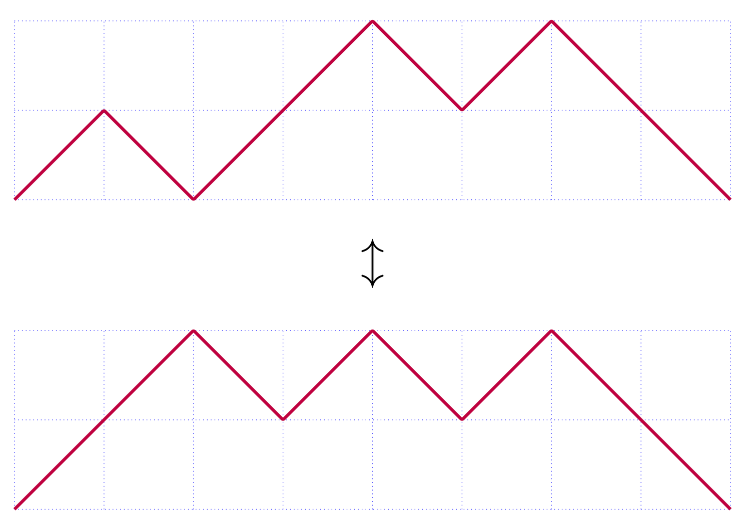
Definition. For $k, m \geq 1$,

$$O(k,m) = \#\{w \in \mathcal{B}(k,m) \mid G(w) \text{ is odd}\}.$$

Proposition. For any $k \geq 2$, we have

$$O(k, 2k-2) = \frac{C_{k-1} + C_{(k-2)/2}}{2}.$$

$(a_1, a_2, a_3, a_4) = (1, 0, 1, 2)$



$(a_1, a_2, a_3, a_4) = (1-1, 0+1, 1, 2) = (0, 1, 1, 2)$

- Parity reversing involution on Dyck paths that have a peak or valley at even height.
- Permutation corresponding to path with all peaks and valleys at odd height is odd.
- Number of Dyck paths of semilength n with all peaks and valleys at odd height is $C_{(n-1)/2}$.

Similar ideas work for general $O(k,m)$ [2].

References

[1] J. B. Gil and J. A. Tomasko. Restricted grassmannian permutations. *Enumer. Combin. Appl.*, 2(S4PP6), 2022.

[2] K. Menon and A. Singh. Grassmannian permutations avoiding identity. *arXiv preprint arXiv:2212.13794*, 2022.

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