Efficient algorithms for generating pattern-avoiding combinatorial objects

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joint work with Petr Gregor (Charles University), Elizabeth Hartung (MCLA), Hung P. Hoang (ETH Zurich), Arturo Merino (TU Berlin), Namrata (University of Warwick), Aaron Williams (Williams College)

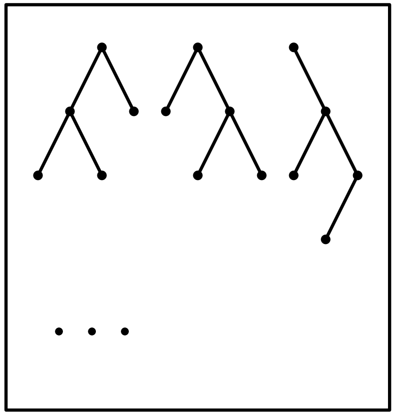





Permutation Patterns 2023

Introduction

- many different classes of combinatorial objects



binary trees

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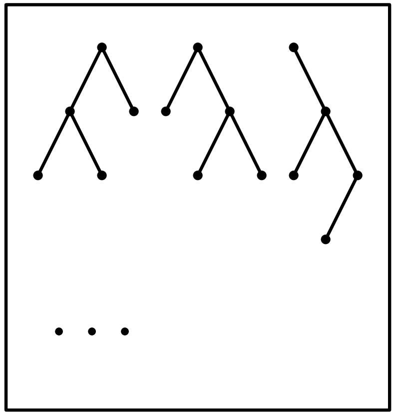

binary trees

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312
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permutations

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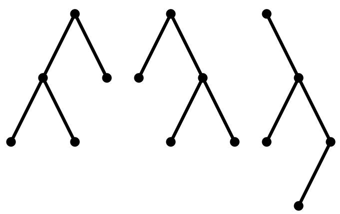

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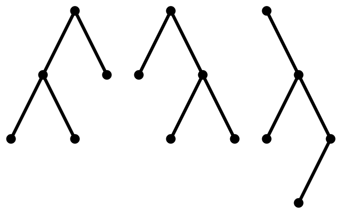

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set partitions

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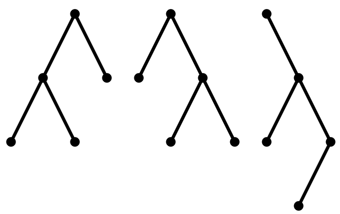

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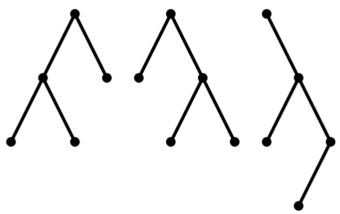

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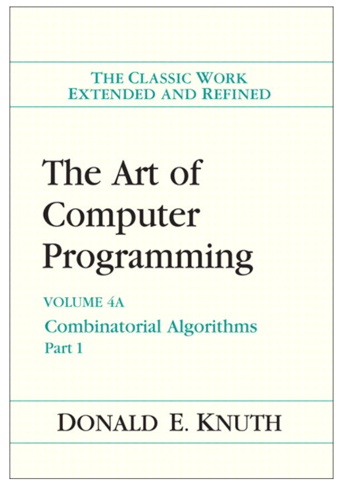
  $+$ **exhaustive generation** [Knuth TAOCP Vol. 4A]



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  - set partitions by **element exchanges** **[Kaye 76]**

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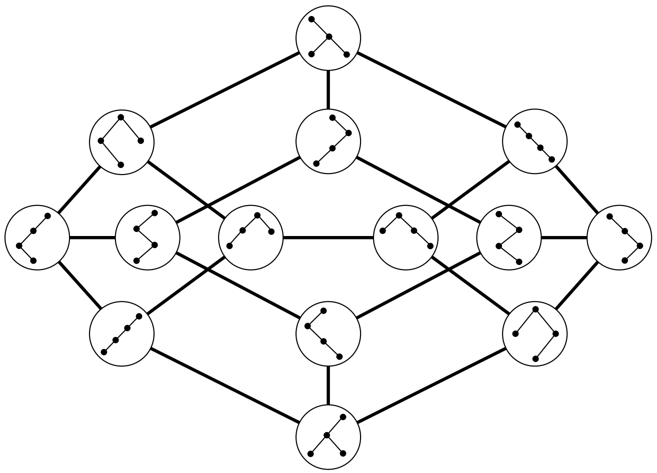

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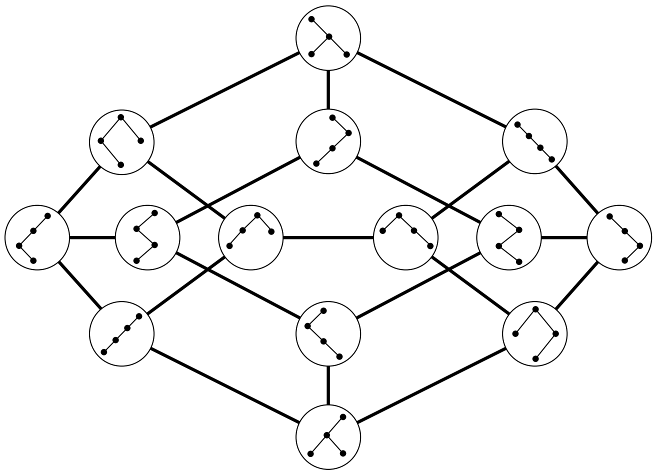

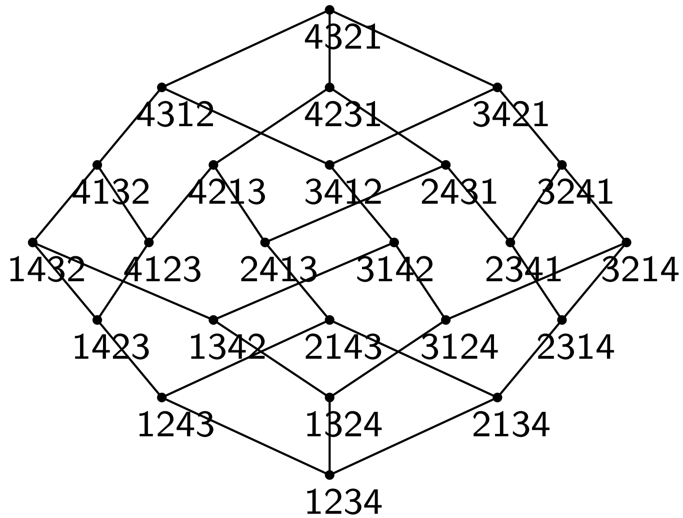

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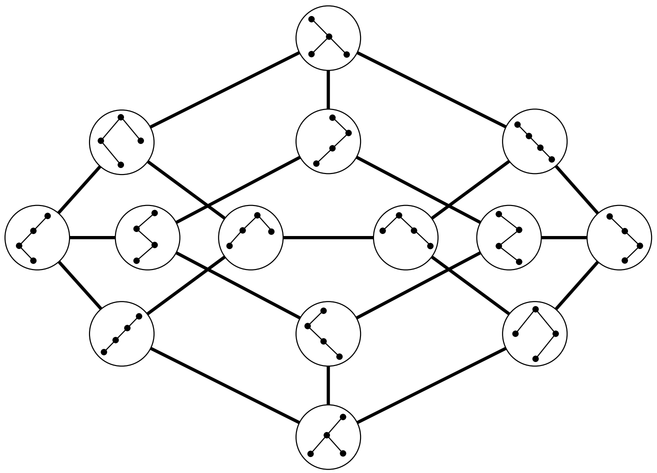

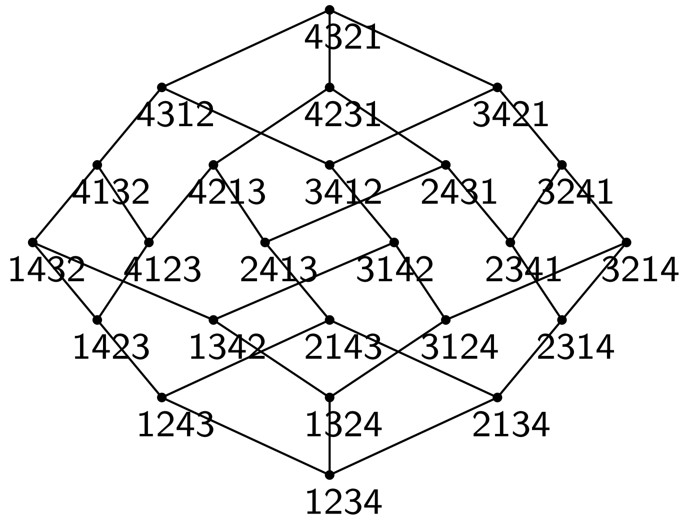

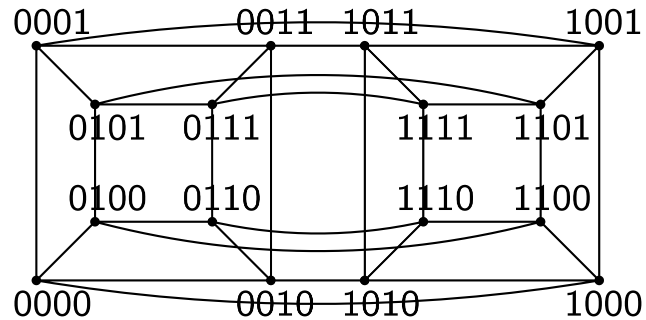

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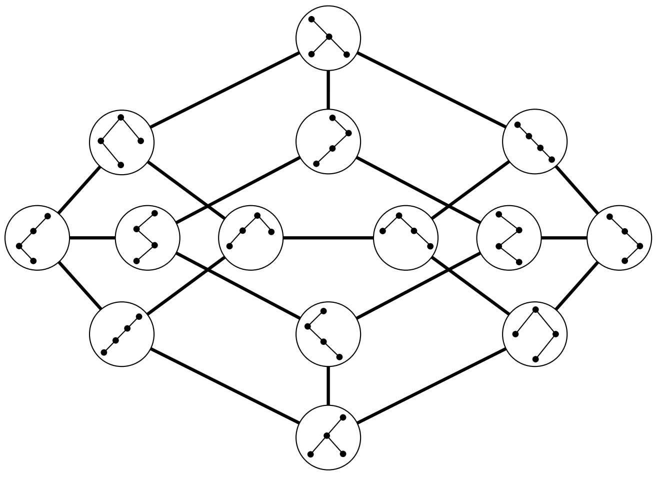

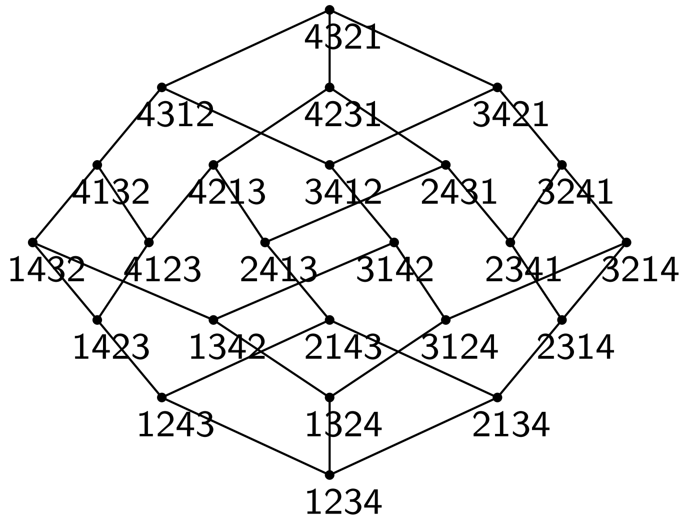

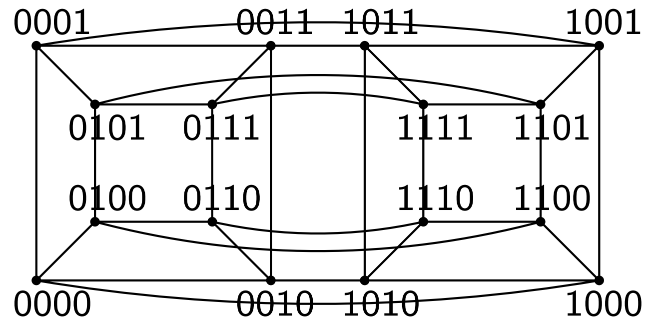

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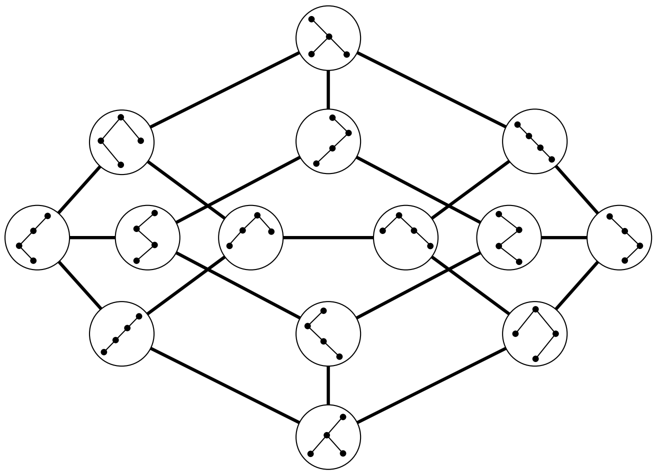
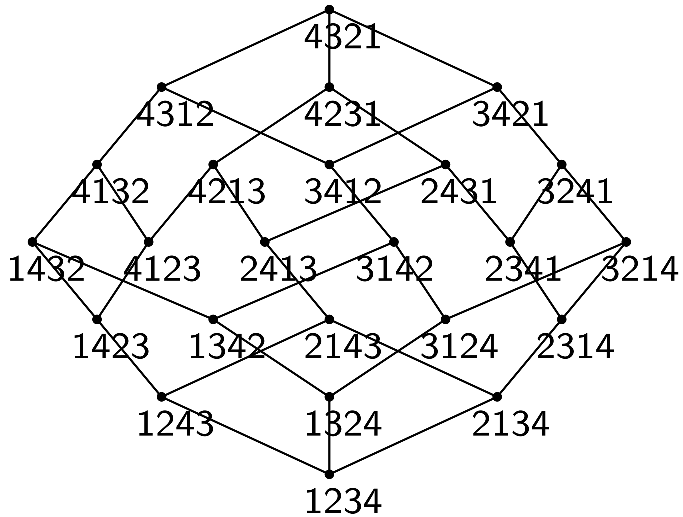
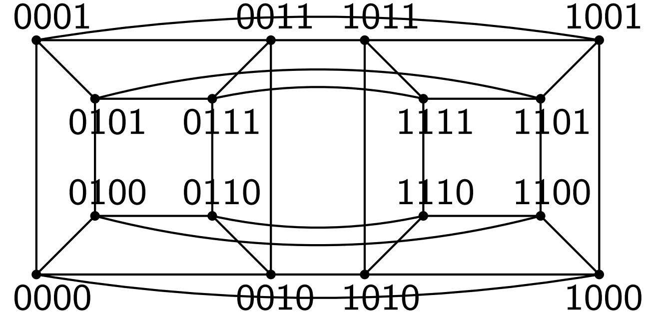


Tamari lattice / associahedron



weak order / permutahedron



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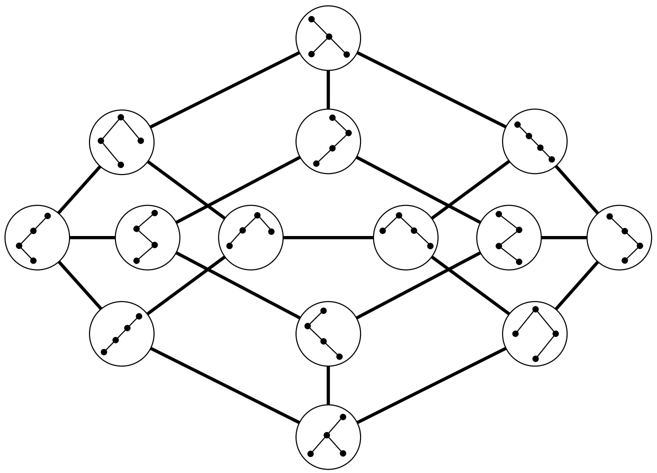

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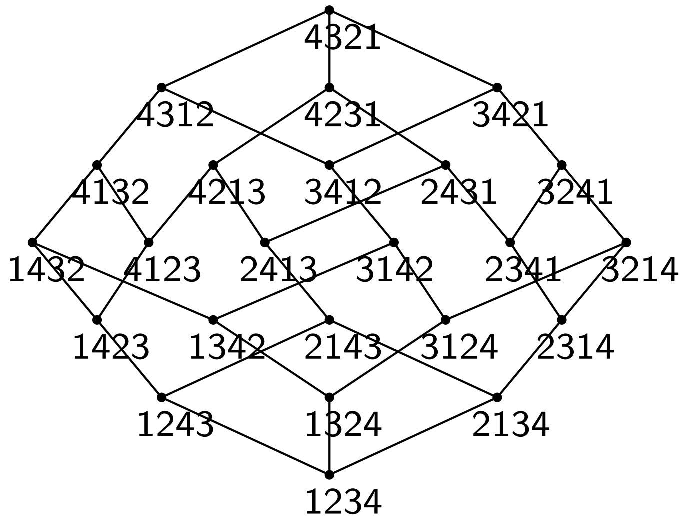

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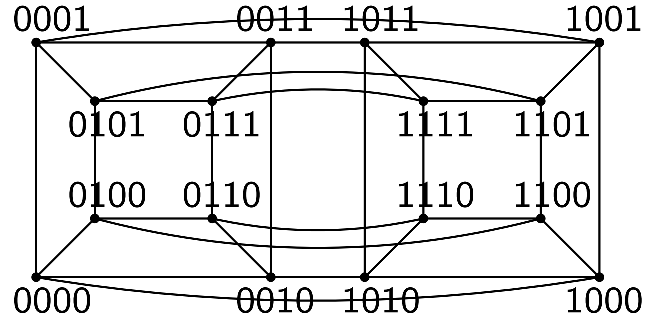

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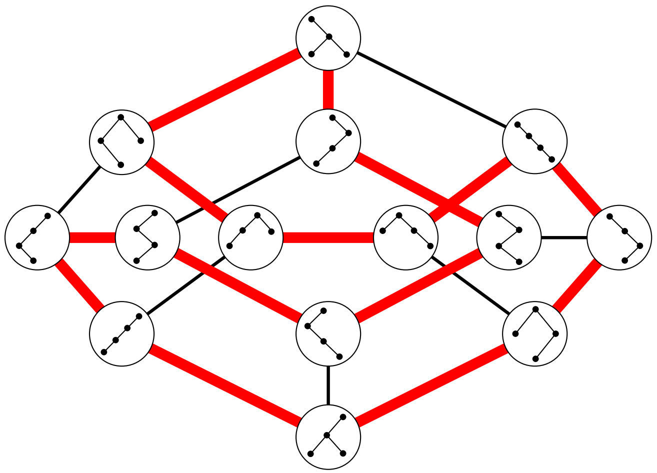

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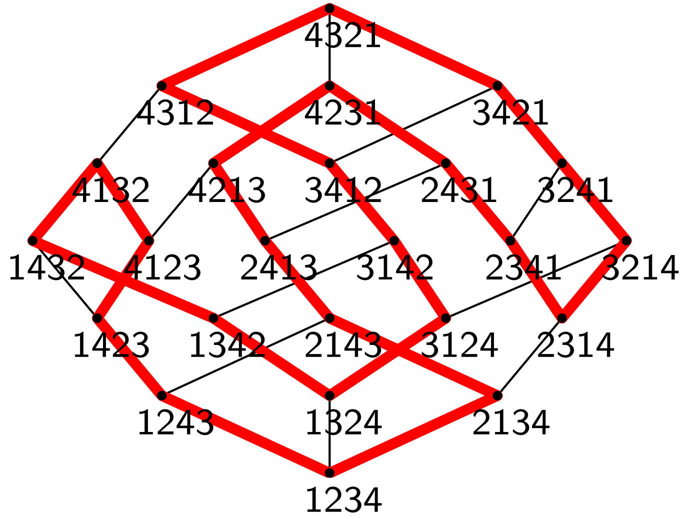

