Increasing Schröder trees and restricted permutations

Permutation Patterns 2023

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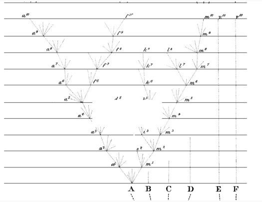
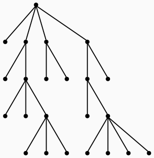
Introduction

- Represent the evolutionary relationship among species
- A Multifurcation means the descendant species have distinguished themselves



- Earliest illustrations made by **Darwin** in his book *iëђ iḓā ëŵÔђ ωĩ ĥʢā!ā'i* (1859)
- Represent divergence in characters



- **Schröder** published *iŜā !ωgƐђ‾ωŵ'!ḓā ĕŵωƐӌāgāi* (1870)
- Asks for the number of ways a string of *ђ* identical letters, can be “bracketed”

Ordinary generating function of a class C to be $G(z) = \sum_{n=0}^{\infty} G_n z^n$ where G_n is # of objects of size n.

Symbolic method systematized by Flajolet and Sedgewick is a grammar used to define (specify) combinatorial classes:

Operation	Notation	Description	OGF
Neutral class		Class consisting of single object of size 0	1
Atomic class	Z	Class consisting of single object of size 1	z
Disjoint Union	$F + G$	Disjoint of objects from F and G	$F(z) + G(z)$
Cartesian product	$F \quad G$	Ordered pairs of objects from F and G	$F(z) \quad G(z)$
Sequence	$\textsc{Seq}\, F$	Sequences of objects from F	$\frac{1}{1 \quad F(z)}$
Substitution	$F \quad G$	Substitute elements of G for atoms of F	$F(G(z))$
Pointing	F	Point an atom from F	$z F'(z)$

Ordinary generating function of a class C to be $G(\_) = \sum_{ɧ=0} G_ɧ \_^{ɧ}$ where $G_ɧ$ is # of objects of size $ɧ$.

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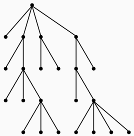
Pointing	F	Point an atom from F	$\_ \{^0(\_)$



Schröder trees

- Size of the tree # Leaves
- Internal nodes arity 2

$$S = Z + \text{SEQ}_{\ 2}(S) =)$$

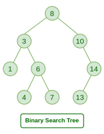
$$\hat{h}(\_) = \_ + \frac{\hat{h}(\_)^2}{1 \quad \hat{h}(\_)}$$

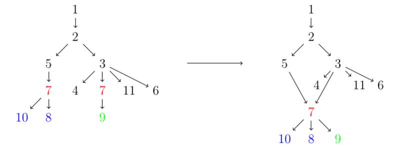
$$\hat{h}(\_) = \frac{1}{4}(1 + \_ \quad p\overline{1 \quad 6\_ + \_^2})$$

$$= \_ + \_^2 + 3\_^3 + 11\_^4 + 45\_^5 + \dots$$

Increasing trees have increasing labels on branches.

- **Unique labels**, analysis of permutations and data structures like BST using increasing binary trees. Drmota, Mahmoud, Flajolet, ...
- **Repeated labellings**, parallel executions of processes. Bodini, Gittenberger, Genitrini, Prodinger, Urbanek...





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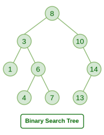
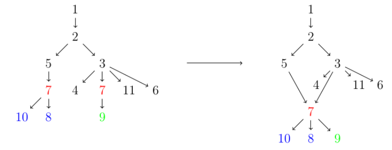
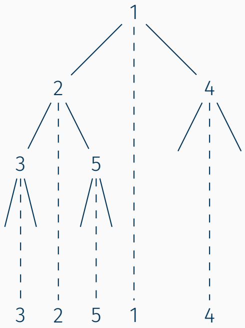


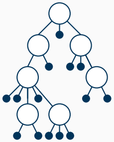




( ) = *h*min( ); (*í*); (*Ĝ*)*i*; ( ) =

Definition: increasingly labelled Schröder trees
Schröder tree t whose internal nodes are labelled with integers such that:

- Root is labelled 1 and along each branch the sequence of labels is strictly increasing.
- If ℓ is the greatest label in the tree, then the set of labels of t is $[\ell]$.
- Label repetitions (allowed or not)



Increasing trees Schröder trees

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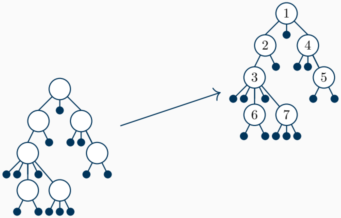

