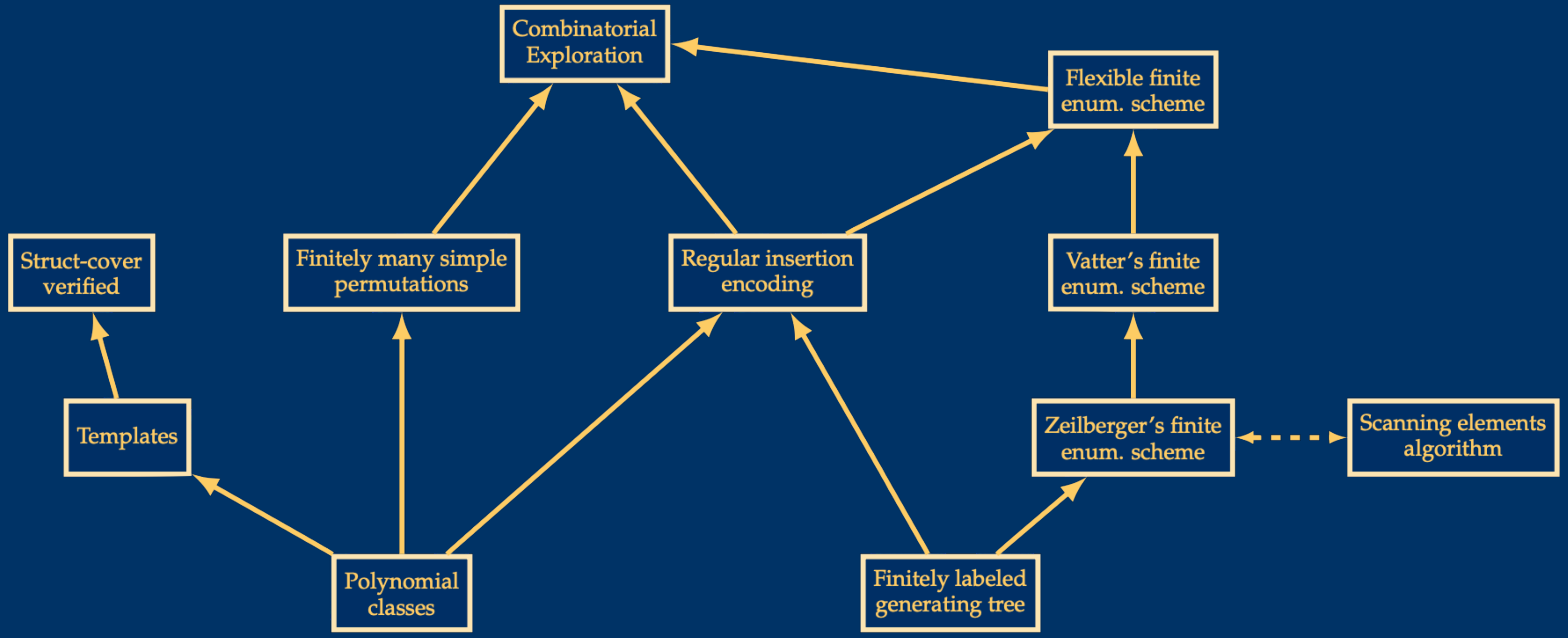
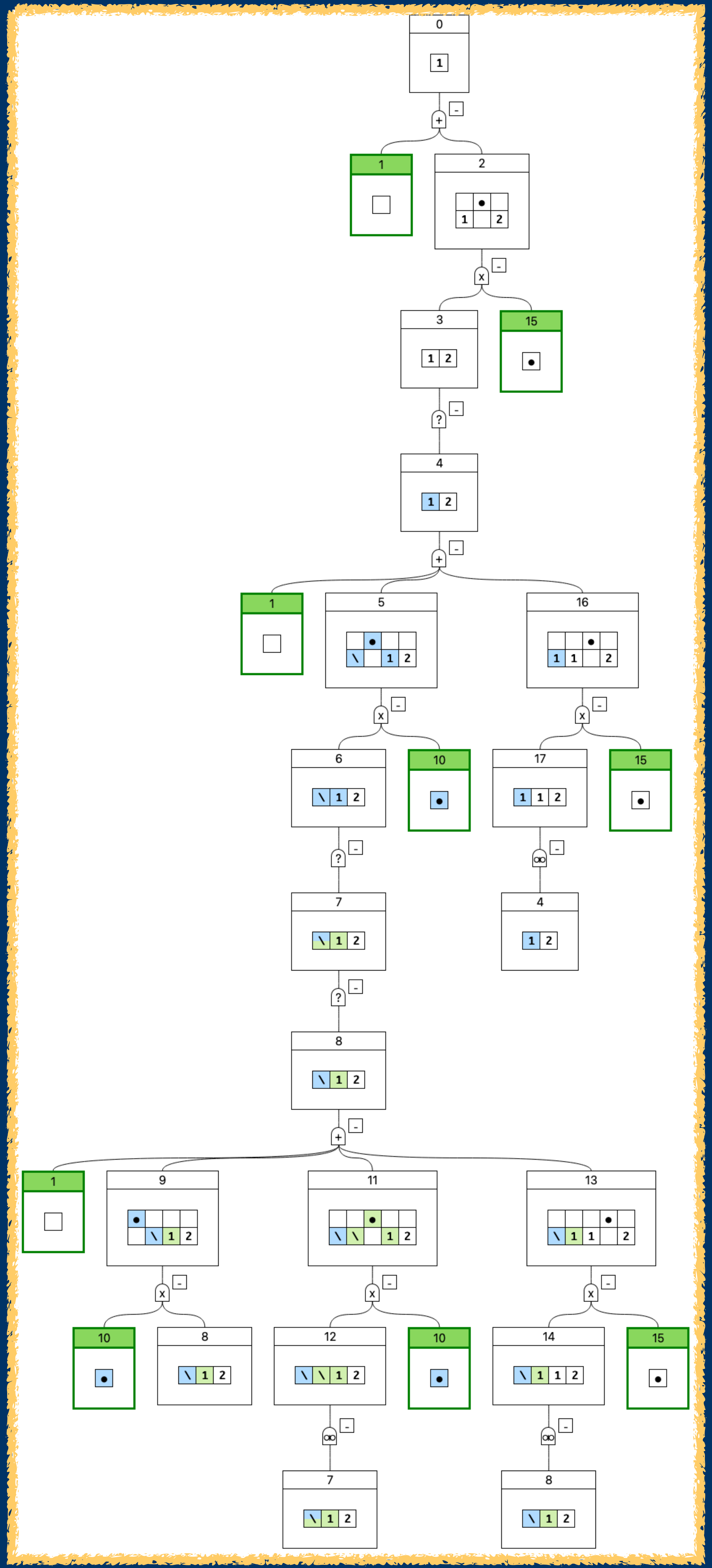
Computational and Experimental Methods in Permutation Patterns