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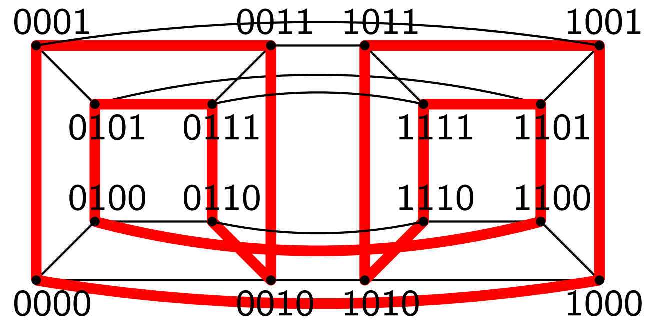

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- **Idea:** Encode objects as a set $F_n \subseteq S_n$ of permutations of length n

Jumps

- **Jump:=** move an entry in the permutation across some neighboring smaller entries (left or right)

$$4 \ 5 \ 1 \ 3 \ 2 \ 6$$

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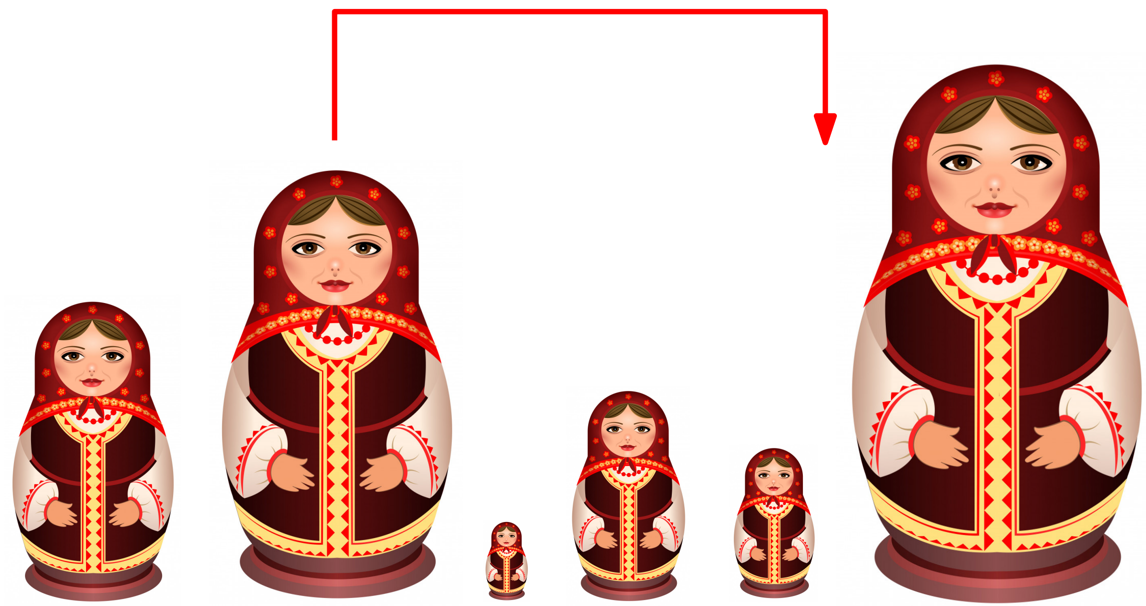

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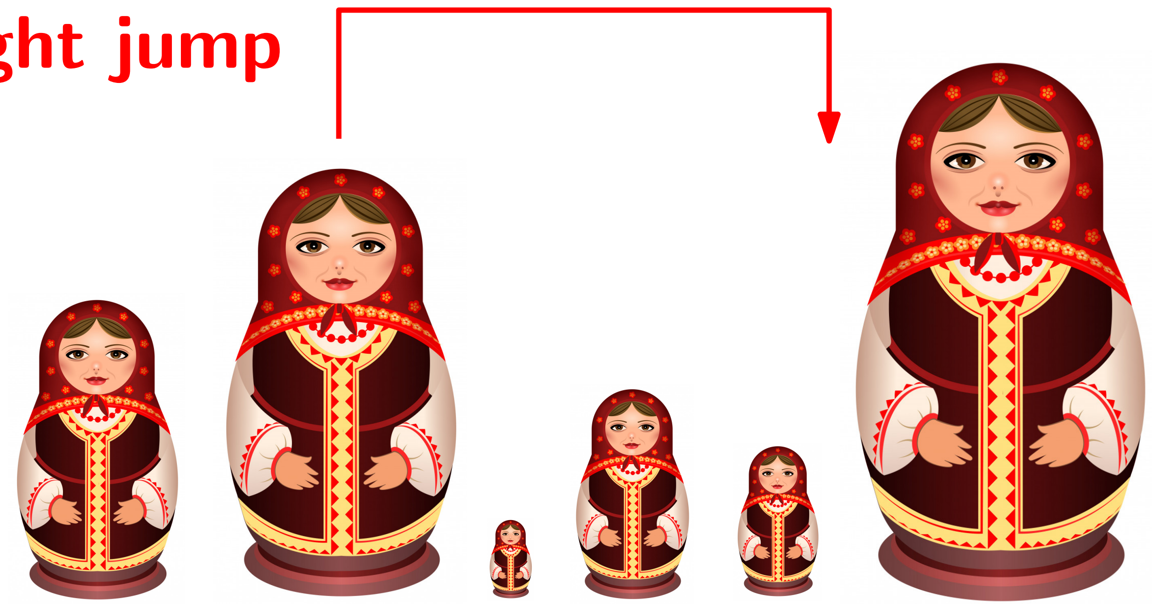
$$\begin{matrix} 4 & 5 & 1 & 3 & 2 & 6 \\ 4 & 1 & 3 & 2 & 5 & 6 \end{matrix}$$

right jump



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$$\begin{array}{cccccc} 4 & \color{red}{5} & \color{blue}{1} & \color{blue}{3} & \color{blue}{2} & 6 \\ 4 & \color{blue}{1} & \color{blue}{3} & \color{blue}{2} & \color{red}{5} & 6 \end{array}$$

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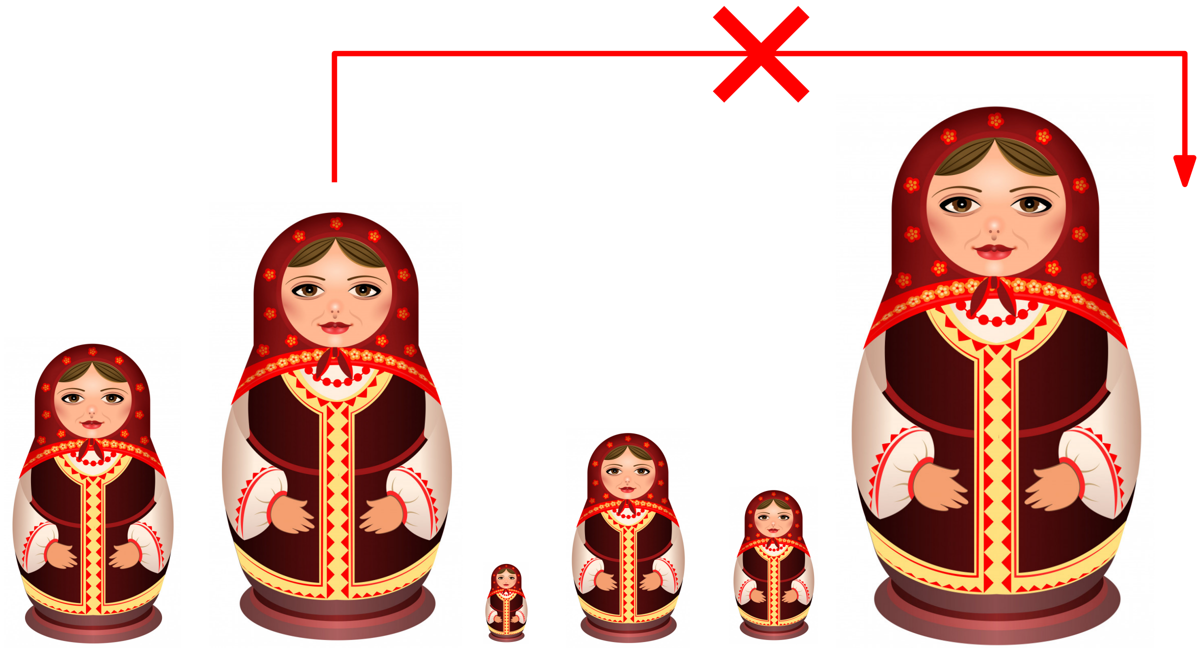

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attempts to generate a set of permutations $F_n \subseteq S_n$

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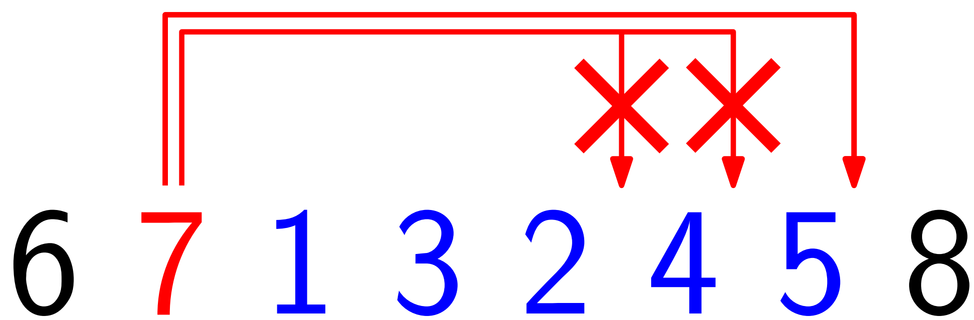

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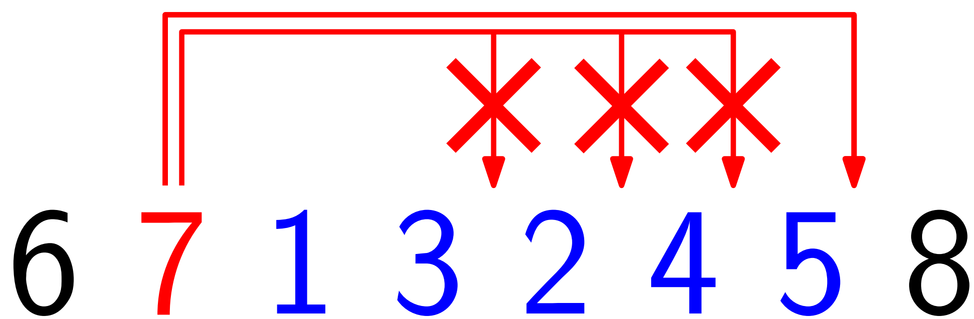

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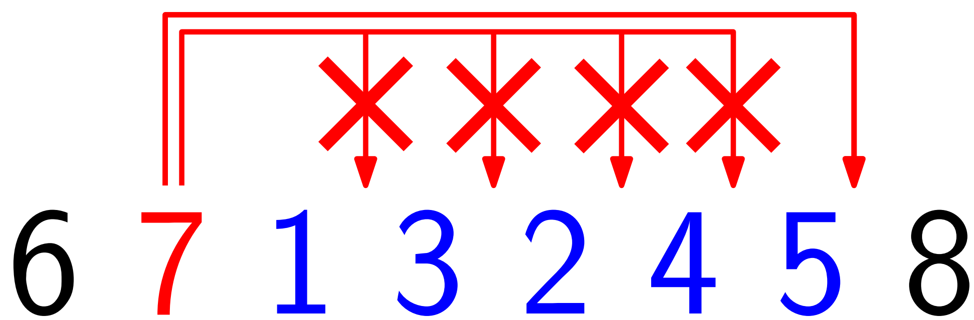

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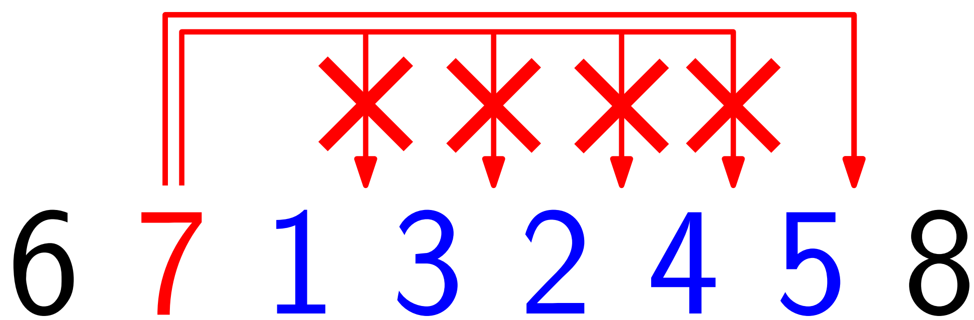

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1243	**4213**
1423	**2134** ✗
4123	no jump possible
4213	
2134 ✓	

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2134 ✘
no jump possible

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direction ambiguous

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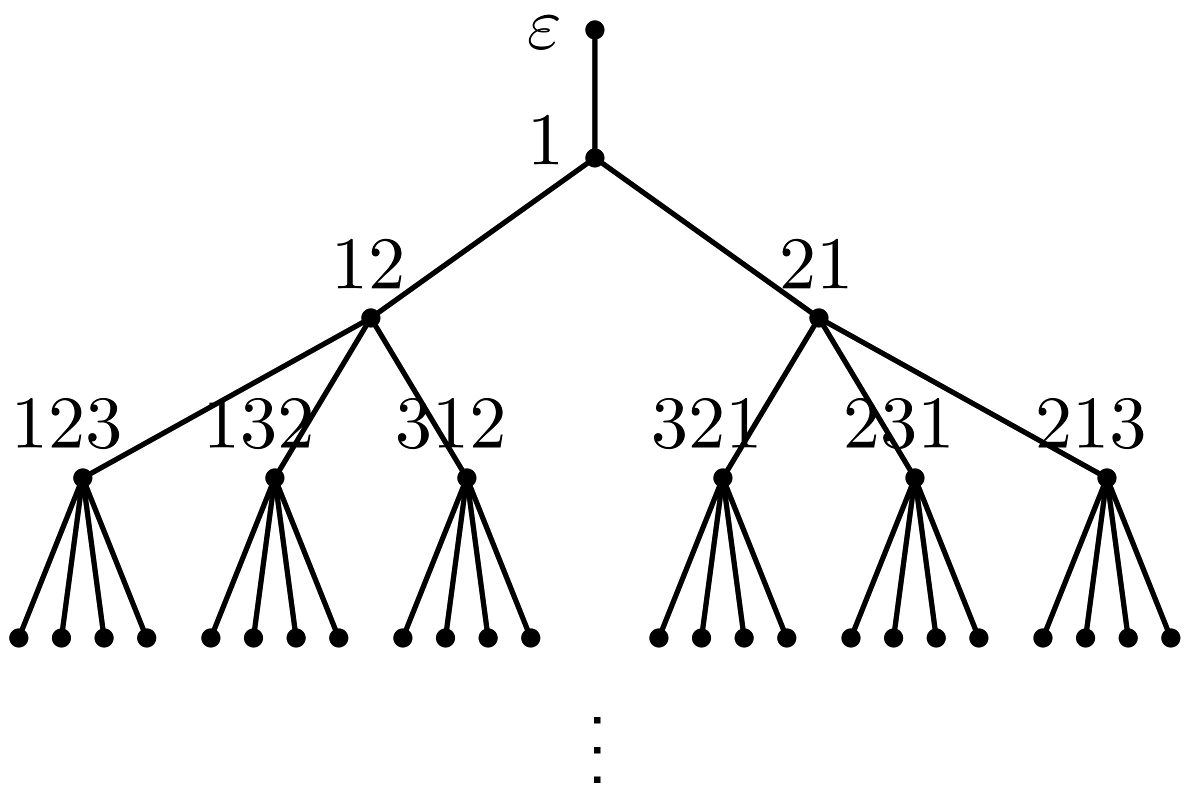
- **Question:** When does Algorithm J generate F_n?

Tree of permutations

- root := empty permutation ε
- given a permutation length $n-1$, its children are obtained by inserting n in every possible position



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all permutations
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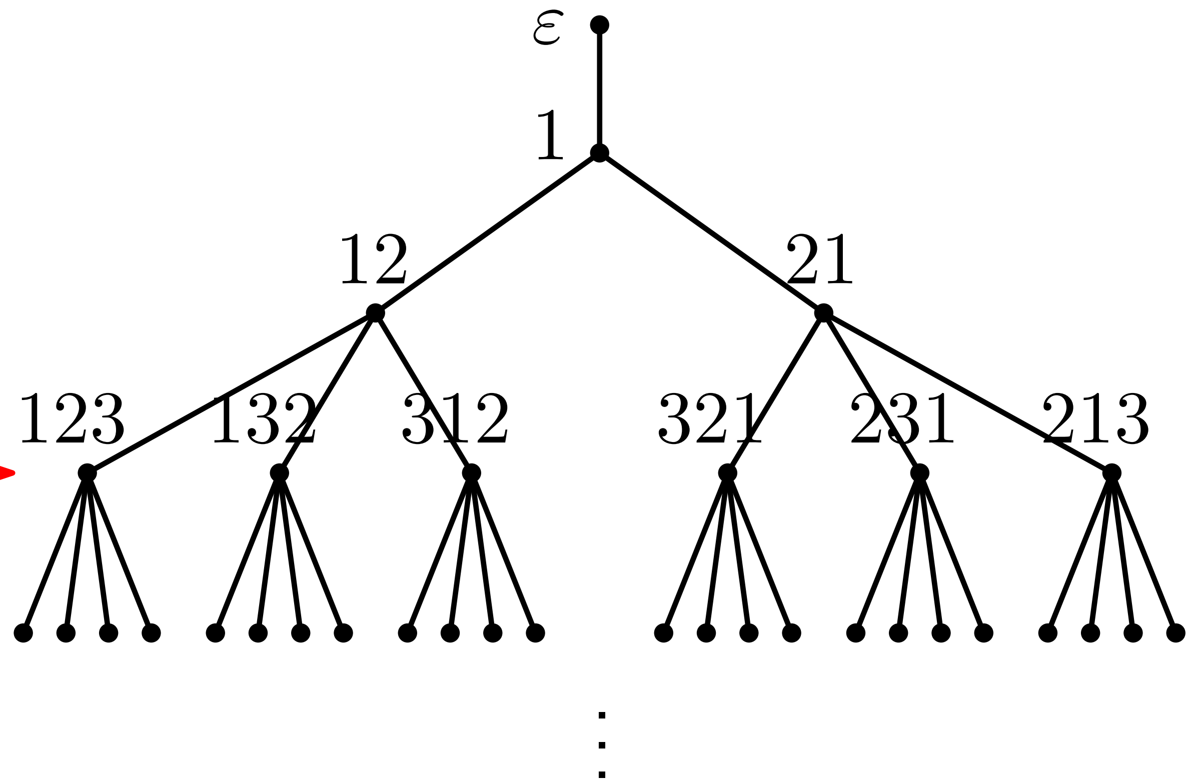

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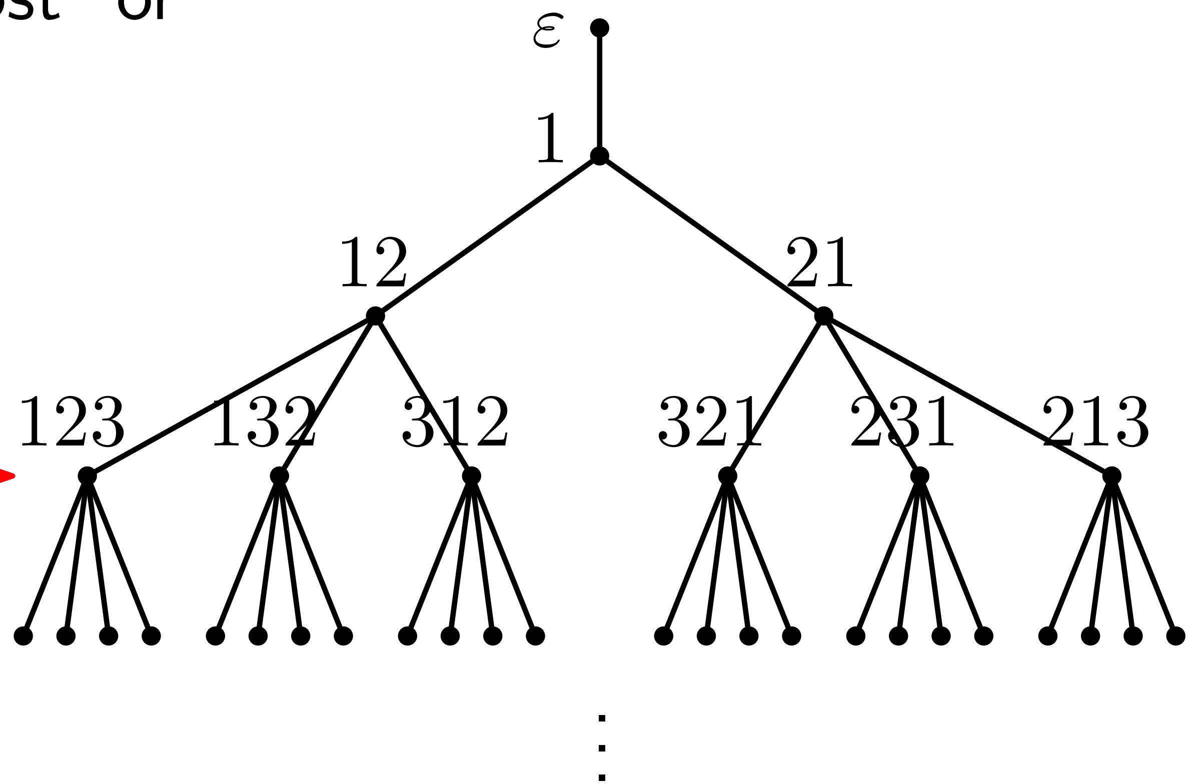

