Characterizing Rothe Diagrams

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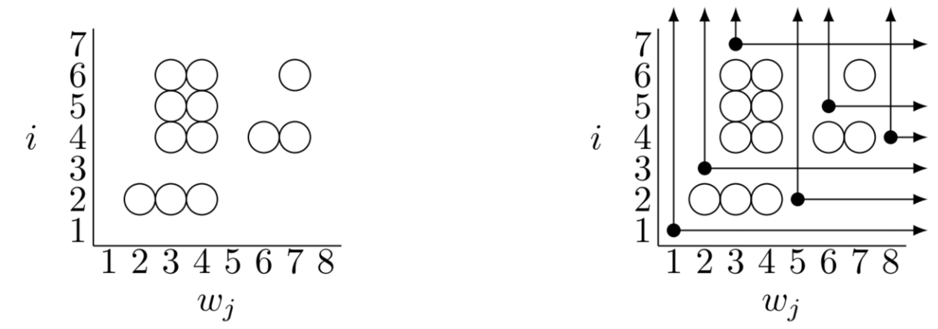
Introduction

Consider a permutation w and it's one-line notation $w = w_1w_2...w_n$. An ***inversion*** of w is a pair of indices $i < j$ such that $w_i > w_j$. For example, if $w = 152869347$, then $w_4 = 8$ and $w_7 = 3$ is an inversion. One way to visualize inversions is with the ***Rothe diagram*** of w, defined to be the following subset of cells in the first quadrant:

$$\mathbb{D}(w) = \{(i, w_j) \colon i < j, w_i > w_j\} \subset \mathbb{Z}^+ \times \mathbb{Z}^+. \tag{1}$$

Graphically, we represent cells in $\mathbb{D}(w)$ by bubbles and use the French convention that the i coordinates are written on the vertical axis and the w_j coordinates are written on the horizontal axis.



Above is the Rothe diagram $\mathbb{D}(152869347)$ and the placement of its "death rays".

Connection with Schubert Varieties

By way of example, consider the nonsingular matrix

$$A = \begin{pmatrix} 6 & 8 & 3 & 1 \\ 8 & 7 & 3 & 1 \\ 5 & 2 & 2 & 1 \\ 6 & 4 & 4 & 2 \end{pmatrix}$$

Denoting row vectors by v_i, there is a nested sequence of subspaces

$$0 \subset \langle v_1 \rangle \subset \langle v_1, v_2 \rangle \subset \langle v_1, v_2, v_3 \rangle \subset \langle v_1, v_2, v_3, v_4 \rangle = \mathbb{C}^4.$$

We call this nested sequence a ***full flag*** in $\mathbb{C}^4$. The full flags are a projective subvariety of a product of Grassmanians:

$$Fl(\mathbb{C}^4) \subset \Pi_{i=1}^4 Gr(i, 4).$$

We can act on these flags by matrix multiplication from $GL_4(\mathbb{C})$. The union of the orbits is the full flag variety, with a coordinate flag in each orbit. These orbits are the ***Schubert cells***, indexed by permutations:

$$Fl(\mathbb{C}^4) = \sqcup_w C_w.$$

Question: what Schubert cell does A live in?

By row echelon moves that do not change the flag, we can obtain a canonical form of the matrix A with 0's above leading 1's:

$$A = \begin{pmatrix} 6 & 8 & 3 & 1 \\ 8 & 7 & 3 & 1 \\ 5 & 2 & 2 & 1 \\ 6 & 4 & 4 & 2 \end{pmatrix} \rightsquigarrow \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 5 & 1 & 0 \\ 1 & 0 & 0 & 0 \\ 3 & 2 & 2 & 1 \end{pmatrix}$$

Then we get a permutation from the leading 1's of the canonical form, in this case $w = 4132$. So the flag of A is contained in the Schubert cell C_{4132}. In general, C_{4132} contains exactly those flags whose canonical form is the Rothe diagram:

$$\begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & * & 1 & 0 \\ 1 & 0 & 0 & 0 \\ * & * & * & 1 \end{pmatrix}$$