Jay Pantone
Marquette University

Permutation Patterns 2023
Dijon, France





25 years ago...

Enumeration Schemes and, More Importantly, Their Automatic Generation

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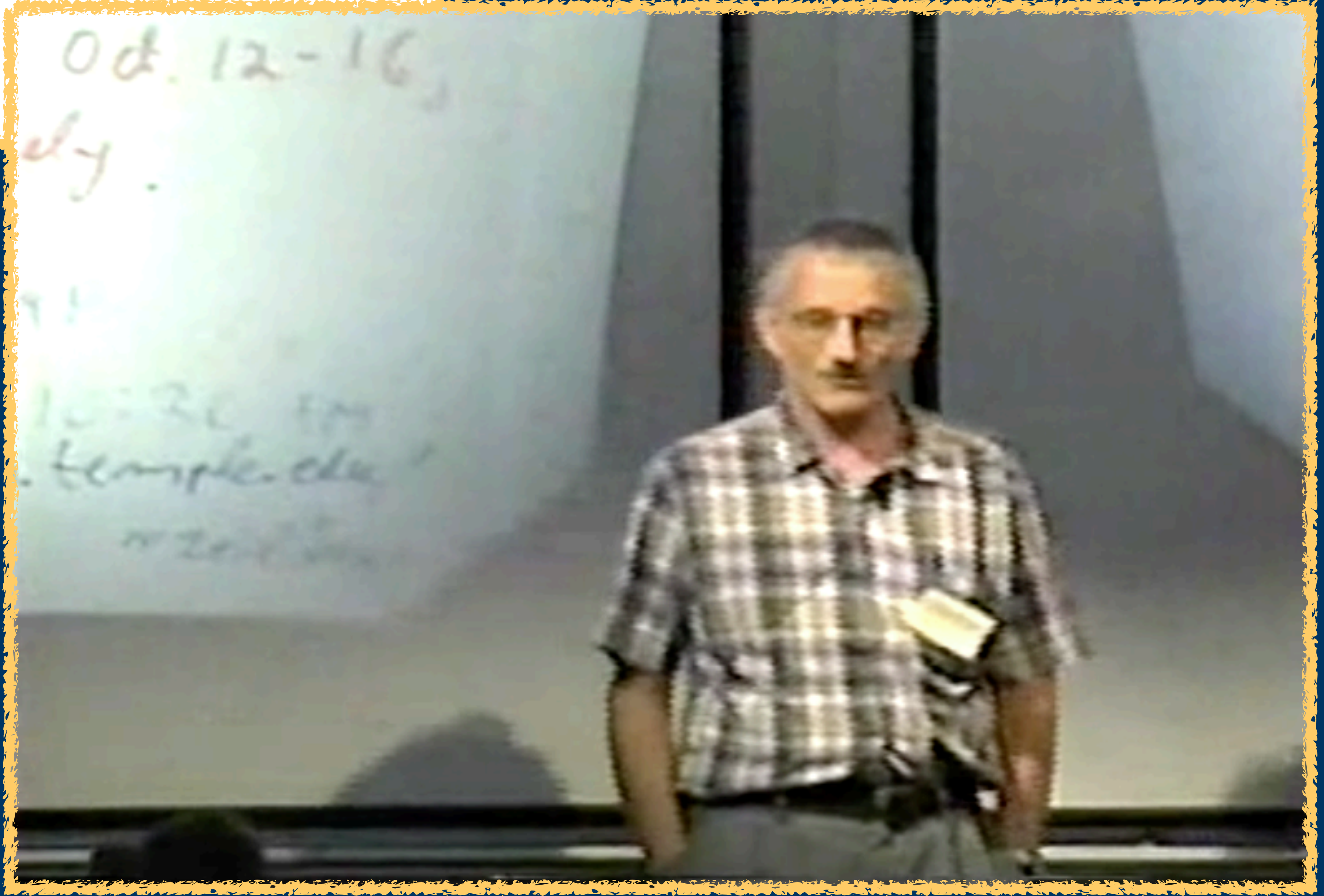

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It is way too soon to teach our computers how to become full-fledged *humans*. It is even premature to teach them how to become *mathematicians*; it is even unwise, at present, to teach them how to become *combinatorialists*. But the time is ripe to teach them how to become experts in a suitably defined and narrowly focused subarea of combinatorics. In this article, the author will describe his efforts in teaching his beloved computer, Shalosh B. Ekhad, how to be an enumerator of Wilf classes.

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With all due respect to Wilf classes and enumeration, and even to combinatorics, the main point of this article is not to enhance our understanding of Wilf classes, but to *illustrate* how much (if not all) of mathematical research will be conducted in a few years. It goes as follows. Suppose a (as of now, human) mathematician has a brilliant idea. Teach that idea to a computer and let the computer 'do research' using that idea.

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What's the state of computation and experimentation in Permutation Patterns?

Enumeration Schemes

Goal: Teach the computer how to search for structure in a permutation class.

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Goal: Teach the computer how to search for structure in a permutation class.

Input: A basis for a permutation class.

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Input: A basis for a permutation class.

Output: An “enumeration scheme” that describes how larger permutations in the class can be recursively described in terms of smaller permutations in the class.