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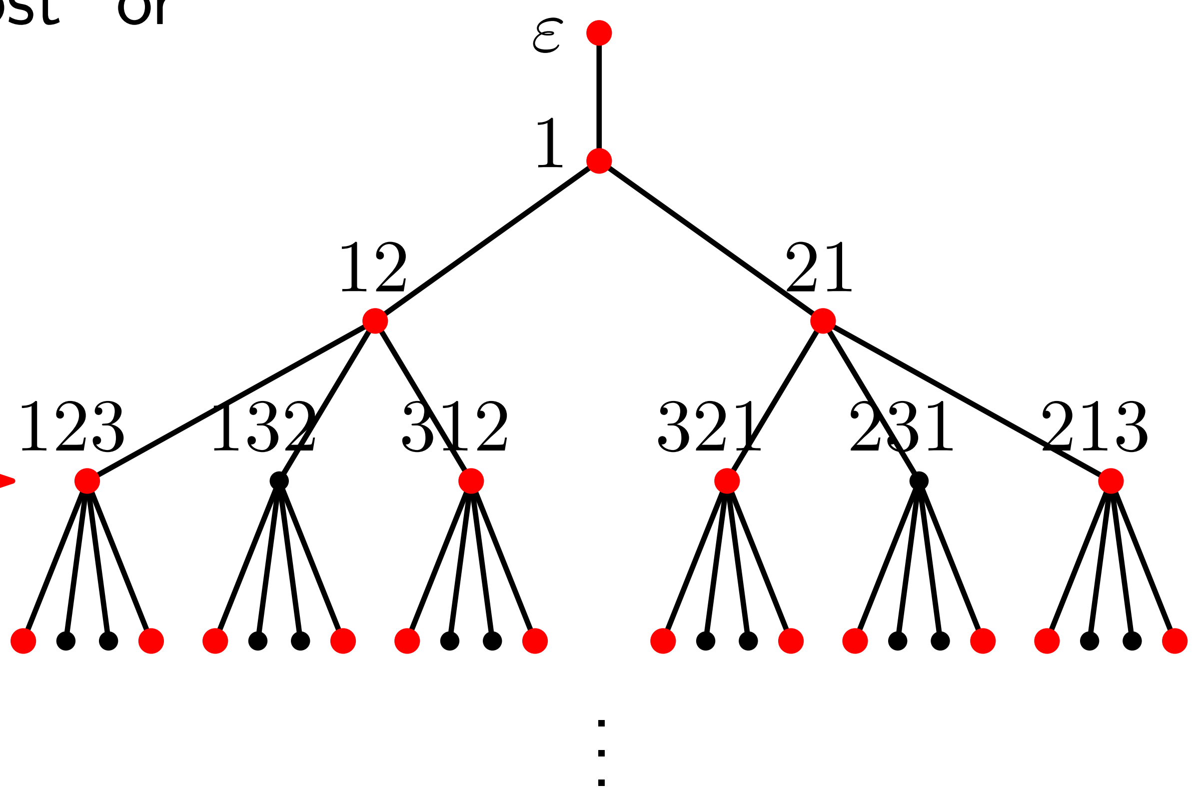

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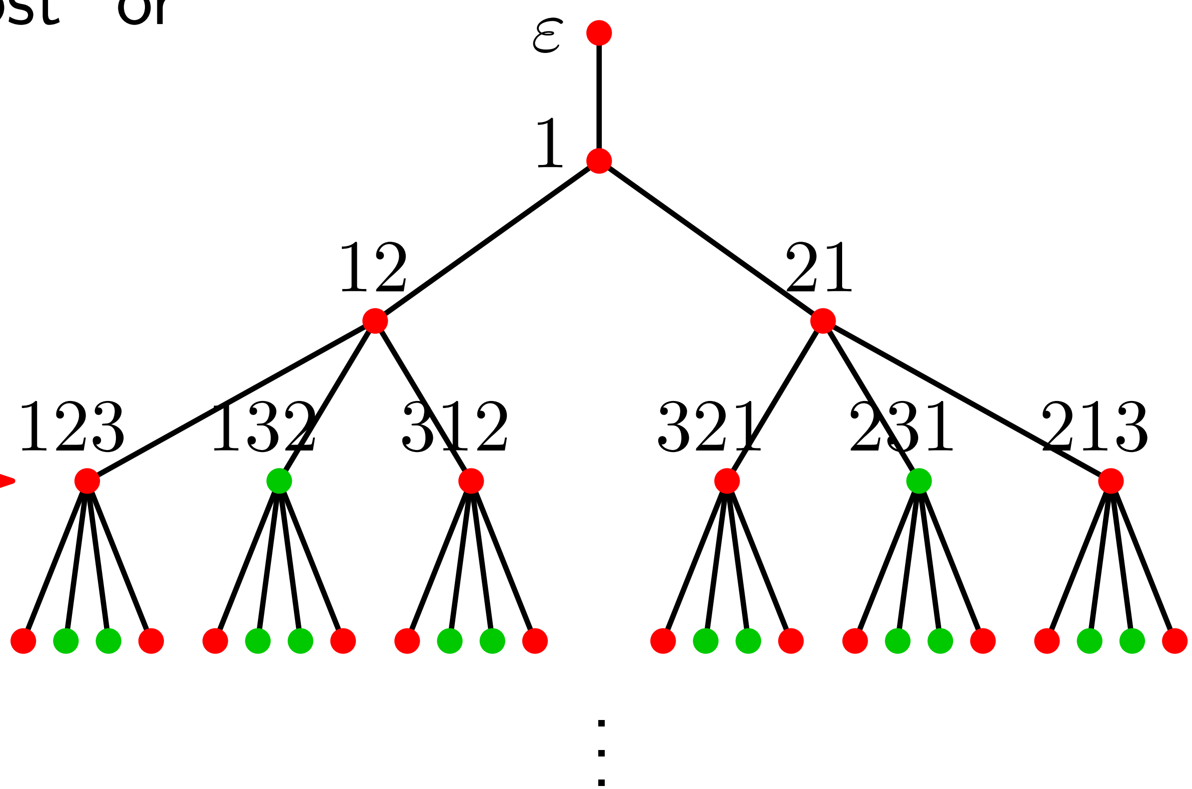

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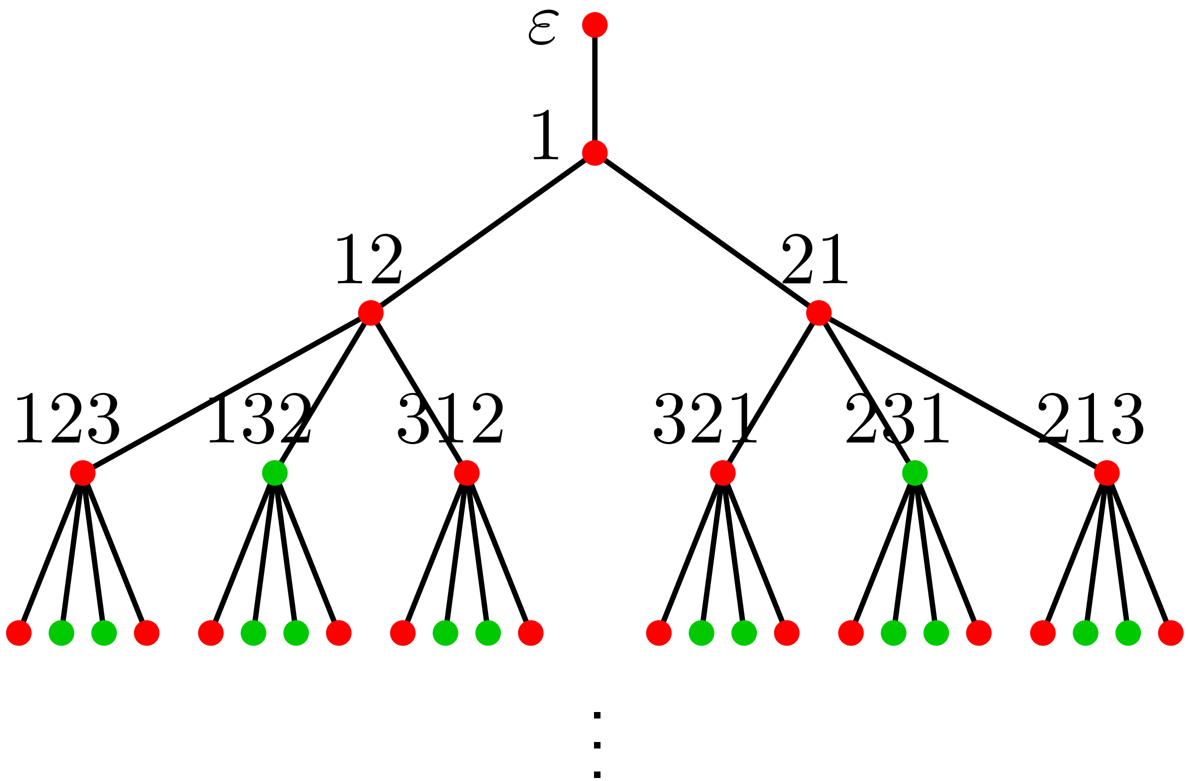
- we may prune subtrees iff their root is ●



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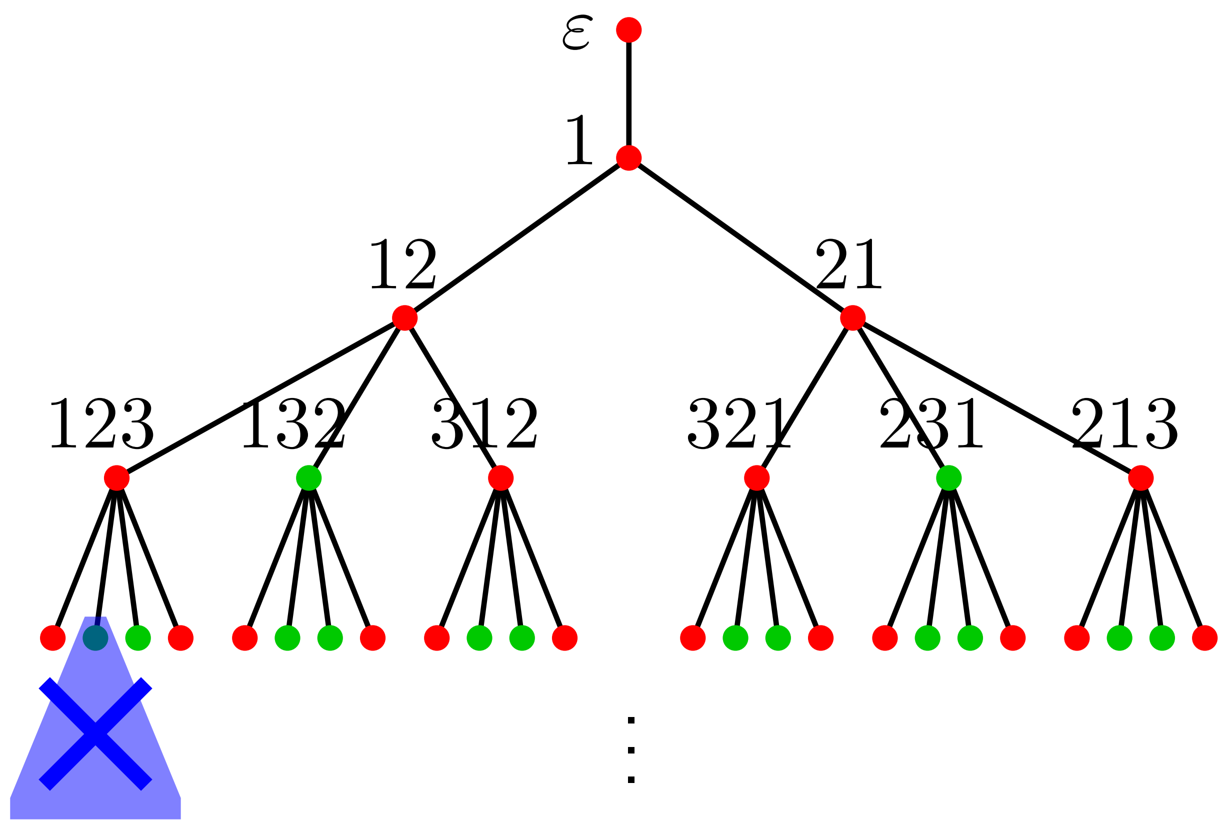

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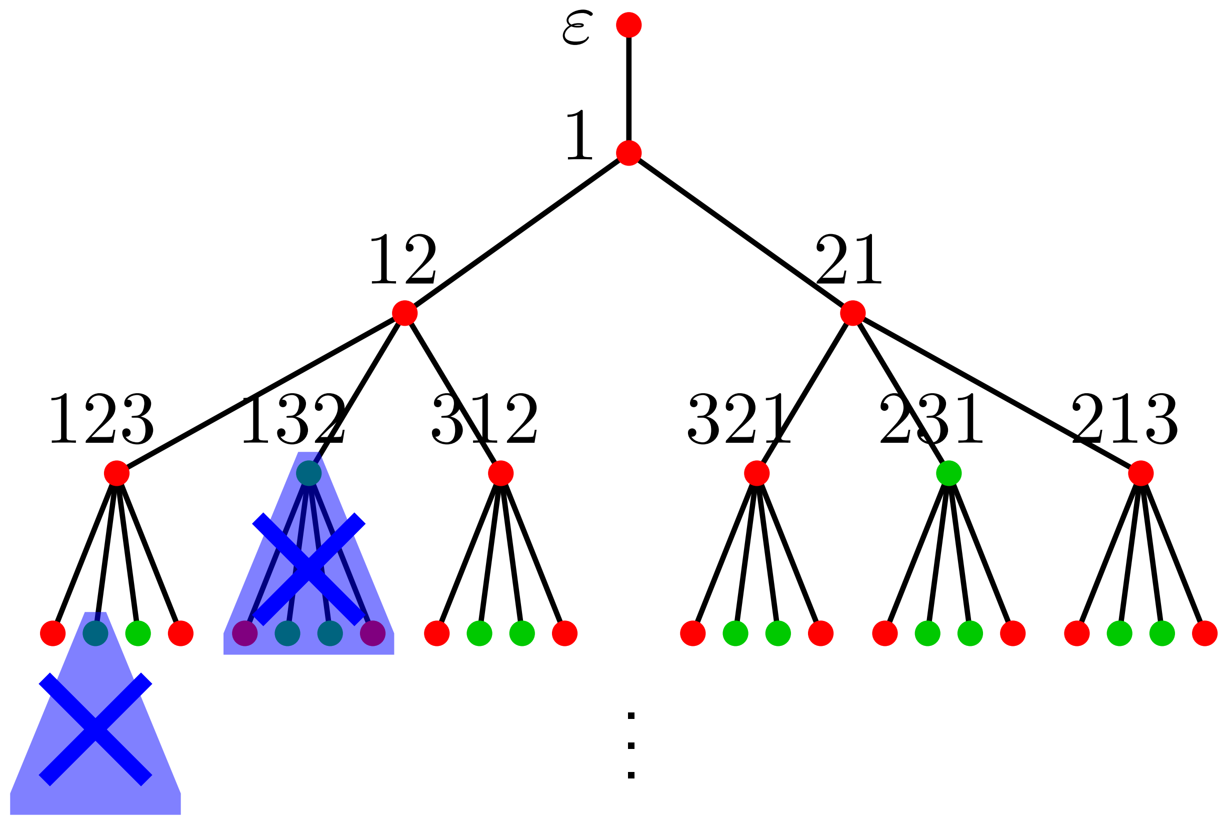

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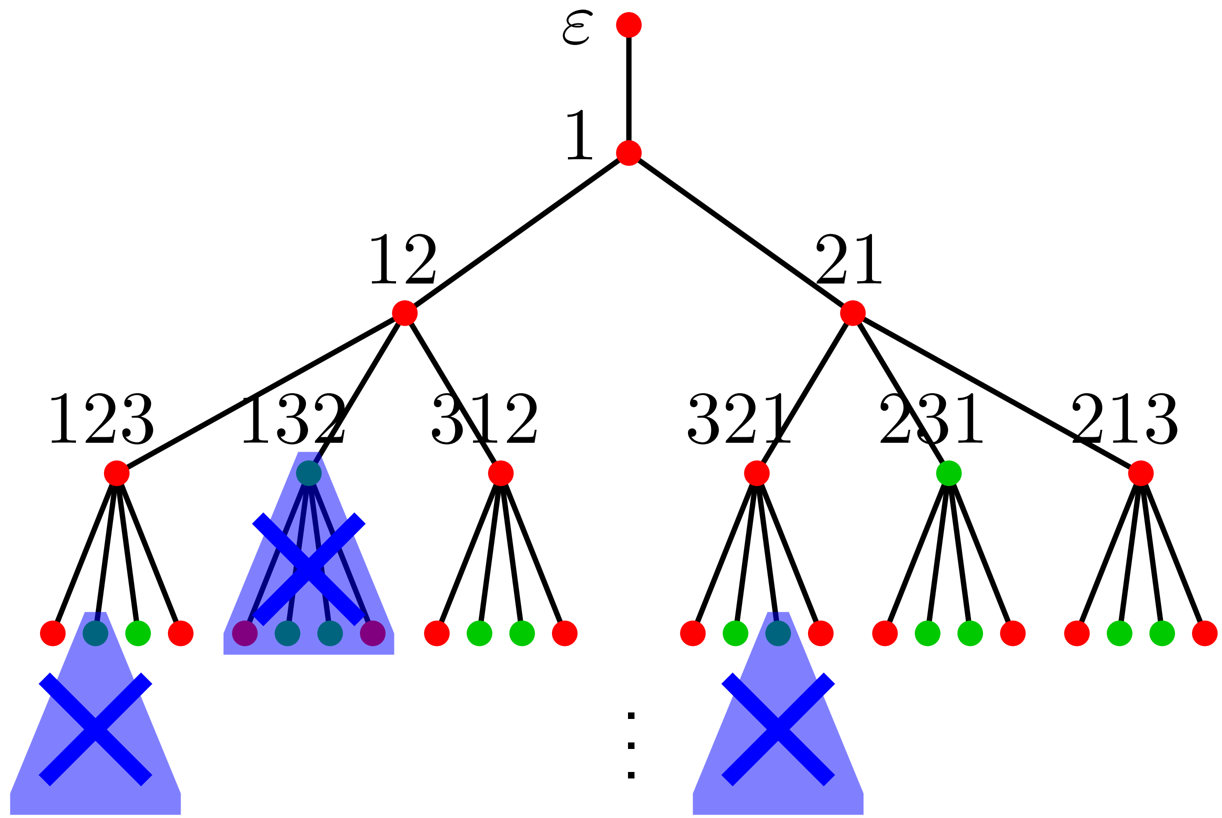