Diagram Properties

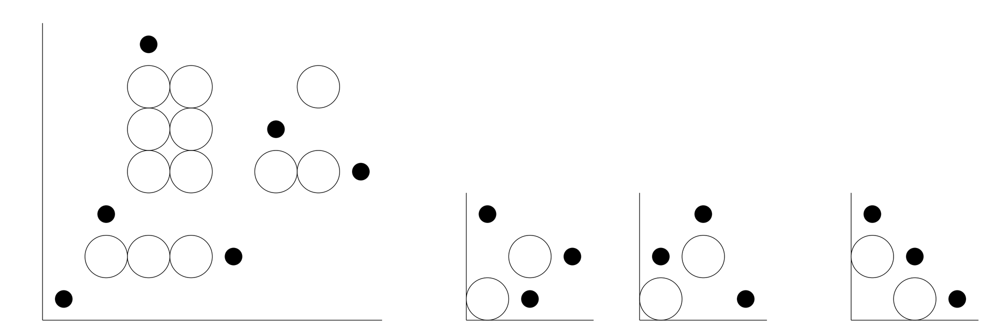
Rothe diagrams satisfy many properties that bubble diagrams in general might not have. We asked what properties give us enough to say that a diagram is the Rothe diagram of some permutation. Thus, we were looking for statements of the form, "D is a Rothe diagram if and only if it satisfies properties X, Y, and Z". We defined the generalized form of the dot rule, the empty cell gap rule, and step-out avoidance to realize such characterizations.

Southwest Rule

A diagram $D \subset \mathbb{Z}^+ \times \mathbb{Z}^+$ is ***southwest*** if $(i,j) \in D$ and $(i', j') \in D$ imply $(\min(i,i'), \min(j,j')) \in D$.

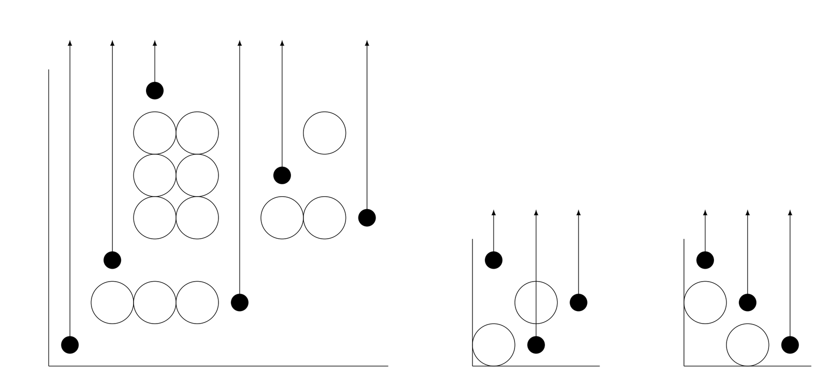
Dot Rule

Define $c_i = \min\{j \in \mathbb{Z}^+ + \colon j > j' \ \forall\ (i,j') \in D, j \neq c_{i'} \ \forall\ i' < i\}$. Then, $R(D) = \{(i, c_i)\}_{i=1}^\infty$ are the ***row dots*** of D. Similarly, define $r_j = \min\{i \in \mathbb{Z}^+ \colon i > i' \ \forall\ (i', j) \in D, i \neq r_{j'} \ \forall\ j' < j\}$. Then, $C(D) = \{(r_j, j)\}_{j=1}^\infty$ are the ***column dots*** of D. A diagram satisfies the ***dot rule*** if $R(D) = C(D)$. In a Rothe diagram, dots correspond to death ray origins.



Popping Rules

The ***vertical popping*** rule states that bubbles are not allowed to be placed above row dots. The ***horizontal popping*** rule states that bubbles are not allowed to be placed to the right of column dots. A diagram satisfies both popping rules if and only if it satisfies the dot rule.



Empty Cell Gap Rule

Include a "column 0" which is filled with a bubble in each row. These bubbles are called ***basement bubbles***. The ***final bubble*** is defined as the last bubble in a row, where all cells afterwards are empty. With basement bubbles, each row must contain a final bubble. Consider bubbles in cells (i_0, w_0) and $(i_0, w_0 + n + 1)$ with no bubbles between them, i.e. with a horizontal gap of length $n \geq 1$. Let $\mathcal{B}_{(i_0, w_0+n+1)} \subset \mathbb{Z}^{\geq 0} \times \mathbb{Z}^+$ be the region below the i_0th row and bounded inclusively by the w_0th and $(w_0 + n)$th columns. A diagram satisfies the ***empty cell gap*** rule if, whenever there is a gap of size n, the box $\mathcal{B}_{(i, w+n+1)}$ contains exactly n final bubbles.