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Enumeration Scheme ⟶ A polynomial-time algorithm to compute the number of permutations of length n.

Enumeration Schemes

"The Most Trivial Non-Trivial Example" – $\mathrm{Av}(123)$

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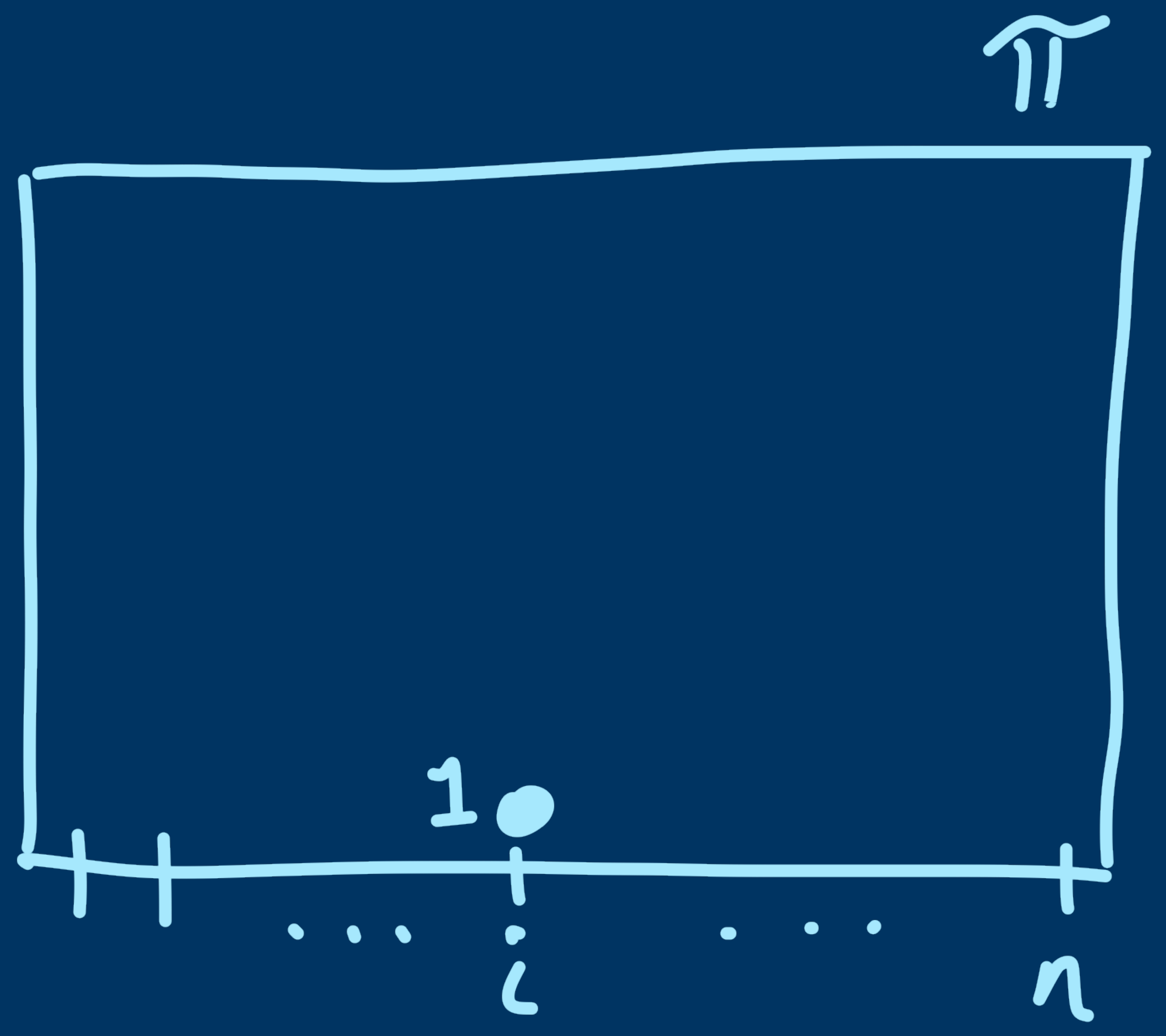