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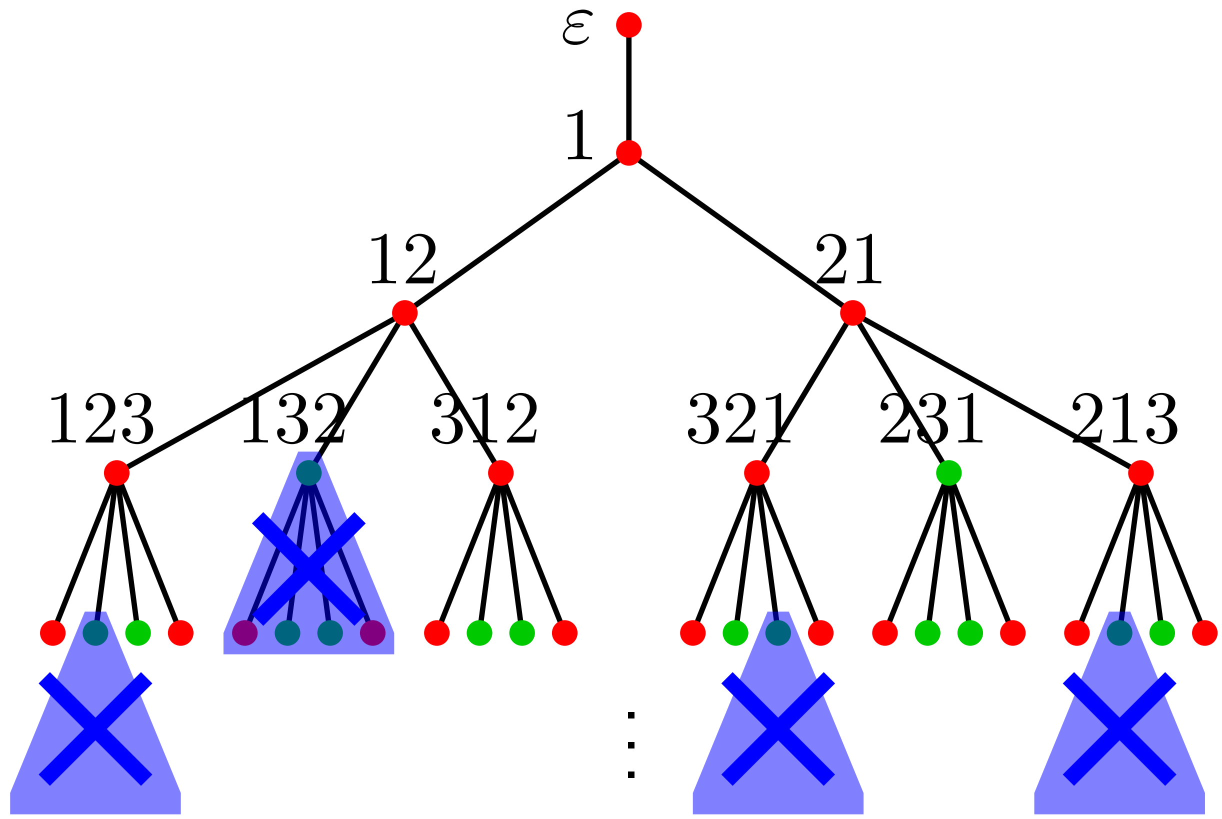

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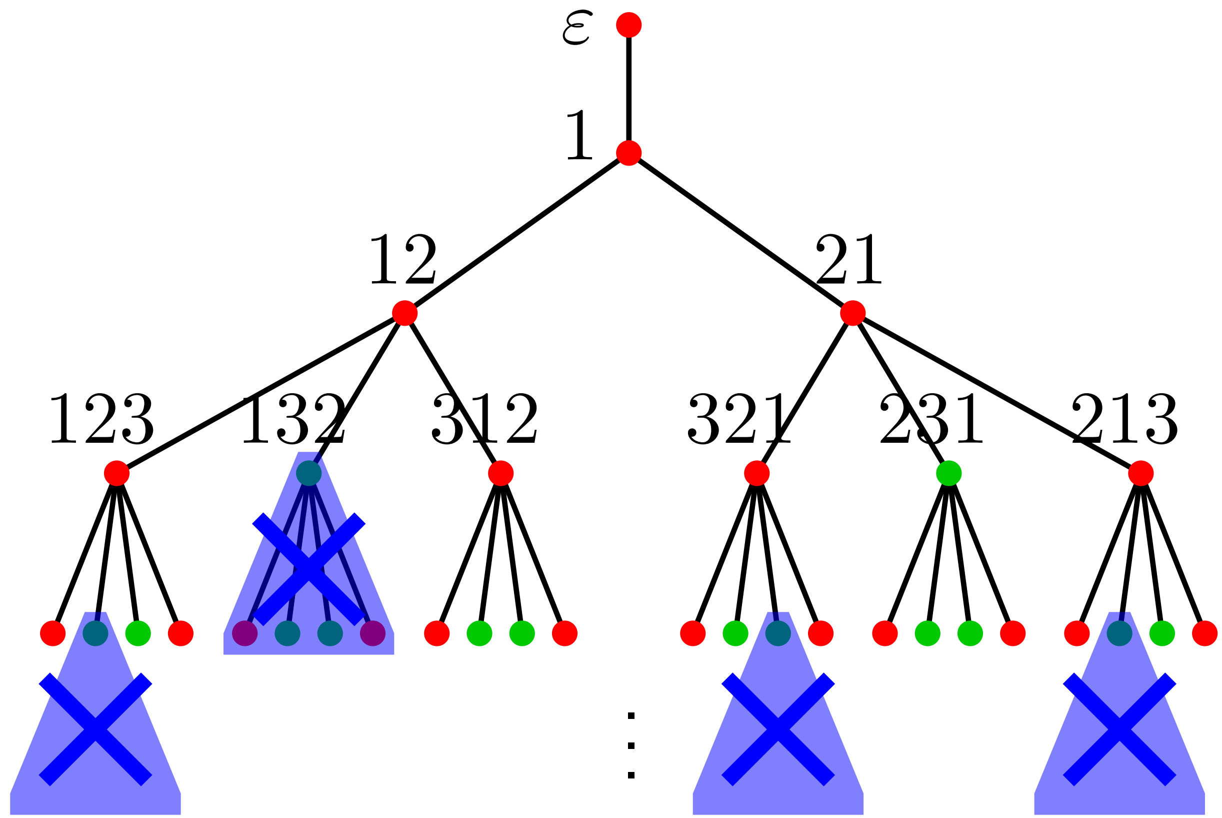
- given any such pruned tree, a set of permutation $F_n \subseteq S_n$ in depth n is called **zigzag language**



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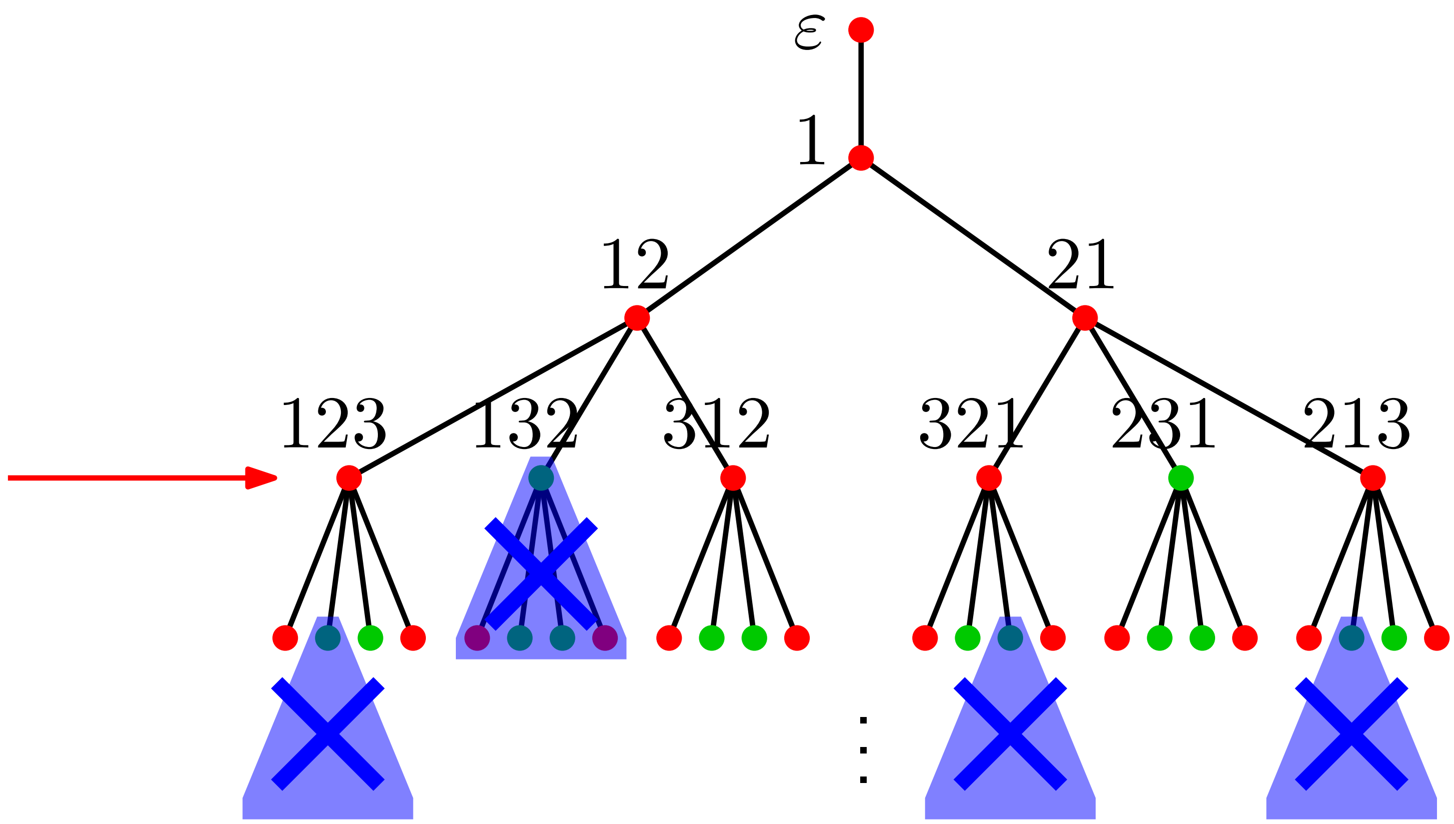

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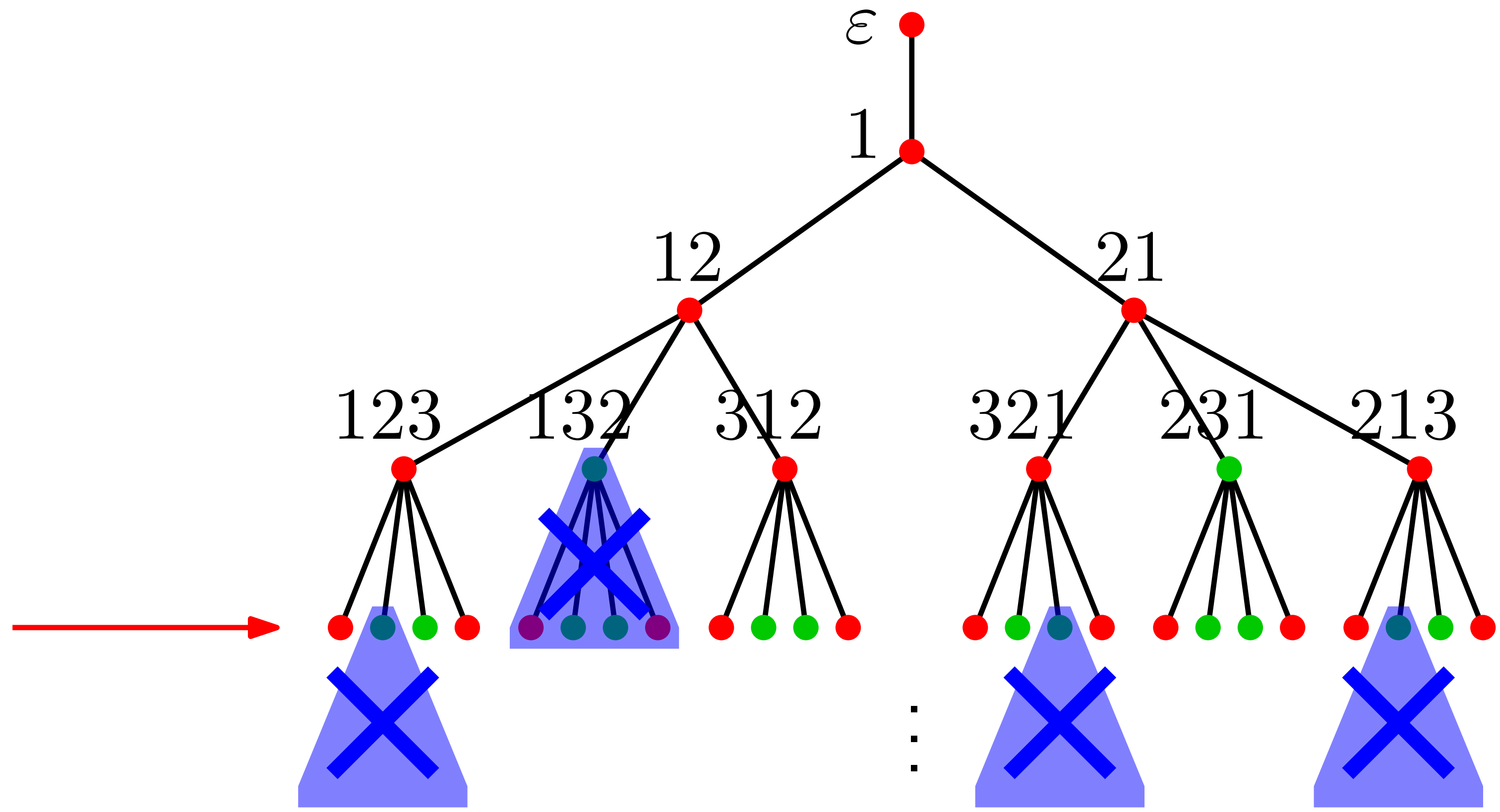

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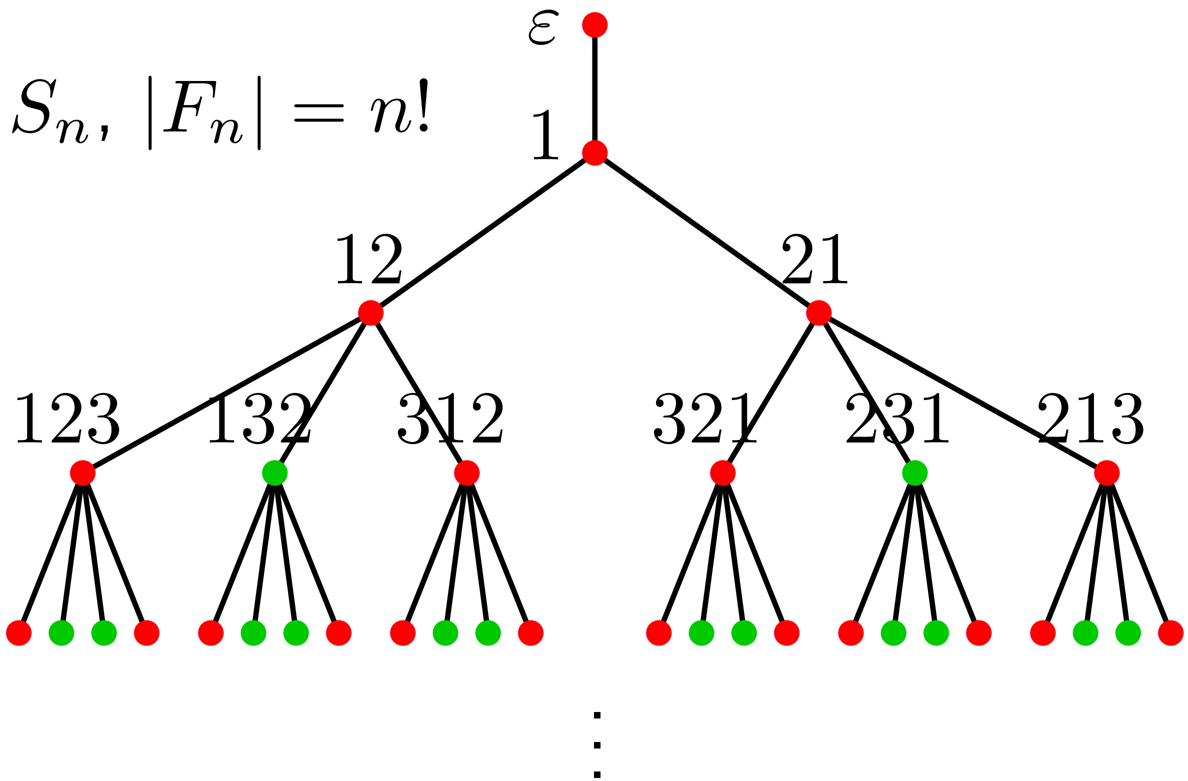

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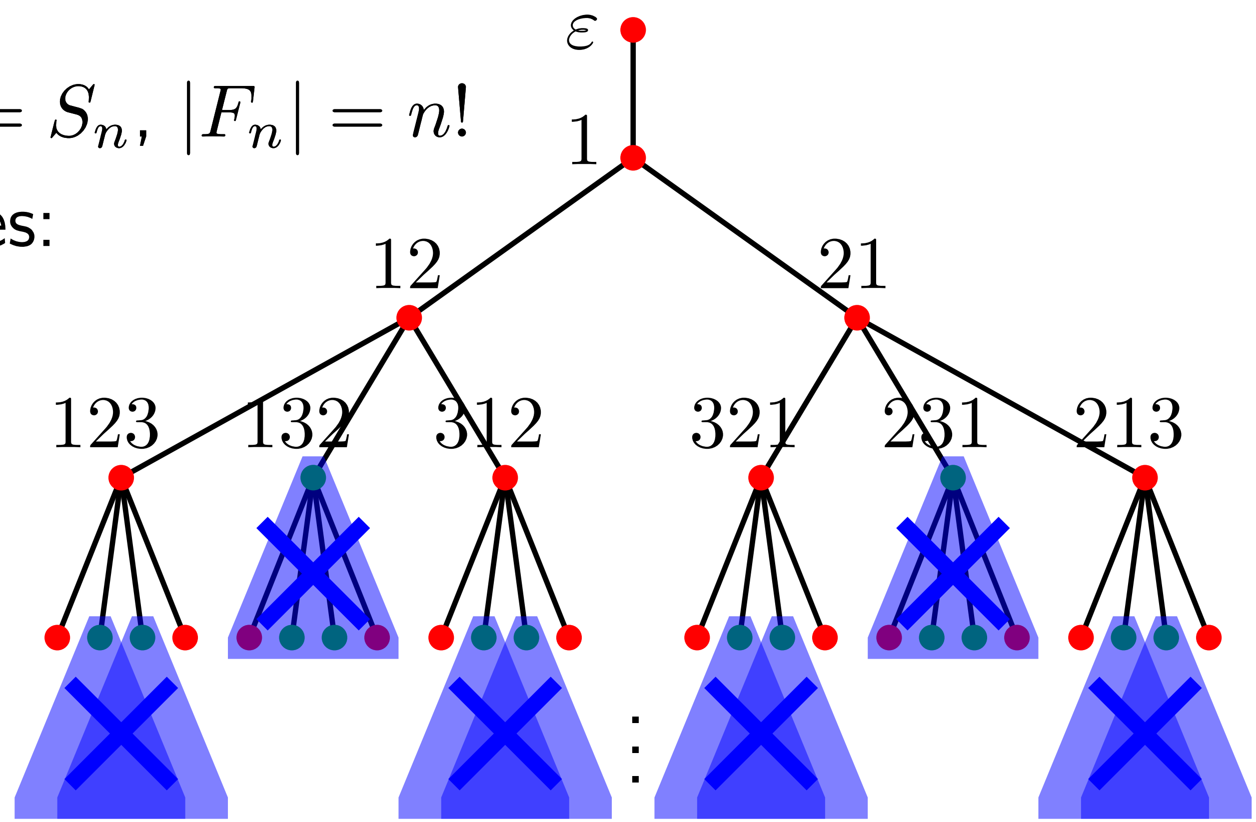

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- **many** of them encode interesting combinatorial objects

Examples

$F_n = S_n$
$|F_n| = n!$

$F_n =$ permutations without peaks
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Examples