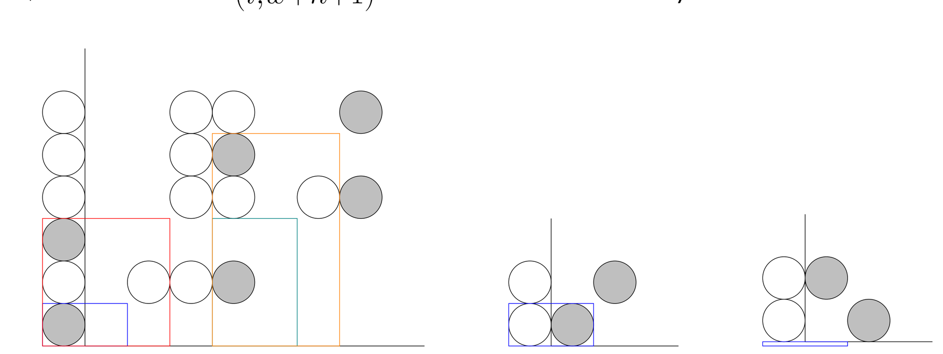
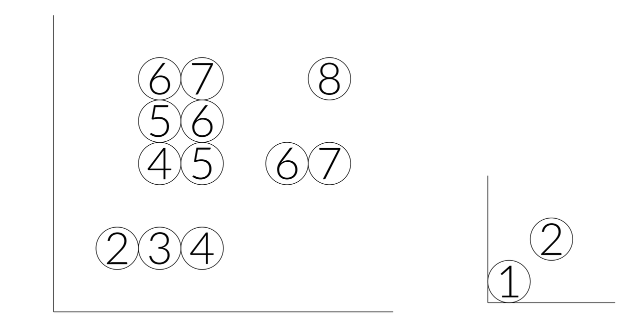


Diagram Properties

Numbering Rule and Step-out Avoidence

A diagram satisfies the ***numbering condition*** if horizontal numbering (labelling the bubbles in the ith row from left to right as $i, i+1, i+2, \cdots$) and vertical numbering (labelling the bubbles in the jth column from bottom to top $j, j+1, j+2, \cdots$) yield the same labels for each bubble.

For a diagram that satisfies the numbering condition, define a ***step-out*** to be a pair of bubbles numbered n and $n+1$ in cells (i, w) and $(i+k, w+\ell)$ respectively. We say that an enumerated diagram is ***step-out avoiding*** if no pair of bubbles is a step-out in the diagram.



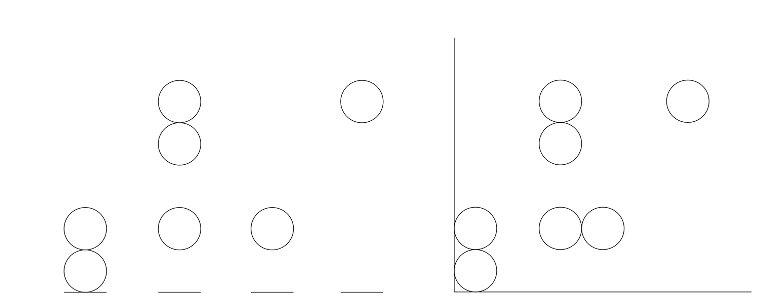
Characterizing Rothe Diagrams

Let D be a diagram. The following are equivalent:

- D is a Rothe diagram.
- D satisfies the vertical popping and emtpy cell gap rules.
- D satisfies the numbering and dot rules.
- D satisfies the dot and southwest rules.
- D satisfies the numbering rule and is step-out avoiding.

Free Columns Variation

A ***collection of free columns*** $C = \alpha_1, ..., \alpha_n$, is an ordered collection of subsets of $\mathbb{Z}^+$. A subset α_i represents a column of bubbles, the rows given by the elements of α_i. Free columns are not allowed to move past each other horizontally. Apply a horizontal numbering to these columns. They satisfy the numbering condition if the columns are labeled with unbroken intervals whose starting values strictly increase moving left to right. There exists a unique placement of free columns into a Rothe diagram if and only if the columns satisfy the numbering condition and are step-out avoiding.



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