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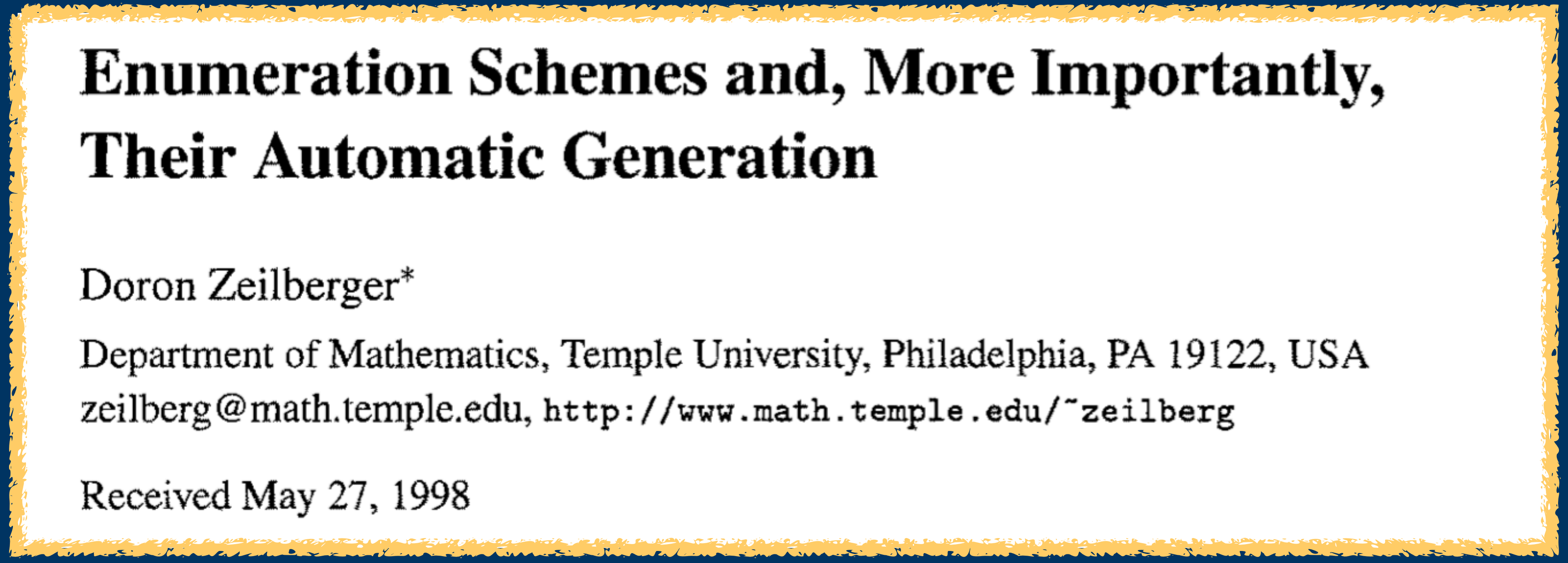