$F_n = S_n$
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1243
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4132
1432
1342
1324
3124
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Steinhaus-Johnson-Trotter!
minimal jumps
$\leftrightarrow$ adjacent transpositions
$\hookrightarrow$ HC on permutahedron

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not a sublist

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1243
1423
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F_n = permutations without peaks
$|F_n| = 2^{n-1}$

<u>234</u>

1234
4123
4312
3124 ↔ f
3214
4321
4213
2134

$$x_i = \begin{cases} 0 & i \text{ right of smaller entries} \\ 1 & i \text{ left of smaller entries} \end{cases}$$

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1243
1423
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	234
1234	0
4123	1
4312	1
3124 $\overset{f}{\longleftrightarrow}$	0
3214	0
4321	1
4213	1
2134	0

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Binary reflected Gray code!
minimal jumps $\longleftrightarrow$ bitflips
$\hookrightarrow$ HC on hypercube

General approach

Combinatorial objects

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Set of permutations $F_n \subseteq S_n$ $\xleftarrow{\;\;f\;\;}$ Combinatorial objects

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- run Algorithm J

  List $=$ **Algo J**(F_n)

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  $\textbf{List} = \textbf{Algo J}(F_n) \longrightarrow f^{-1}(\textbf{List})$

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  $\hookrightarrow$ walks on lattices / polytopes

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- in many cases **polynomial-time algorithms** for concrete objects, sometimes even **loopless**

Applications

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- **VI. pattern-avoiding binary trees**

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Pattern-avoiding permutations

- $S_n(\tau_1, \ldots, \tau_k) \subseteq S_n$:= set of permutations avoiding each of the patterns $\tau_1, \ldots, \tau_k$

Pattern-avoiding permutations

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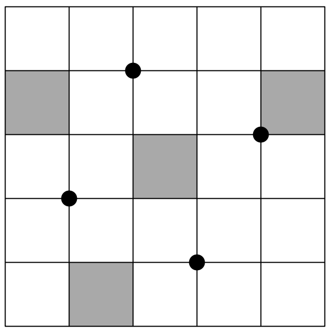 ✓

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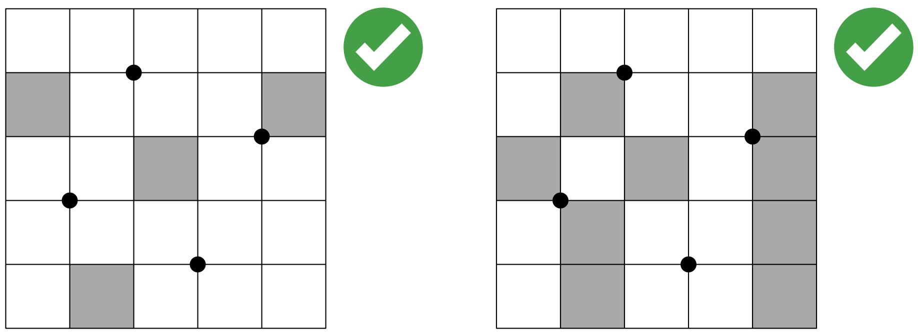

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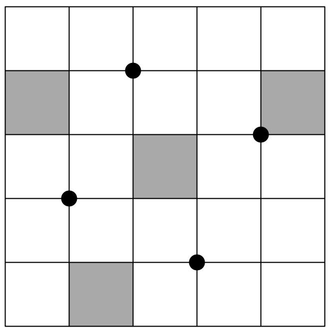
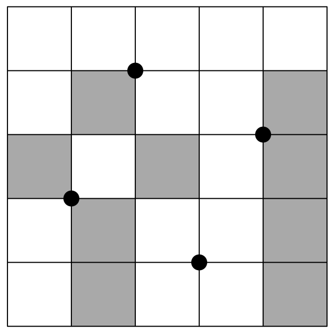
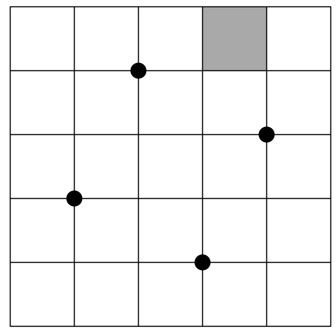
 ✓  ✓  ✗

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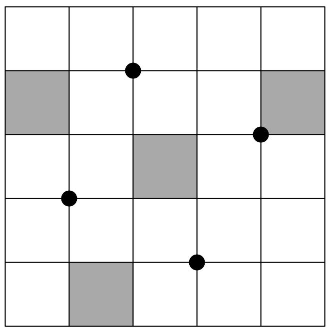

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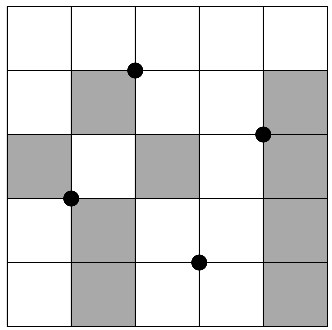

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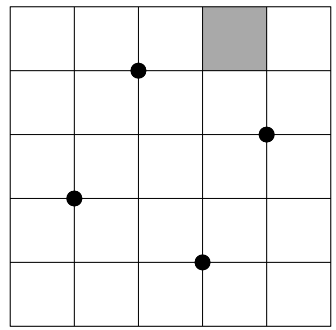

✗

Theorem: If $\tau_1, \ldots, \tau_k$ are tame patterns, then $S_n(\tau_1, \ldots, \tau_k)$ is a zigzag language.

Pattern-avoiding permutations

Tame patterns $\xleftrightarrow{f}$ Combinatorial objects

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Tame patterns $\xleftrightarrow{f}$	Combinatorial objects
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Tame patterns $\overset{f}{\longleftrightarrow}$ Combinatorial objects		
231	Catalan families	• binary trees by **rotations** • triangulations by **flips** • Dyck paths by **hill flips**

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$\rightarrow$ see the Combinatorial Object Server: `www.combos.org/jump`

Grid classes

- monotone grid class $\mathrm{Grid}_n(M)$ **[Huczynska, Vatter 06]**
- geometric grid class $\mathrm{Geo}_n(M)$ **[Albert et al. 13]**

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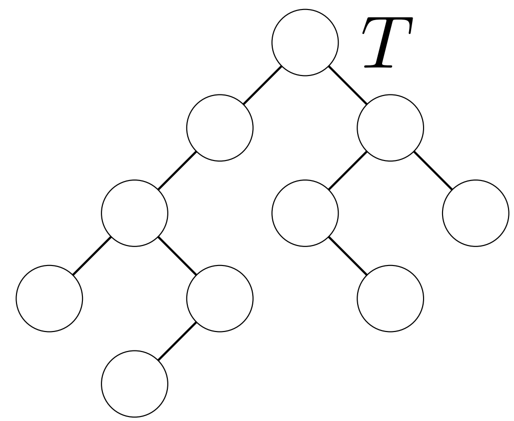
Theorem: If $M = \begin{array}{|c|c|c|c|}\hline -1 & & & +1 \\ \hline & & & \\ \hline & & & \\ \hline \end{array}$, then both $\mathrm{Grid}_n(M)$ and $\mathrm{Geo}_n(M)$ are zigzag languages.

Binary trees



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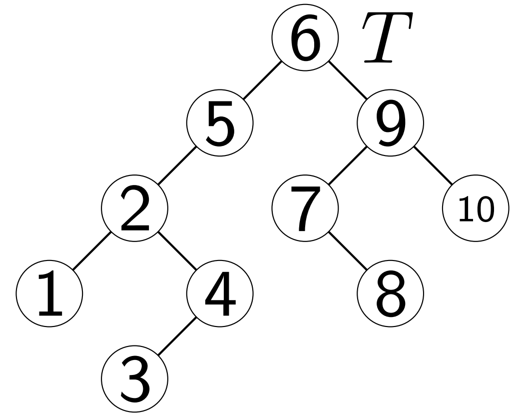

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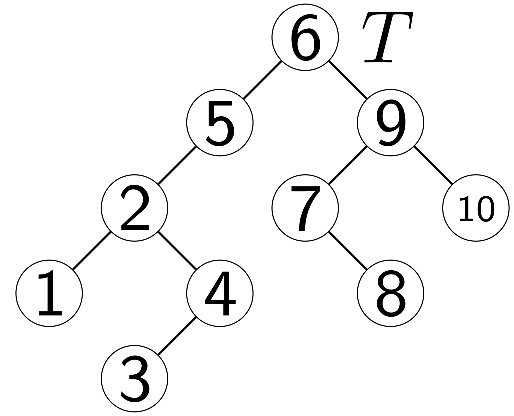

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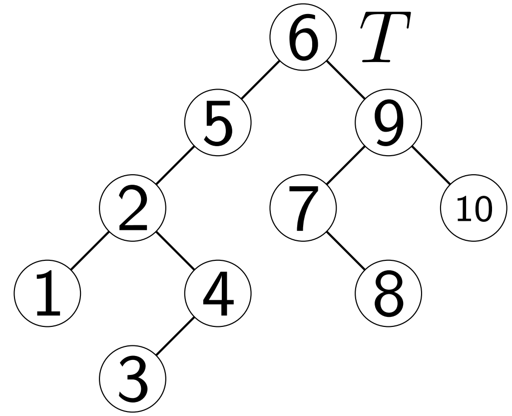

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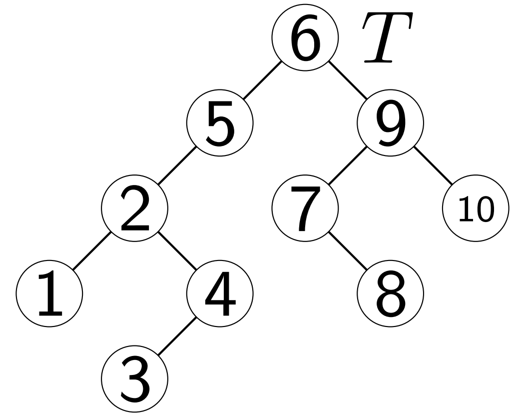

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'preorder traversal'

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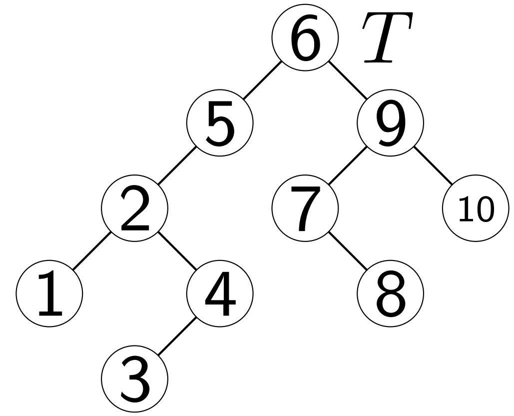

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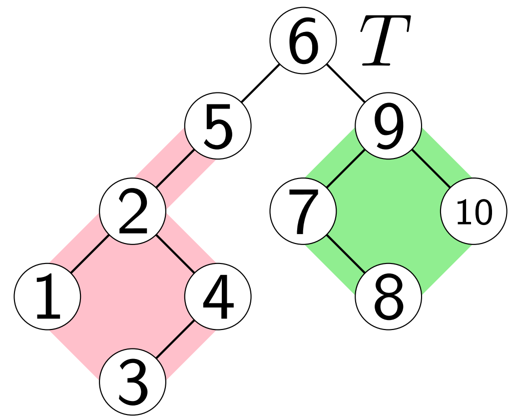

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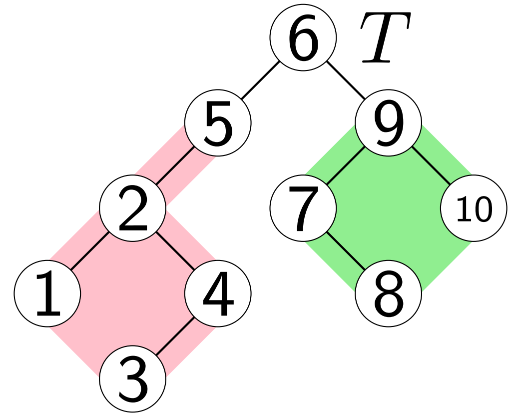

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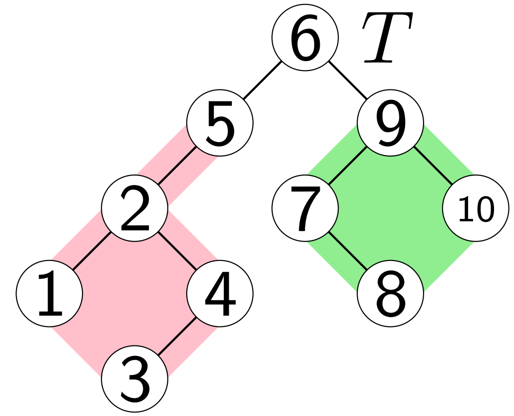

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Theorem: Under f^{-1}, minimal jumps of Algorithm J translate to tree rotations, i.e., we obtain a rotation Gray code for binary trees ($\hookrightarrow$ HP on associahedron).

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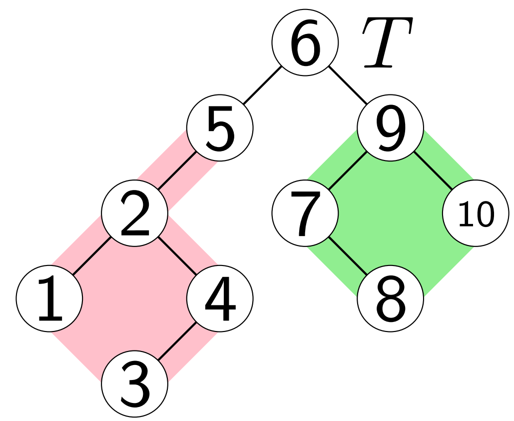

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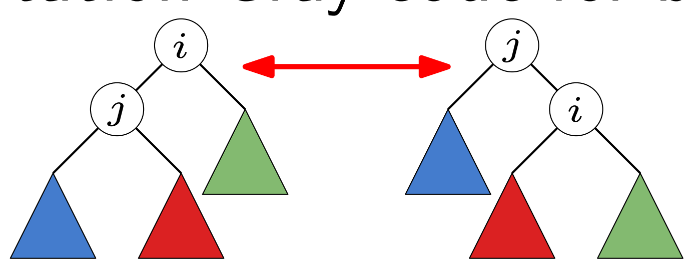
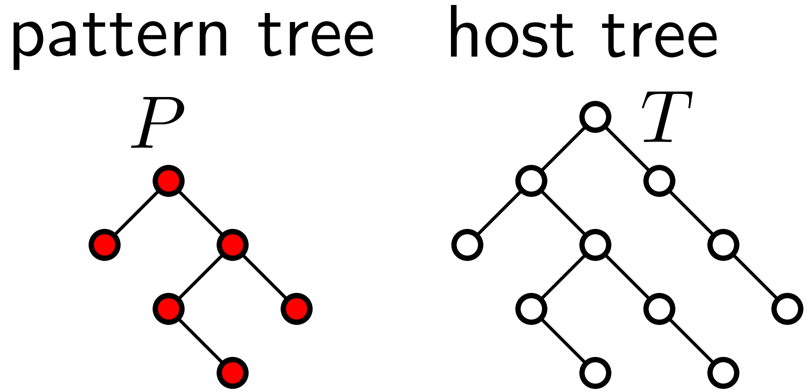


Patterns in binary trees



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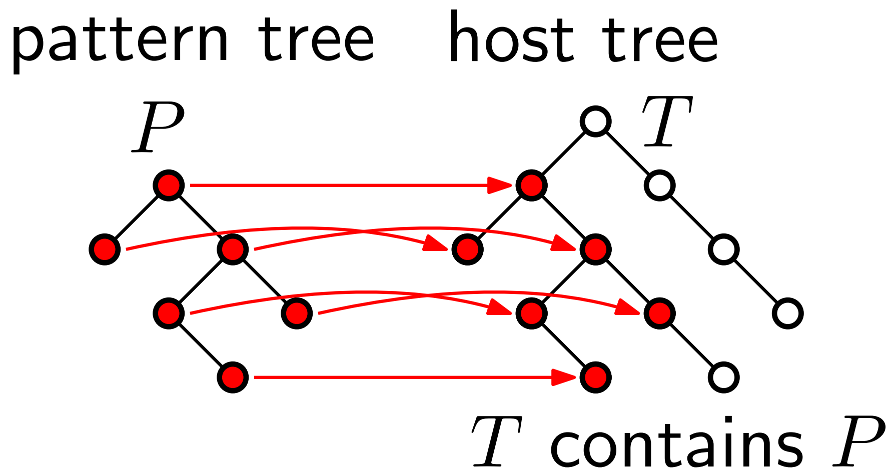

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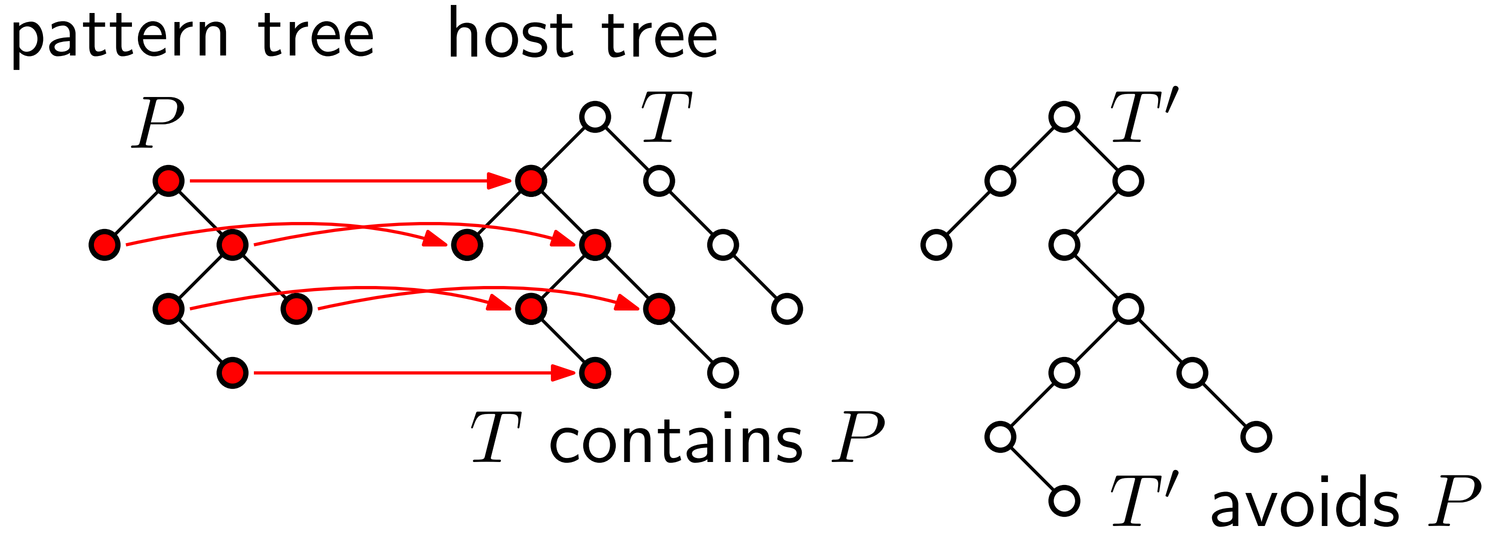
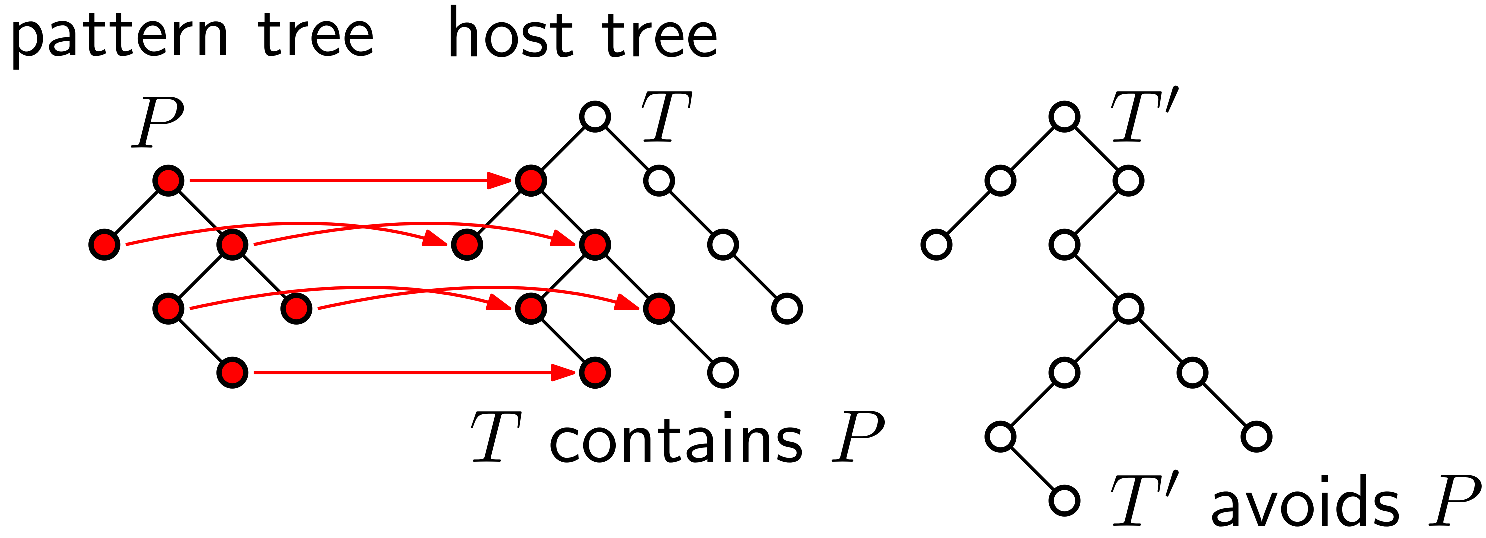