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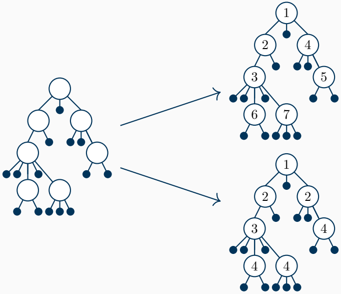

Permutations

Cycles

$$\begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 3 & 4 & 6 & 2 & 5 & 1 \end{pmatrix}$$

↕ cycle nota.

$(1;3;6)(2;4)(5)$

- Cycle canonical order $=)$ smallest integer 1st
- $^{1}(ʝ)$ returns $(\mathring{0};?)$, $\mathring{0}$ the cycle number that contains $ʝ$ and $?$ its index

Runs, Rises, Descents

- A rise (resp. descent) two successive increasing (resp. decreasing) elements
- A run is maximal increasing subsequence $\mathbf{qtmr}(3;4;6;2;5;1) = 3$
- # of $ɦ$-permutations with $ʝ$ des. Eulerian numbers $\genfrac{}{}{0pt}{}{ɦ}{ʝ}$
- $\mathbf{cdr}(Ç) = \mathbf{qtmr}(Ç) \quad 1$
- $\mathbf{qhr}(Ç) = ɦ \quad 1 \quad \mathbf{cdr}(Ç)$

Set Partitions

- A set partition is a partition of a set into a non-empty set of subsets
- $\genfrac{}{}{0pt}{}{ɦ}{ʝ}$ # of set partitions of a set $ɦ$ elements into $ʝ$ non-empty subsets
- **Ordered set partition** adds an order on the subsets, $ʝ!\genfrac{}{}{0pt}{}{ɦ}{ʝ}$
- (Ordered Bell numbers) $\Sigma_{ɦ} = \sum_{ʝ=1}^{ɦ} ʝ! \genfrac{}{}{0pt}{}{ɦ}{ʝ}$

1 subset	2 subsets	3 subsets
$(f1;2;3g)$	$(f1;2g;f3g)$	$(f1g;f2g;f3g)$
	$(f3g;f1;2g)$	$(f2g;f1g;f3g)$
	$(f1;3g;f2g)$	$(f1g;f3g;f2g)$
	$(f2g;f1;3g)$	$(f2g;f3g;f1g)$
	$(f2;3g;f1g)$	$(f3g;f1g;f2g)$
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Increasing Schröder without repetitions

Evolution process

Start at step 0 with a leaf; at each step i 1 do:

Evolution process

Start at step 0 with a leaf; at each step [illegible] 1 do:

1. Choose one leaf of the so-far built tree.

Evolution process

Start at step 0 with a leaf; at each step $i \geq 1$ do:

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Evolution process

Start at step 0 with a leaf; at each step $i \geq 1$ do:

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3. Replace the leaf with an internal node labelled i and attach to it j new leaves.

Evolution process

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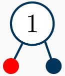

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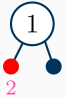

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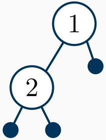

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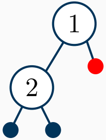

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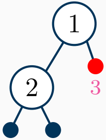

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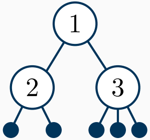

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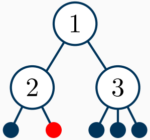

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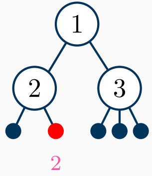

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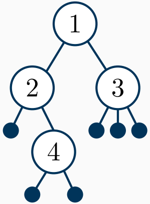

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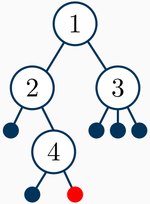

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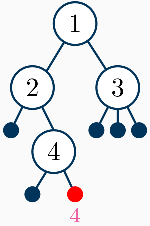

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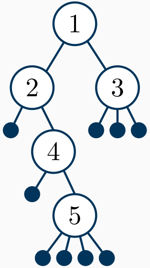