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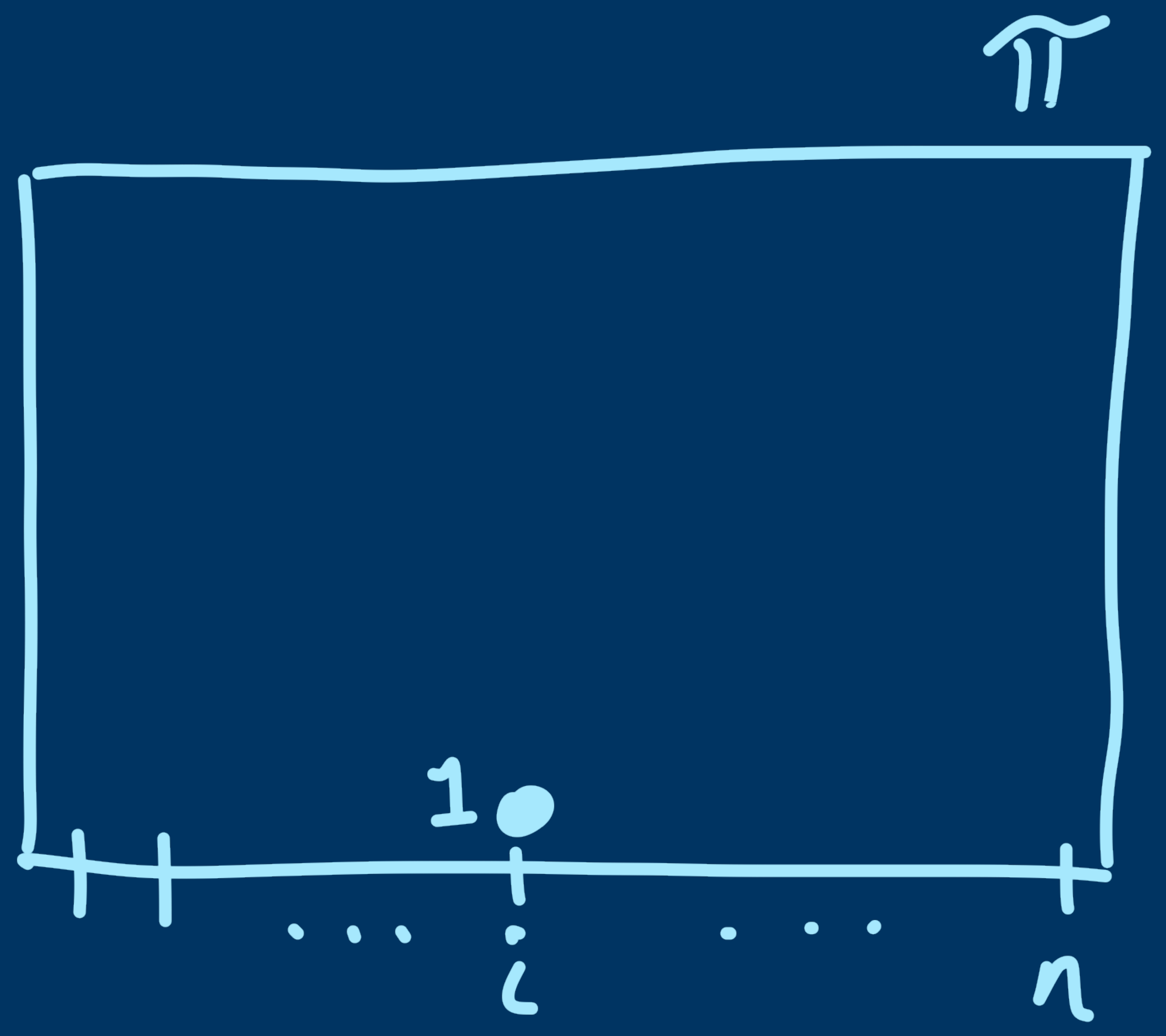
Obviously $A(n) = \bigcup_{i=1}^{n} A_1(n,i)$.