Patterns in binary trees

contiguous
[Rowland 10]



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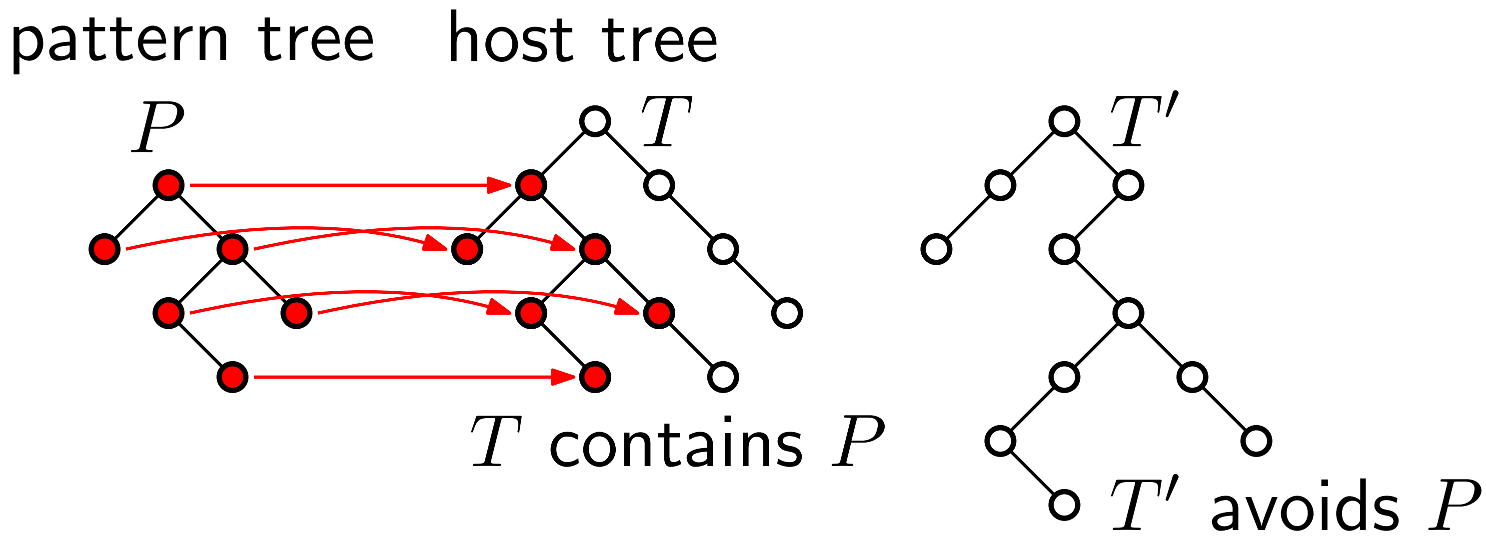

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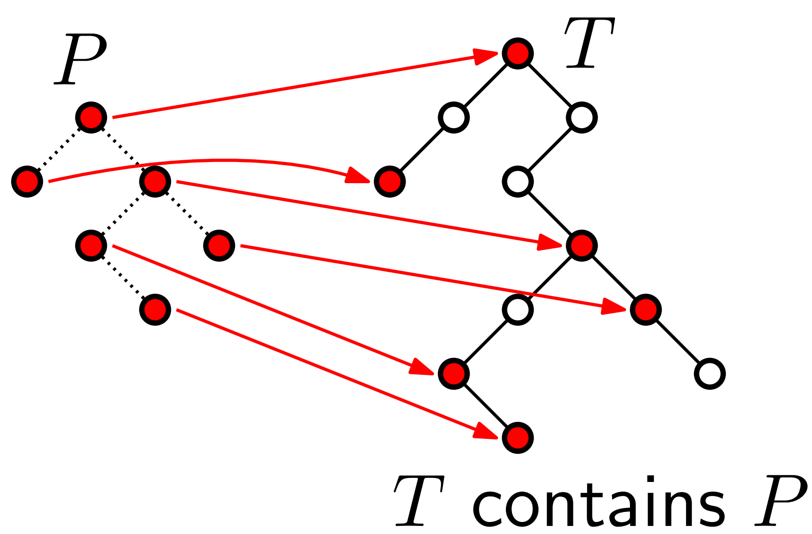
non-contiguous

[Dairyko, Tyner, Pudwell, Wynn 12]



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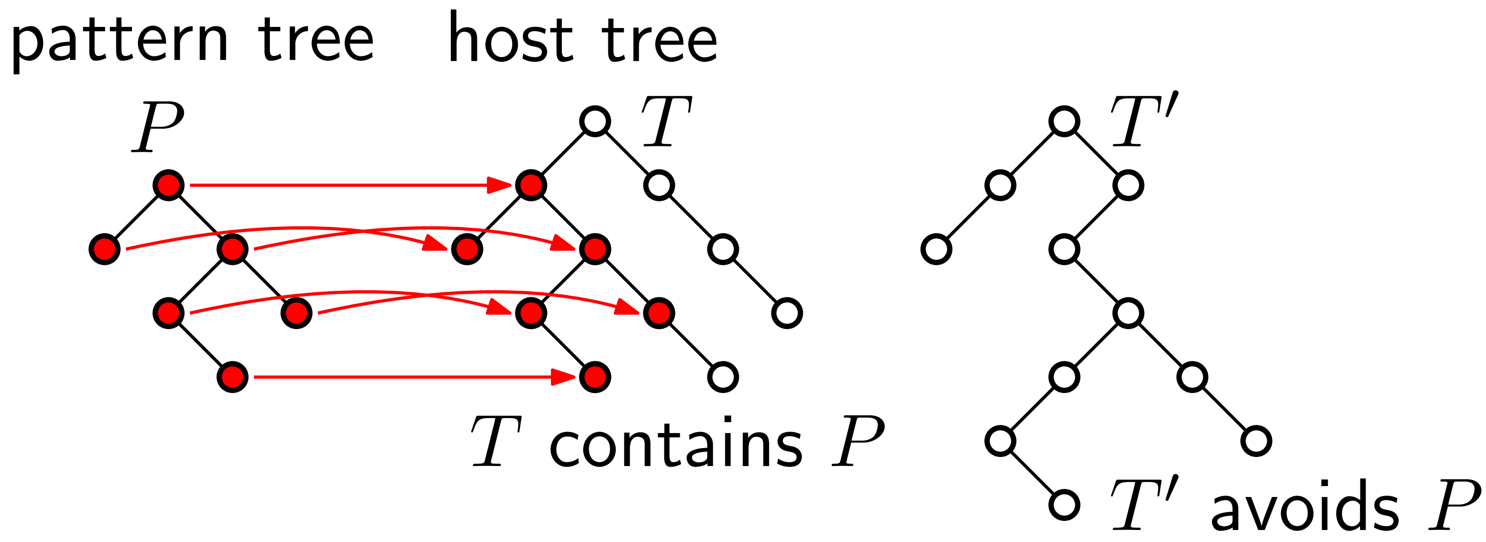

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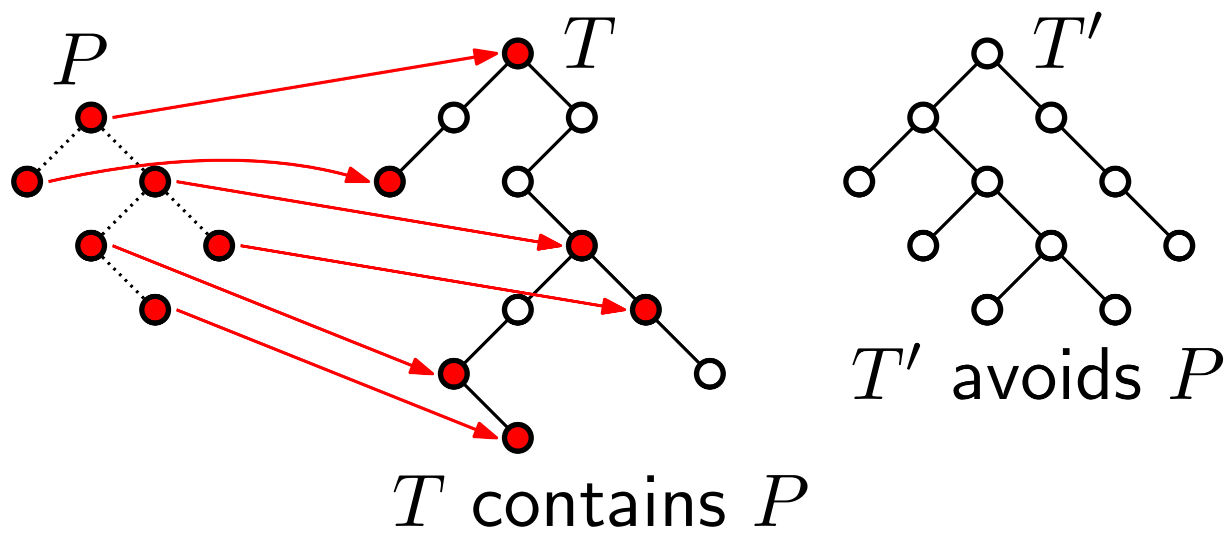
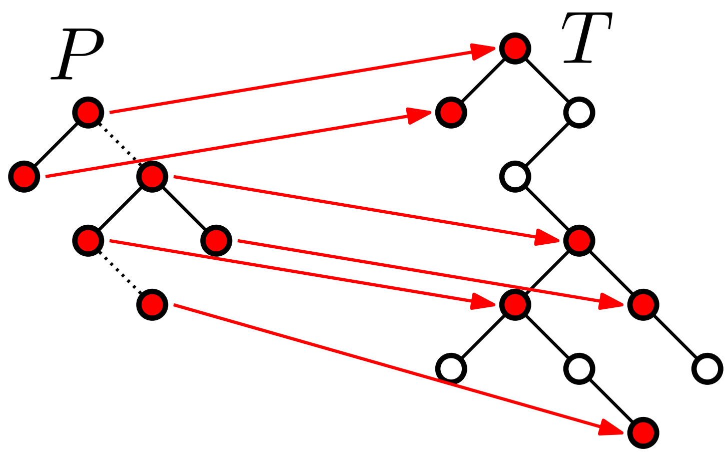


Mixed tree patterns

mixed (new)



T contains P

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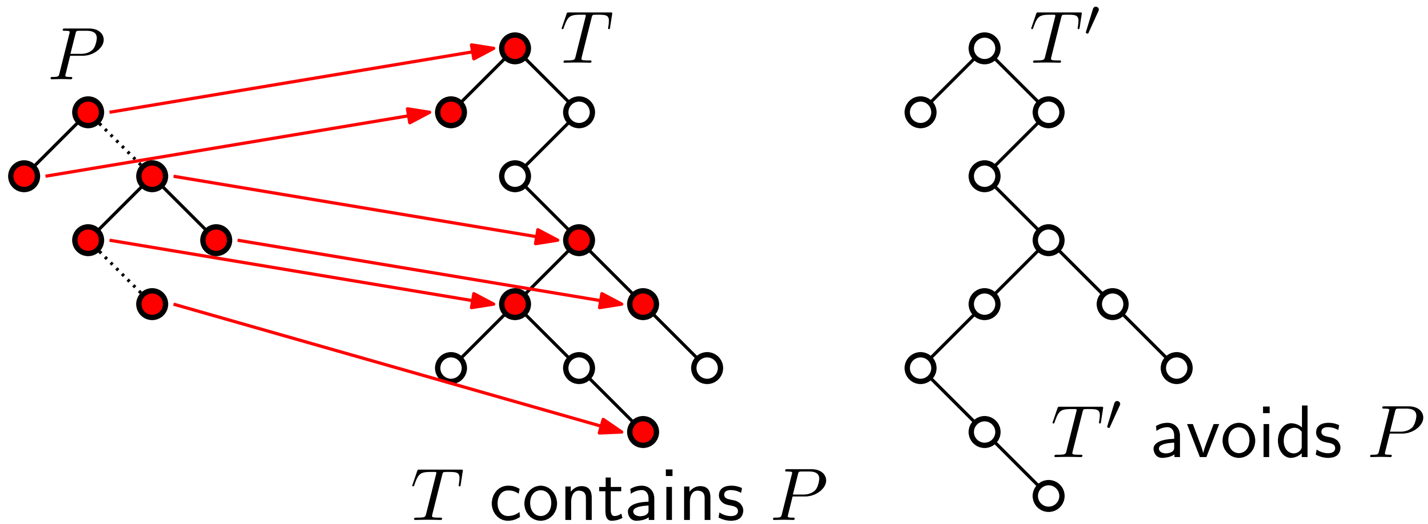

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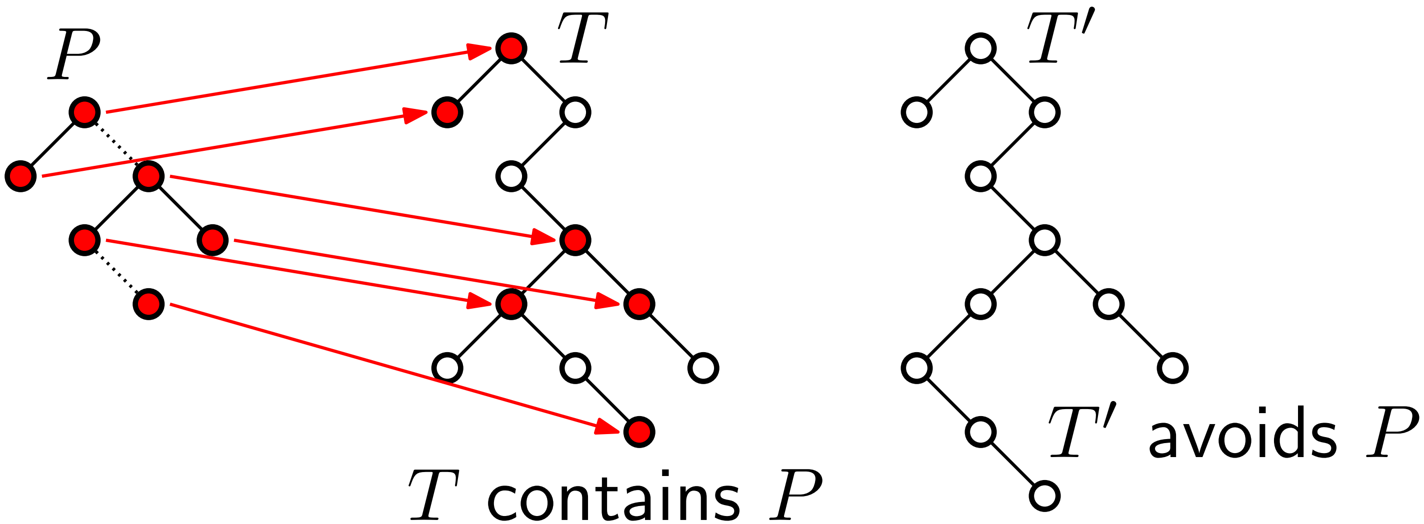

Theorem: For every (mixed) tree pattern, there is a permutation mesh pattern $\tau(P) = (f(P), C)$ such that $f : T_n(P) \to S_n(231, \tau(P))$ is a bijection.

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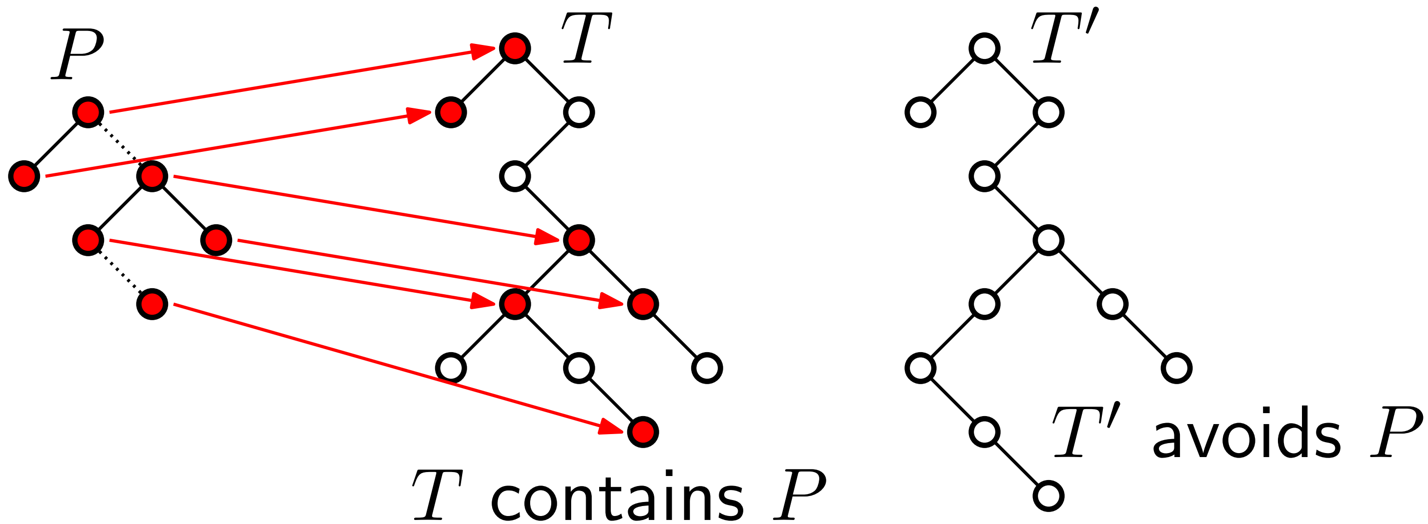

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- generalizes result of [Pudwell, Scholten, Schrock, Serrato 14]

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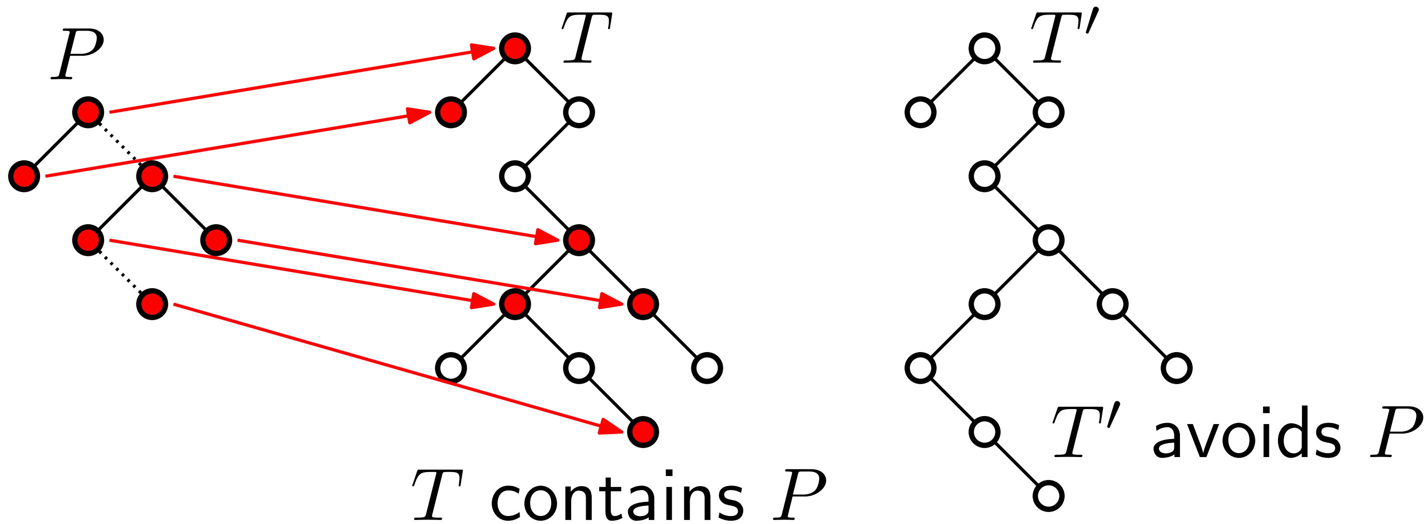
mixed (new)

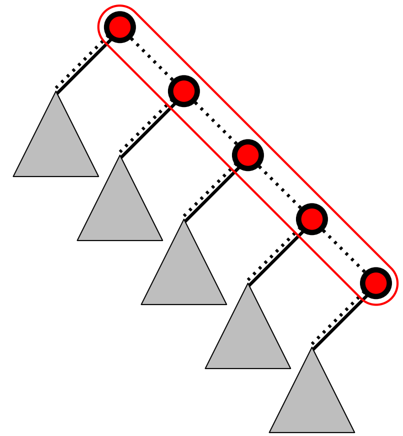


Theorem: For every (mixed) tree pattern, there is a permutation mesh pattern $\tau(P) = (f(P), C)$ such that $f : T_n(P) \to S_n(231, \tau(P))$ is a bijection.