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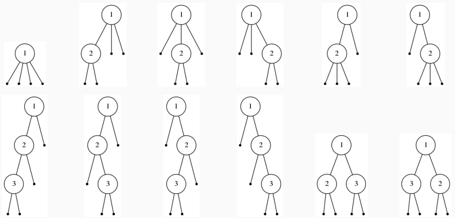
$$T = Z + \; T \; \mathrm{SEQ}_{\geq 1} Z \;\Rightarrow\; \phi(z) = z + \frac{z^2}{1-z}\phi'(z).$$

$$\tau_n = \begin{cases} 1, & n = 1, 2 \\ n\,\tau_{n-1}, & n \geq 3 \end{cases}$$

$$(\tau_n)_{n \geq 0} = 0, 1, 1, 3, 12, 60, 360, 2520, 20160, 181440, 1814400, \ldots$$

Theorem: The number of trees of size n for all $n \geq 2$ is

$$\tau_n = \frac{n!}{2}.$$



	Mean	Variance	Limit law
Internal nodes	$n - \ln n$	$\ln n$	Normal
Number of binary nodes	$n - 2\ln n$	$4\ln n$	Normal
Number of ternary nodes	$\ln n$		
Depth of the leftmost leaf	$\ln n$	$\ln n$	Normal
Height of the tree	$(\ln n)$		
Degree of the root	$2e - 3$	$14e - 4e^2 - 8$	modified Poisson

Let PR set of permutations with elements 1 and 2 in different cycles. Define $N : PR \to T$

- If $\sigma = (1)(2)$ then $N(\sigma)$ is a cherry labelled 1.
- Else, Let $(i, j) = \sigma^{-1}(n)$ where n largest element in σ.
    - If $|\omega_i| = 1$ then, we set $N(\sigma)$ to be the tree $N(\sigma \setminus n)$ and add a rightmost leaf to the last internal node.
    - Else, let $j = \omega_{i,j-1}$, we set $N(\sigma)$ to be the tree $N(\sigma \setminus n)$ and add a binary node labelled next integer with 2 leaves. Insert this binary node in $N(\sigma \setminus n)$ by placing in the j-th leaf of $N(\sigma \setminus n)$.

Let PR set of permutations with elements 1 and 2 in different cycles. Define $N : PR \to T$

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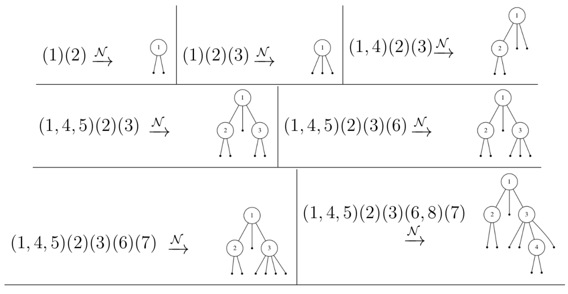
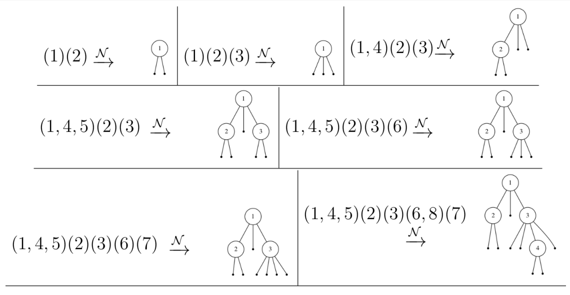
- Else, Let $(\mathring{\imath}; ?) = \ ^{1}(ɧ)$ where ɧ largest element in .
  - If $j!_{\mathring{\imath}}j = 1$ then, we set $N(\ )$ to be the tree $N(\ nɧ)$ and add a rightmost leaf to the last internal node.
  - Else, let $ɉ = !_{\mathring{\imath};?}\ \ _1$, we set $N(\ )$ to be the tree $N(\ nɧ)$ and add a binary node labelled next integer with 2 leaves. Insert this binary node in $N(\ nɧ)$ by placing in the ɉ-th leaf of $N(\ nɧ)$.



Let PR set of permutations with elements 1 and 2 in different cycles. Define $N : PR \to T$

- If $= (1)(2)$ then $N(\ )$ is a cherry labelled 1.
- Else, Let $(\mathring{0}; ?) = \ ^{1}(\eta)$ where η largest element in .
  - If $j\,!_{\mathring{0}}j = 1$ then, we set $N(\ )$ to be the tree $N(\ n\eta)$ and add a rightmost leaf to the last internal node.
  - Else, let $j = !_{\mathring{0};?}\ _{1}$, we set $N(\ )$ to be the tree $N(\ n\eta)$ and add a binary node labelled next integer with 2 leaves. Insert this binary node in $N(\ n\eta)$ by placing in the j-th leaf of $N(\ n\eta)$.