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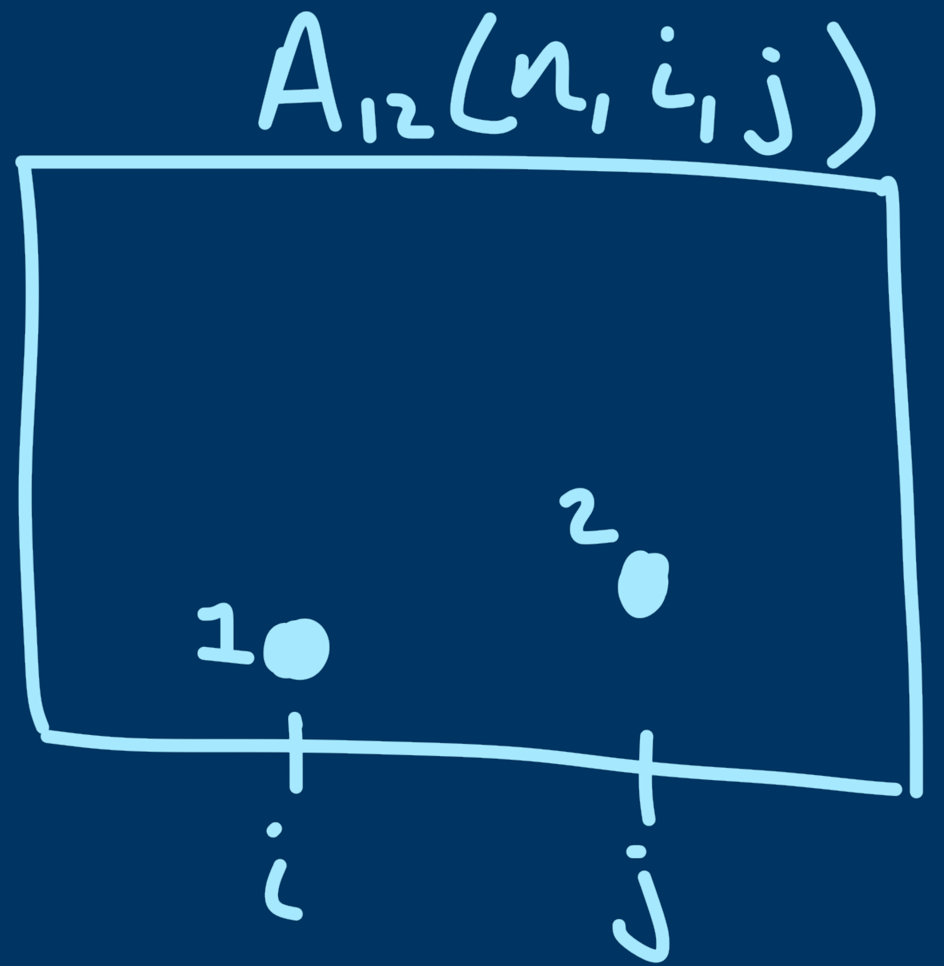
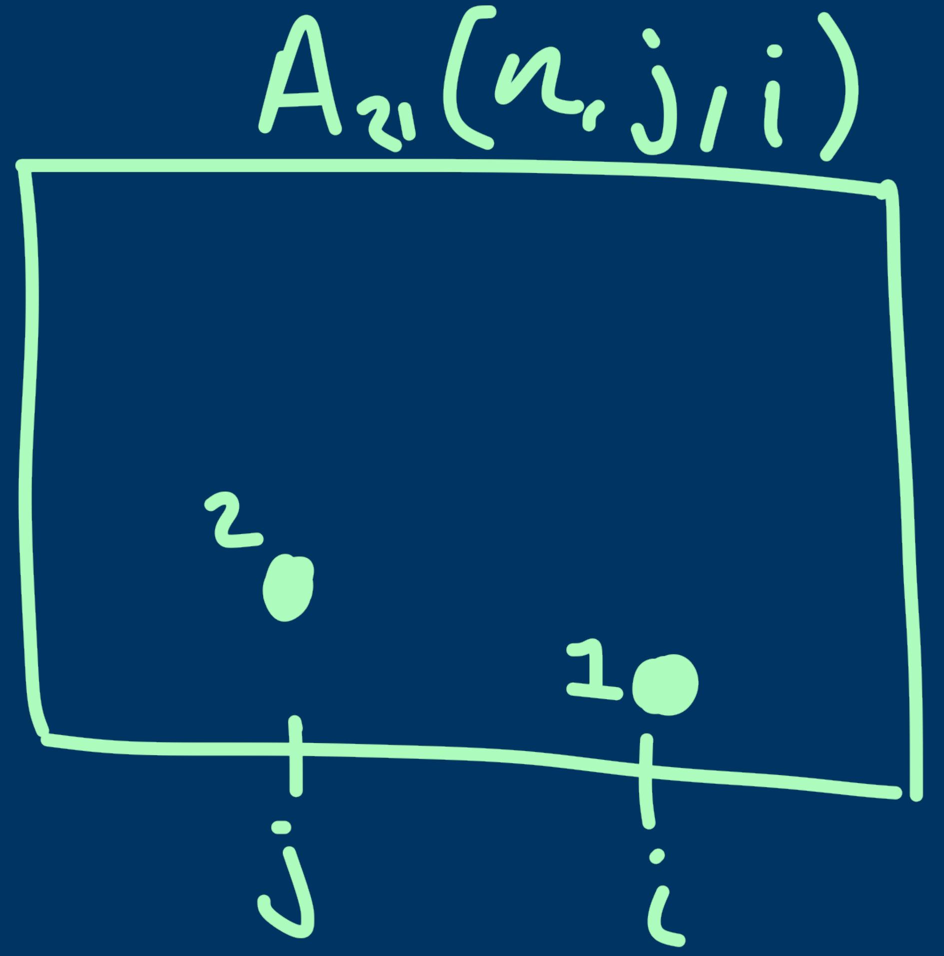
Define $A_{12}(n, i, j) := \{\pi \in A(n) : \pi(i) = 1,\ \pi(j) = 2,\ i < j\}$
and $A_{21}(n, j, i) := \{\pi \in A(n) : \pi(i) = 1,\ \pi(j) = 2,\ i > j\}$