- generalizes result of [Pudwell, Scholten, Schrock, Serrato 14]
- classified all tree patterns on ≤ 5 vertices; interesting bijections to pattern-avoiding lattice paths and set partitions

Tame patterns

- A pattern P is **tame**, if the largest node is neither root nor leaf, and the right branch from the root is non-contiguous



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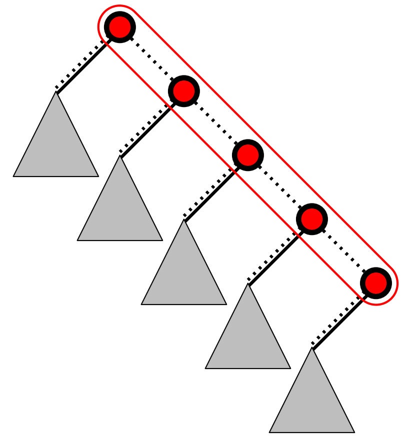

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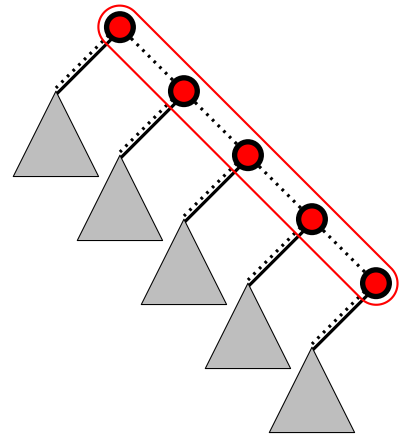

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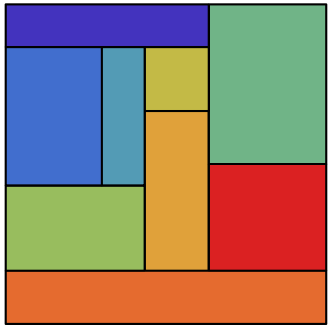
$\rightarrow$ see `www.combos.org/btree`

Generic rectangulations

- **Generic rectangulation:** subdivision of a square into n rectangles s.t. no four rectangles meet



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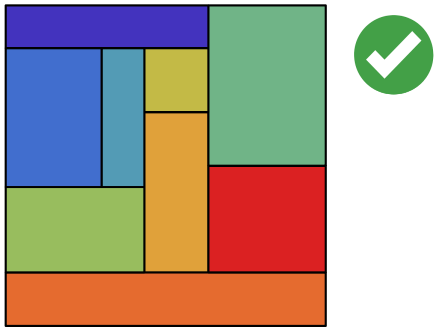

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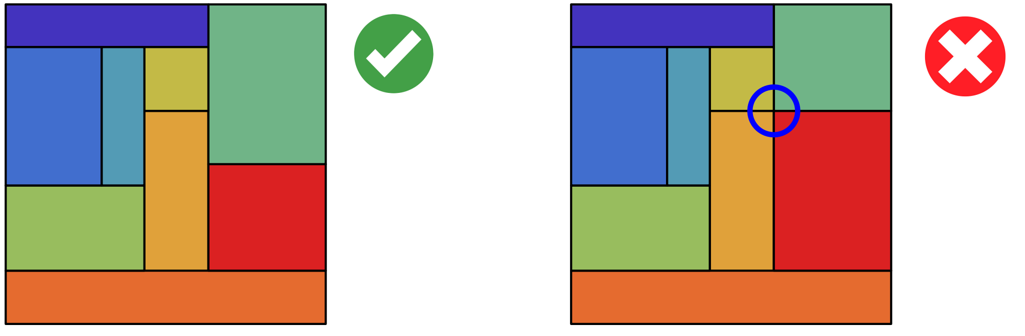

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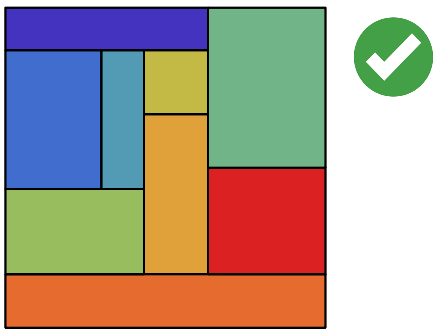

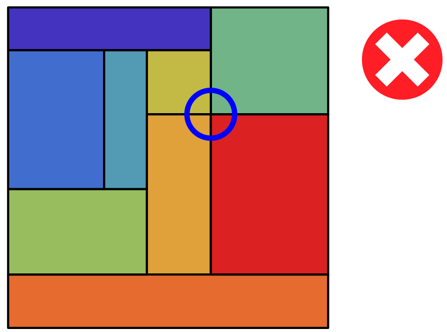

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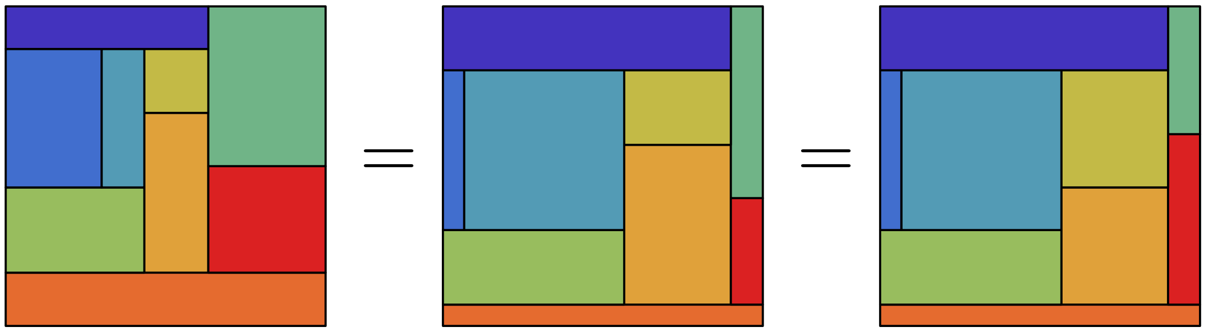

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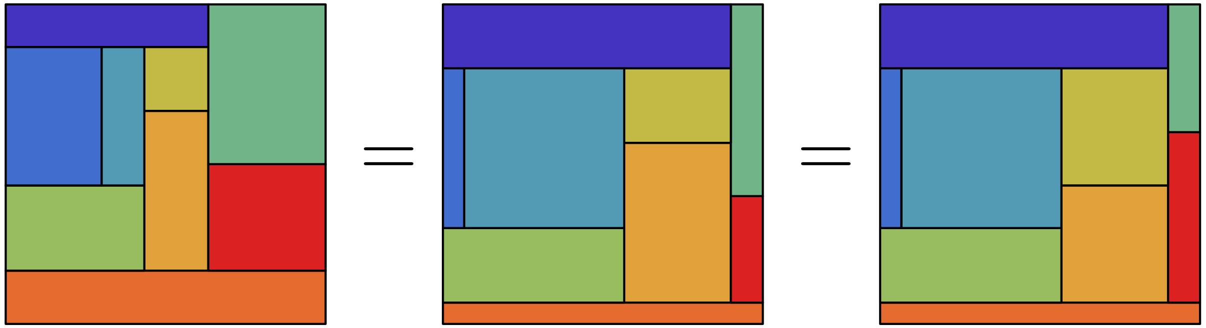

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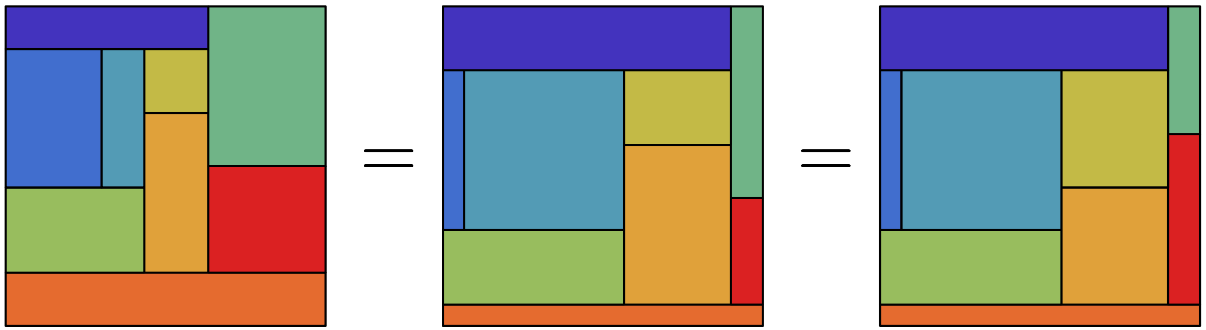

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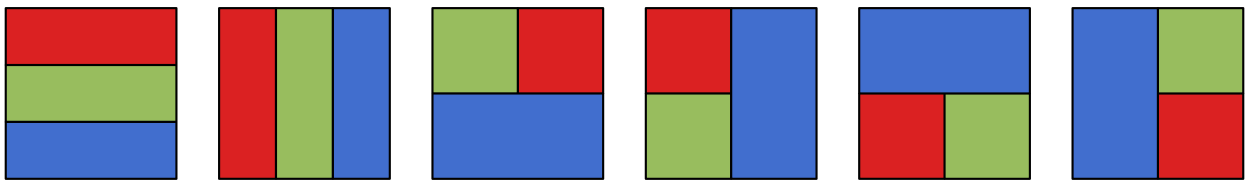
$R_3 =$ 

Generic rectangulations

Theorem [Reading 12]**:** There is a bijection f between R_n and $S_n(3\underline{51}24, 3\underline{51}42, 24\underline{51}3, 42\underline{51}3)$ (2-clumped permutations).

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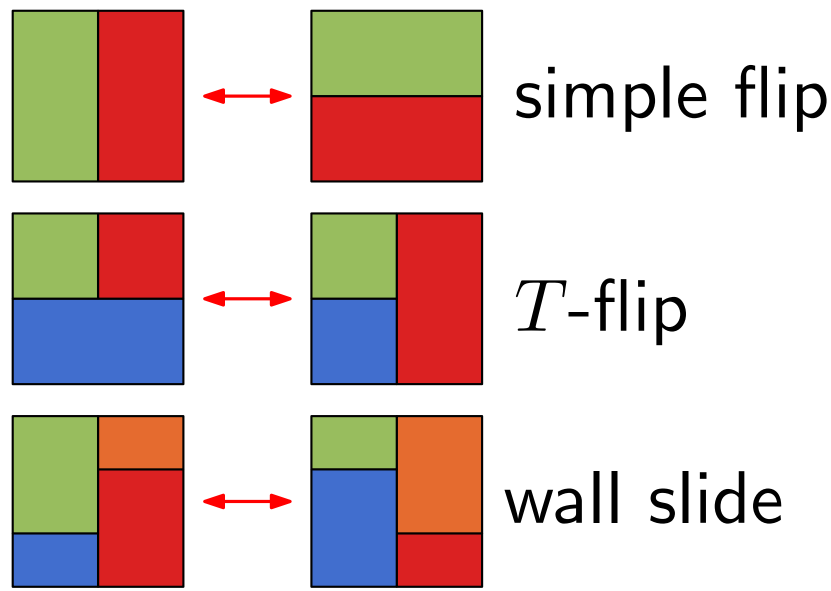
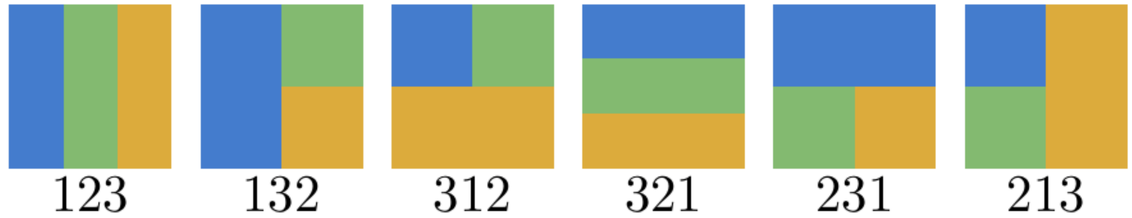
rectangle flips



Flip Gray code

$n = 3$



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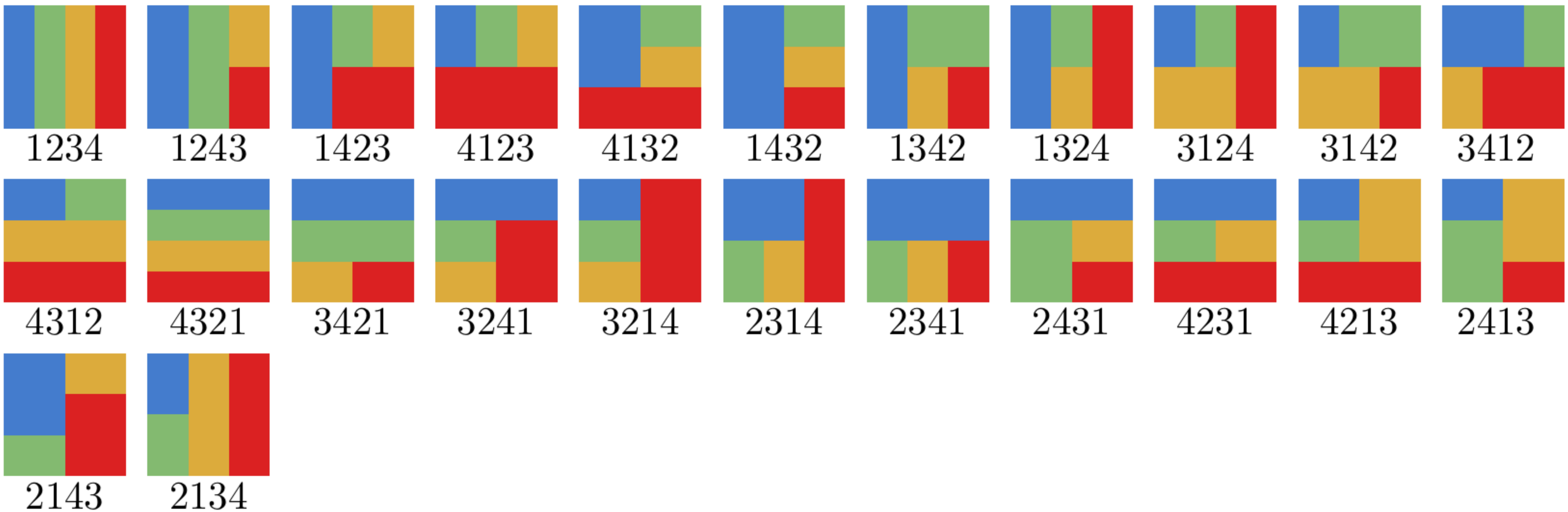
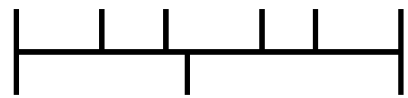
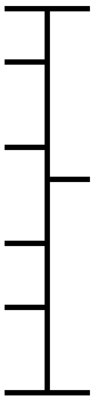


Patterns in rectangulations

- **Segment:** maximal sequence of inner edges





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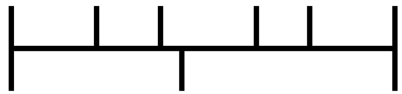

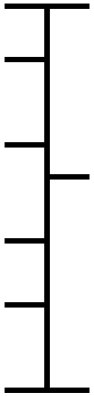

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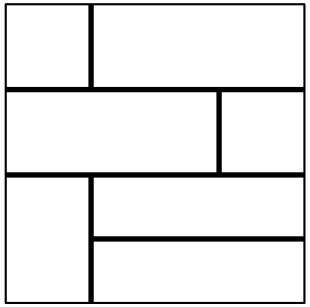
$P =$ 

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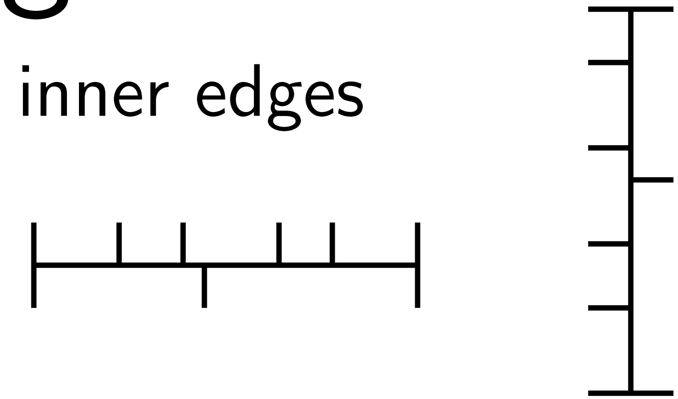

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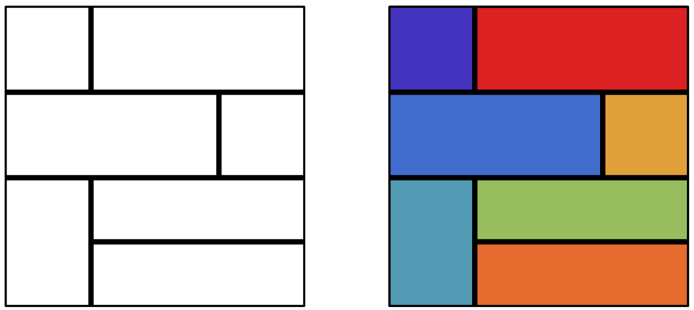
$P =$ 

can be seen as a rectangulation itself

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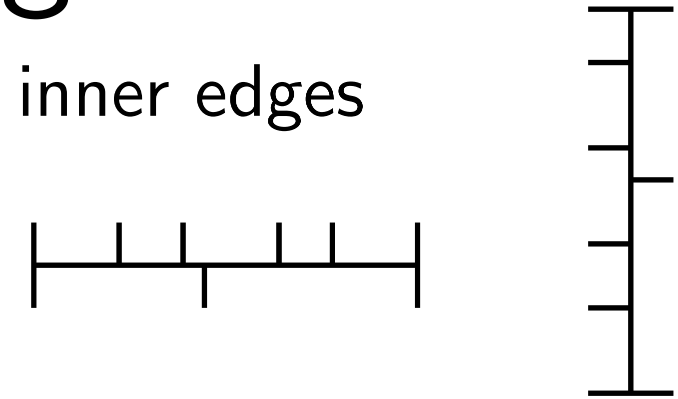

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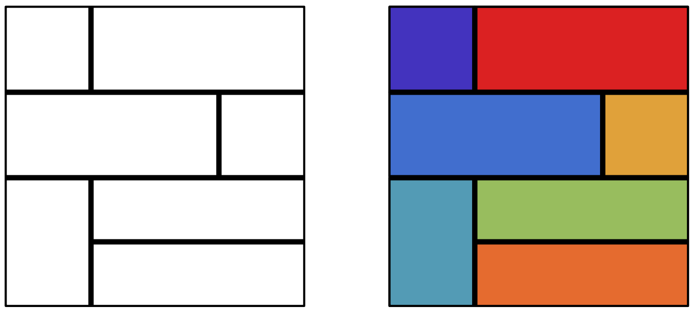