Theorem: Let $^{-}$ be an increasing Schröder tree with η leaves and j internal nodes, and let $\varsigma = N\ ^{1}(^{-})$ be its corresponding permutation. Let $\mathring{0}$ be the number of cycles of ς, then

$$j = \eta + 1 \quad \mathring{0}:$$

Increasing Schröder with repetitions

Evolution process

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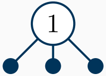

1. Choose a non-empty subset S of leaves of the so-far built tree.
2. For each $\ell \in S$ choose an integer $k_\ell > 1$.
3. Replace ℓ with an internal node labelled i and attach to it k_ℓ new leaves.

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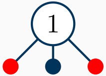

1. Choose a non-empty subset S of leaves of the so-far built tree.
2. For each $\ell \in S$ choose an integer $k_\ell \geq 1$.
3. Replace ℓ with an internal node labelled n and attach to it k_ℓ new leaves.

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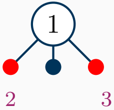
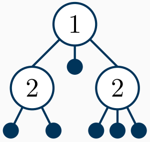


1. Choose a non-empty subset S of leaves of the so-far built tree.
2. For each $\ell \in S$ choose an integer $k > 1$.
3. Replace ℓ with an internal node labelled i and attach to it k new leaves.

Evolution process

Start at step 0 with a leaf; at each step $k \geq 1$ do:



1. Choose a non-empty subset L of leaves of the so-far built tree.
2. For each $\ell \in L$ choose an integer $j > 1$.
3. Replace ℓ with an internal node labelled k and attach to it j new leaves.

Evolution process

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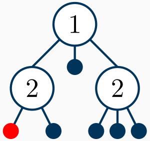

1. Choose a non-empty subset S of leaves of the so-far built tree.
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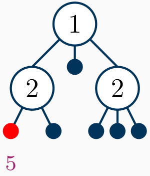

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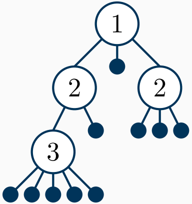

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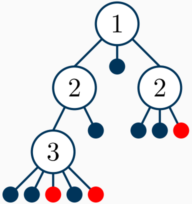
2. For each $\ell \in L$ choose an integer $k > 1$.
3. Replace ℓ with an internal node labelled i and attach to it k new leaves.

Evolution process

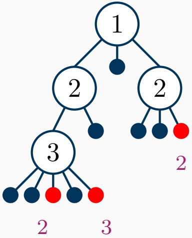
Start at step 0 with a leaf; at each step $k \geq 1$ do:



1. Choose a non-empty subset S of leaves of the so-far built tree.
2. For each $\ell \in S$ choose an integer $j_\ell > 1$.
3. Replace ℓ with an internal node labelled k and attach to it j_ℓ new leaves.

Evolution process

Start at step 0 with a leaf; at each step $i \geq 1$ do:



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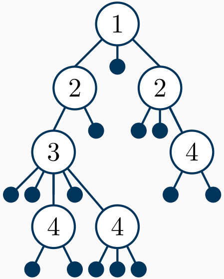

1. Choose a non-empty subset L of leaves of the so-far built tree.
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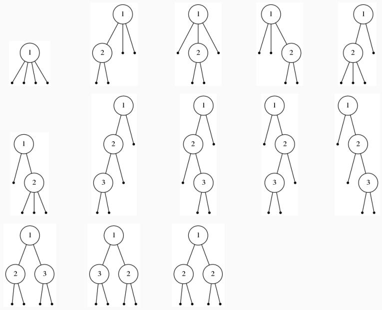
$$\sim(\_) = \_ + \sim \frac{\_^2}{1 - \_} + \_ \quad \sim(\_):$$

$$Ô_ɧ = \begin{cases} 1 & \text{if } ɧ = 1 \\ \sum_{j=1}^{ɧ-1} \binom{ɧ-1}{j-1} Ô_j: & \text{otherwise.} \end{cases}$$

$$(Ô_ɧ)_{ɧ \geq 0} = 0; 1; 1; 3; 13; 75; 541; 4683; \ldots$$

Theorem: The number of trees of $Ô_ɧ$:

$$Ô_ɧ =_{ɧ \to \infty} \frac{(ɧ - 1)!}{2\,(\ln 2)^ɧ}.$$



	Mean	Variance	Limit law
Internal nodes	ɧ ln 2 ln ɧ		
Distinct labels	$\frac{1}{2\ln 2}$ ɧ	$\frac{(1 \ln 2)}{(2\ln 2)^2}$ ɧ	Normal
Degree of the root	2 ln 2 + 1	2 ln 2 (ln 2 1)	(shifted) zero-truncated Poisson
Depth of the leftmost leaf	ln ɧ	ln ɧ	Normal