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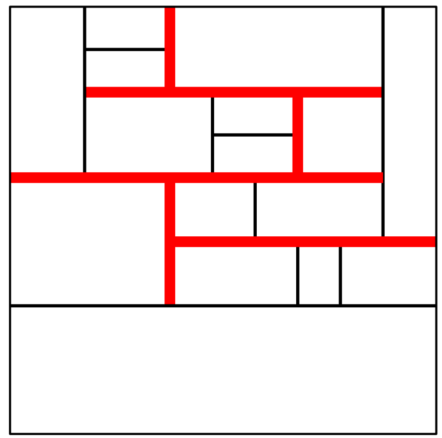

contains P

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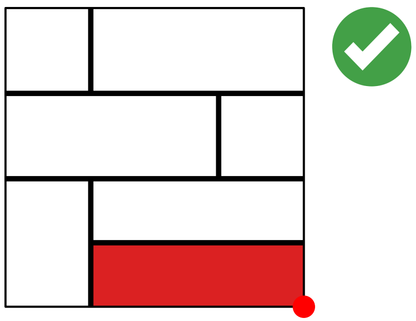

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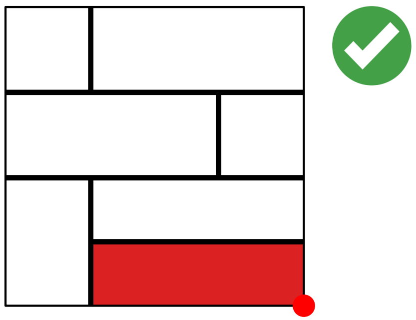

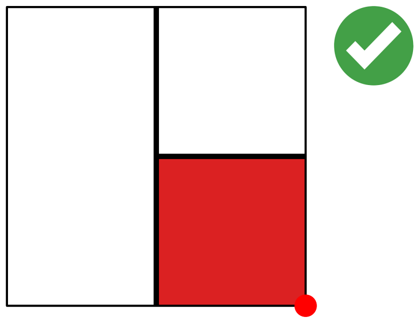

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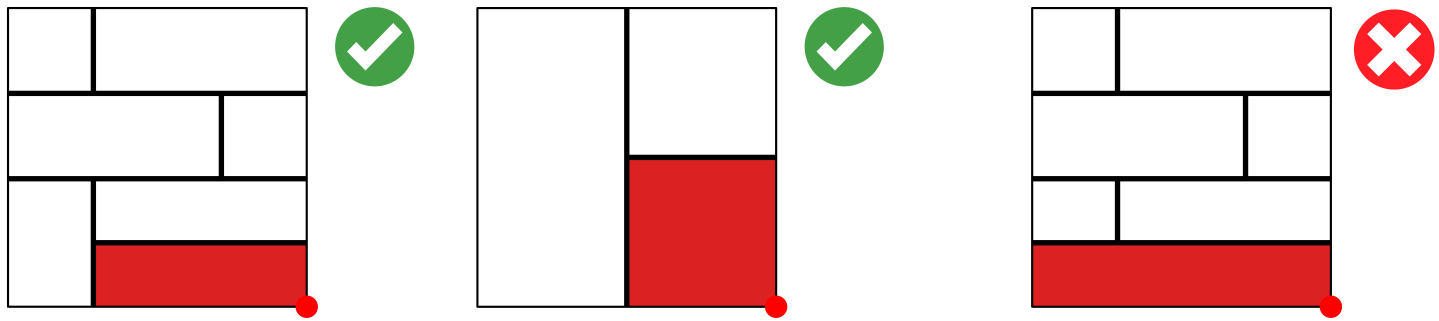

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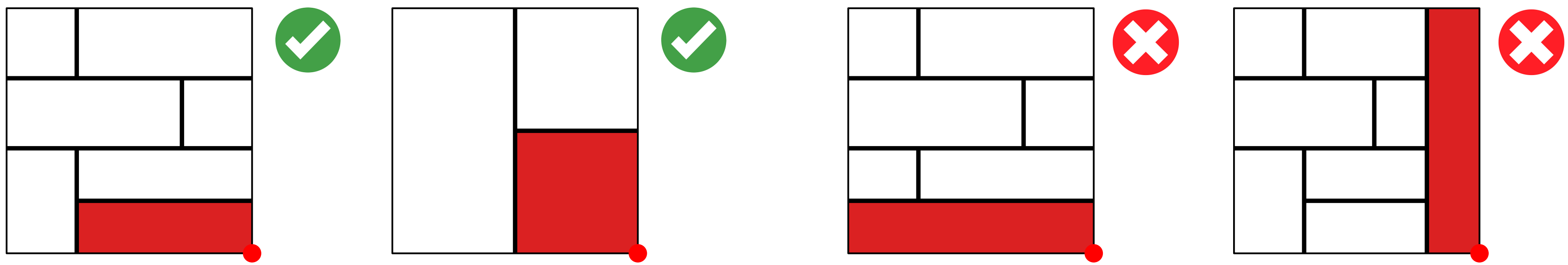

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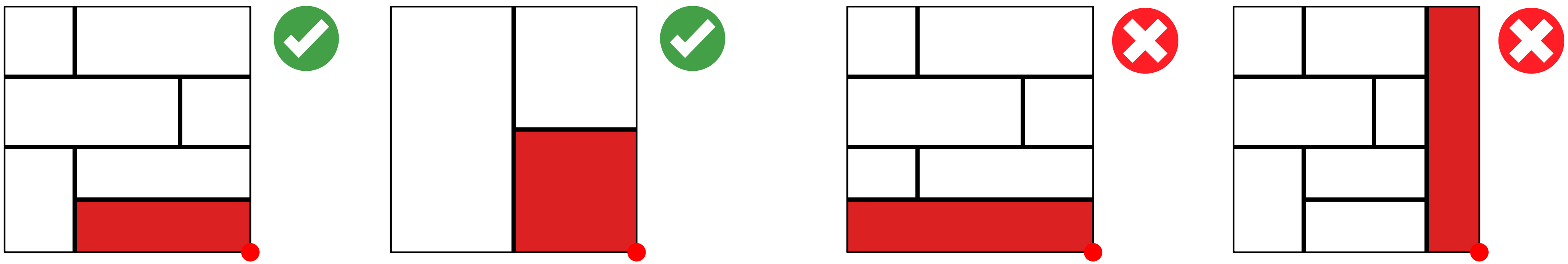

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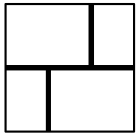
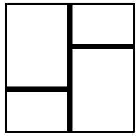
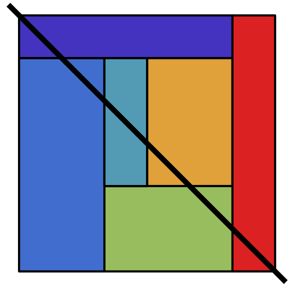
$R_n($  $,$  $)$

diagonal rectangulations



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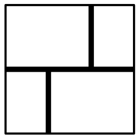
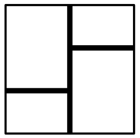
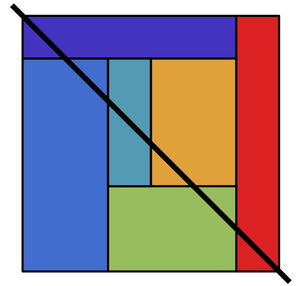
$R_n($  $,$  $)$

diagonal rectangulations
$\hookrightarrow$ HC on quotientope



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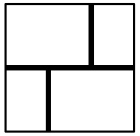
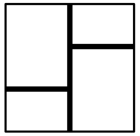
$R_n($  ,  $)$

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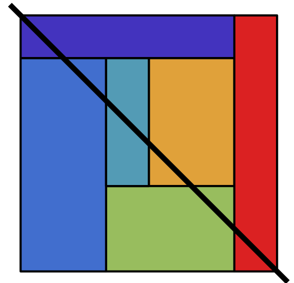
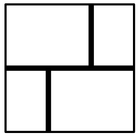
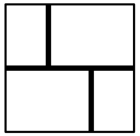
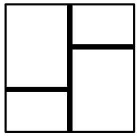
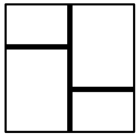


$R_n($  ,  ,  ,  $)$

area-universal rectangulations
[Eppstein, Mumford, Speckmann, Verbeek 2012]

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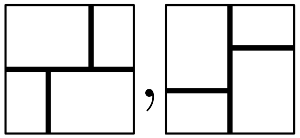
$R_n($  $)$

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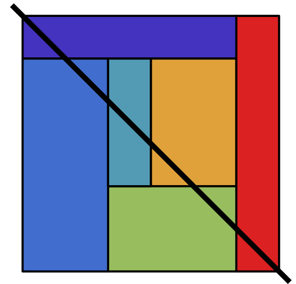
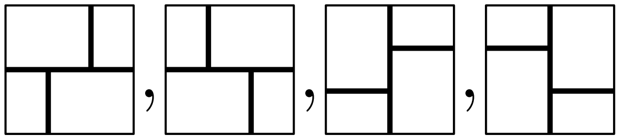


$R_n($  $)$

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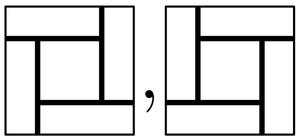
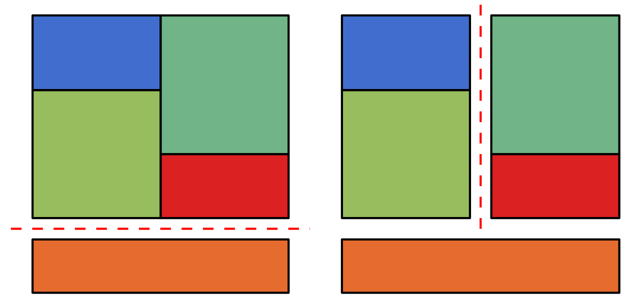
$R_n($  $)$

guillotine rectangulations



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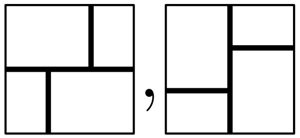
$R_n($  $)$

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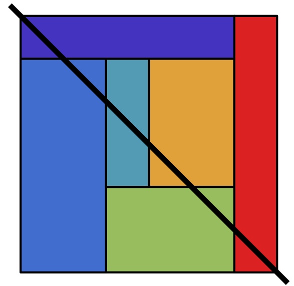
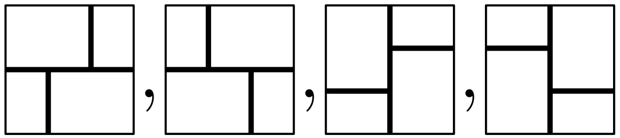


$R_n($  $)$

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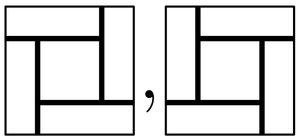
$R_n($  $)$

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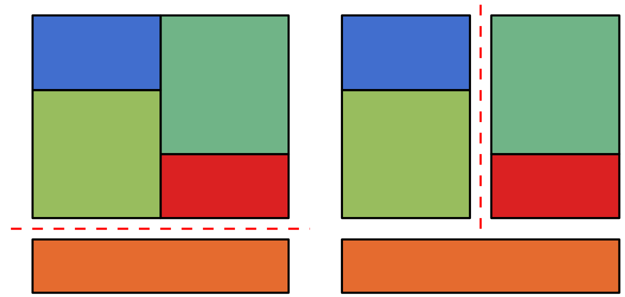
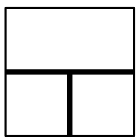
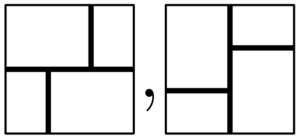


$R_n($  $)$

Catalan staircases C_n
[Downing, Einstein, Hartung, Williams 2023]

Examples

$R_n($  $)$

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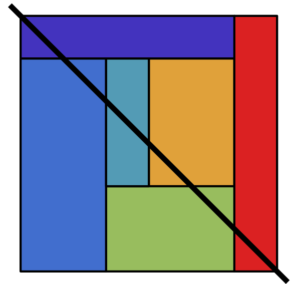
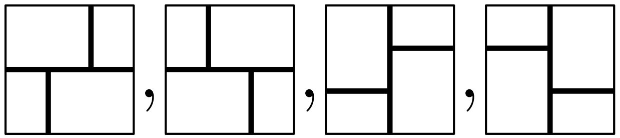


$R_n($  $)$

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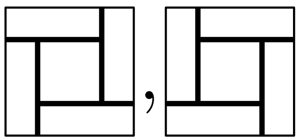
$R_n($  $)$

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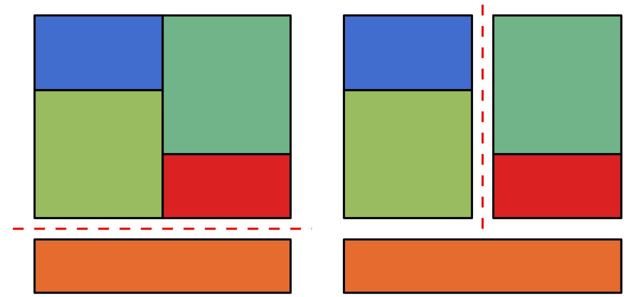
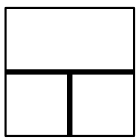
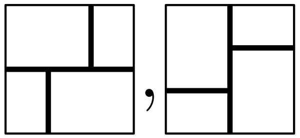


$R_n($  $)$

Catalan staircases $\quad C_n$
[Downing, Einstein, Hartung, Williams 2023]
$\hookrightarrow$ HP on associahedron

Examples

$R_n($  $)$

diagonal rectangulations

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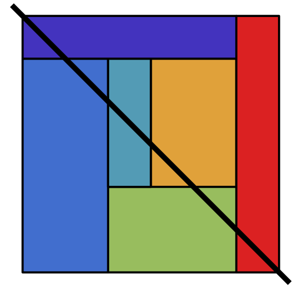
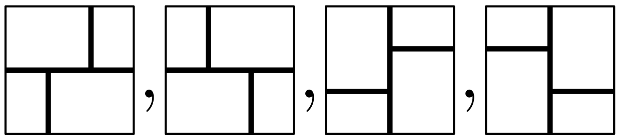
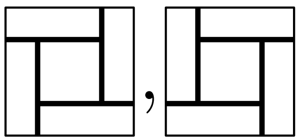


$R_n($  $)$

area-universal rectangulations

[Eppstein, Mumford, Speckmann, Verbeek 2012]

$R_n($  $)$

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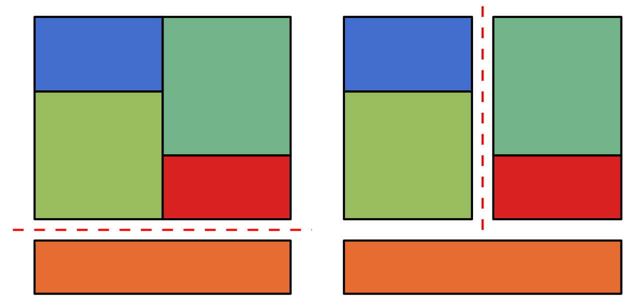
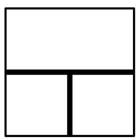


$R_n($  $)$

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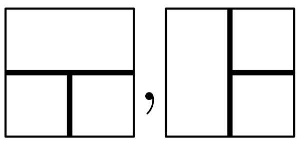
$R_n($  $)$

stacked rectangulations $\quad 2^n$

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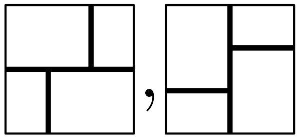
$R_n($  $)$

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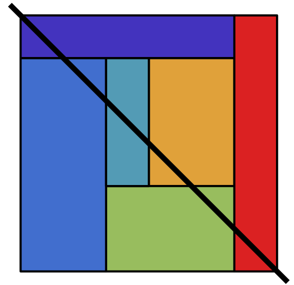
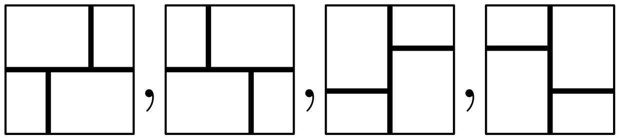


$R_n($  $)$

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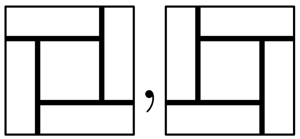
$R_n($  $)$

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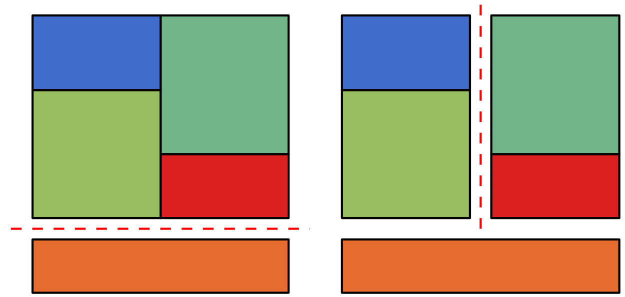
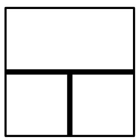


$R_n($  $)$

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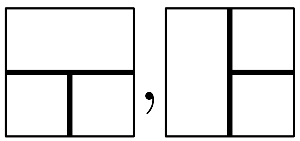
$R_n($  $)$

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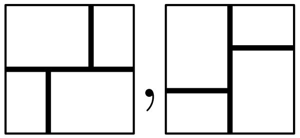
$R_n($  $)$

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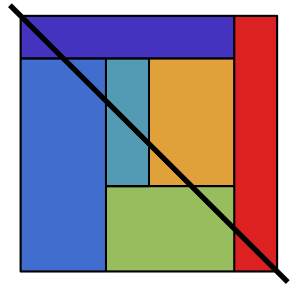
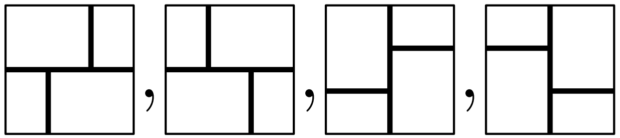


$R_n($  $)$

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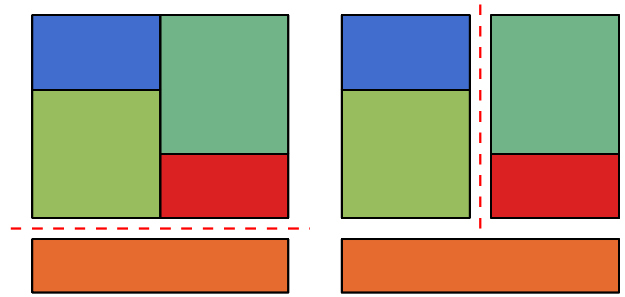
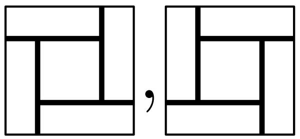


$R_n($  $)$

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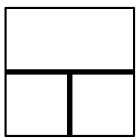
$R_n($  $)$

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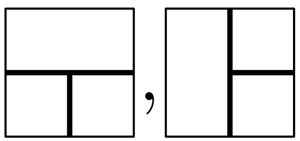
$R_n($  $)$

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$\rightarrow$ see `www.combos.org/rect`

Open questions

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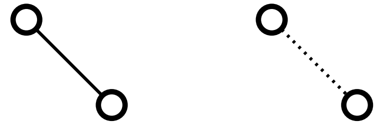
- Third notion of edge type in tree patterns 

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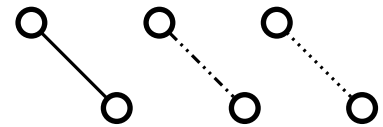
- Third notion of edge type in tree patterns 

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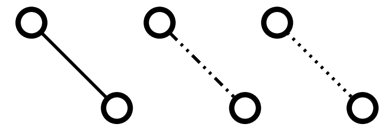
- Third notion of edge type in tree patterns 
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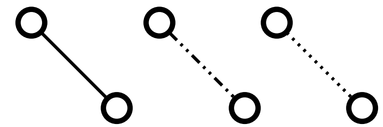
- Third notion of edge type in tree patterns 
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- Applications of the generation framework to other (pattern-avoiding) combinatorial objects

Thank you!