An ordered partition is denoted $\pi = (\pi_1, \ldots, \pi_k)$. We define **L&**

- If π contains one subset, associate a tree with one internal node labelled by 1 to which are attached $|\pi_1| + 1$ leaves.
- Else if π has k subsets, construct the tree T on $(\pi_1, \ldots, \pi_{k-1})$. Denote by $\pi'_1, \ldots, \pi'_k$ the ordered subsets of the renormalization of $(\pi_1, \ldots, \pi_k)$, i.e. $\mathbf{norm}((\pi_1, \ldots, \pi_k)) = (\pi'_1, \ldots, \pi'_k)$. We denote by $w_1, \ldots, w_r$ the runs of π'_k, i.e. $\mathbf{runs}(\pi'_k) = (w_1, \ldots, w_r)$, then for i from 1 to r, take the leaf whose index is the first element of w_i and replace it with an internal node with label k attached to $|w_i| + 1$ leaves.

Bijection with Ordered set partitions

An ordered partition is denoted $\pi = (\pi_1, \ldots, \pi_k)$. We define **L&**

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An ordered partition is denoted $\pi = (\pi_1, \dots, \pi_k)$. We define **M**

- If π contains one subset, associate a tree with one internal node labelled by 1 to which are attached $|\pi_1| + 1$ leaves.
- Else if π has k subsets, construct the tree τ on $(\pi_1, \dots, \pi_{k-1})$. Denote by $\pi'_1, \dots, \pi'_k$ the ordered subsets of the renormalization of $(\pi_1, \dots, \pi_k)$, i.e. $\mathbf{norm}((\pi_1, \dots, \pi_k)) = (\pi'_1, \dots, \pi'_k)$. We denote by $r_1, \dots, r_m$ the runs of π'_k, i.e. $\mathbf{runs}(\pi'_k) = (r_1, \dots, r_m)$, then for i from 1 to m, take the leaf whose index is the first element of r_i and replace it with an internal node with label k attached to $|r_i| + 1$ leaves.

Theorem: The number of trees of size n with k distinct labels:

$$T_{n,k} = k! \left\{ {n-1 \atop k} \right\} \implies T_n = \sum_{k=0}^{n-1} k! \left\{ {n-1 \atop k} \right\}.$$

From unique labels to repetitions

- $\mathbf{qhr}((4; 6; 1; 2; 5; 3)) = 3$
- Distinguish rises

$$4 \; j \; 6 \qquad 1 \; j \; 2 \; j \; 5 \qquad 3$$

- Selected consec. rises merge in a set

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  (*f*4*g*; *f*6*g*; *f*1; 2; 5*g*; *f*3*g*)

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$4 \mid 6 \qquad 1 \mid 2 \mid 5 \qquad 3$

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$(\{4\},\{6\},\{1\},\{2\},\{5\},\{3\})$	$(\{4,6\},\{1\},\{2\},\{5\},\{3\})$
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- (Velleman and Call 95) showed that

$$\varepsilon_\eta = \sum_{j=0}^{\eta-1} 2^j \left\langle {\eta \atop j} \right\rangle .$$

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- Distinguish rises

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- (Velleman and Call 95) showed that

$$\mathcal{E}_n = \sum_{j=0}^{n-1} 2^j \left\langle {n \atop j} \right\rangle.$$

- A tree $\bar{\tau}$ (without repetitions) of size n with j descents, makes

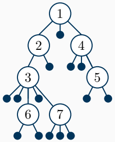
$$2^{n-j-1} + 2^j,$$

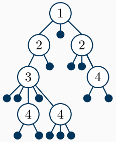
  trees with repetitions of size $n-1$
- **Open question**. Give an interpretation on the tree level

Conclusion

Increasing versions of classical Schröder trees

No repetitions ——— **With repetitions**





- Bijection with classes of Permutations
- Cycles relates to internal nodes

- Bijection with Set partitions
- Subset number relates to the max label
- Step toward a more general framework of increasing label repetitions