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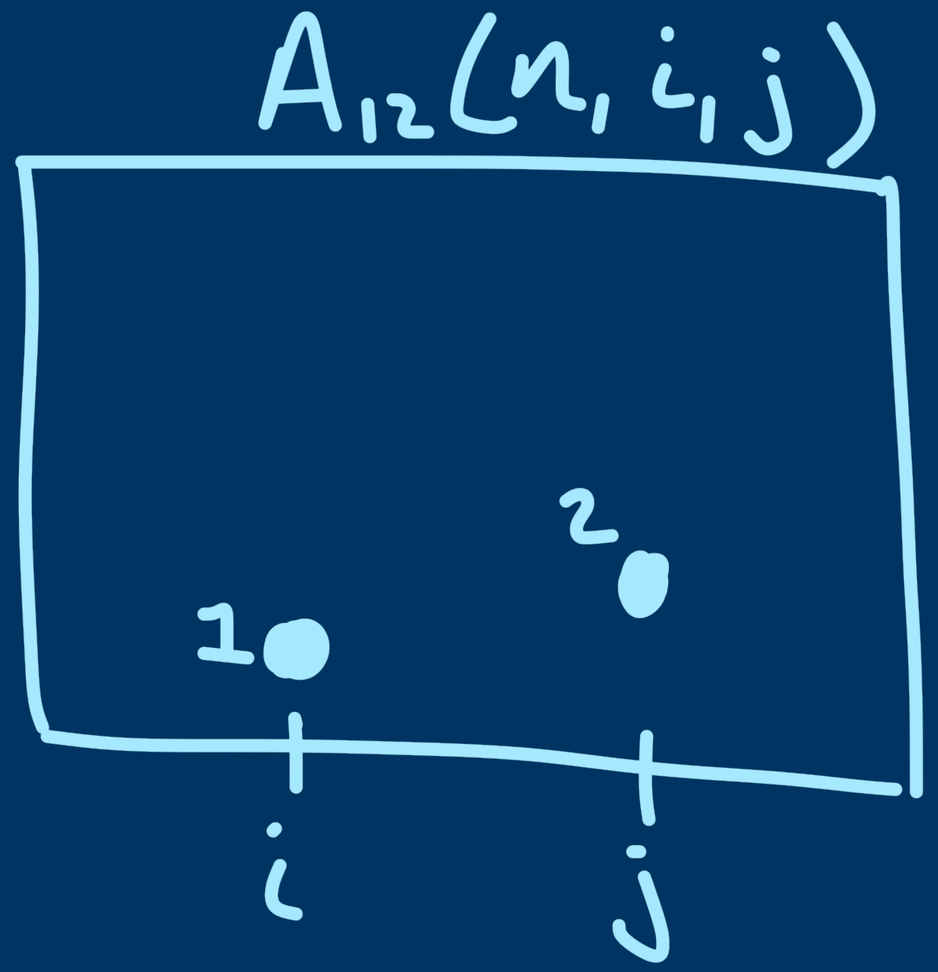

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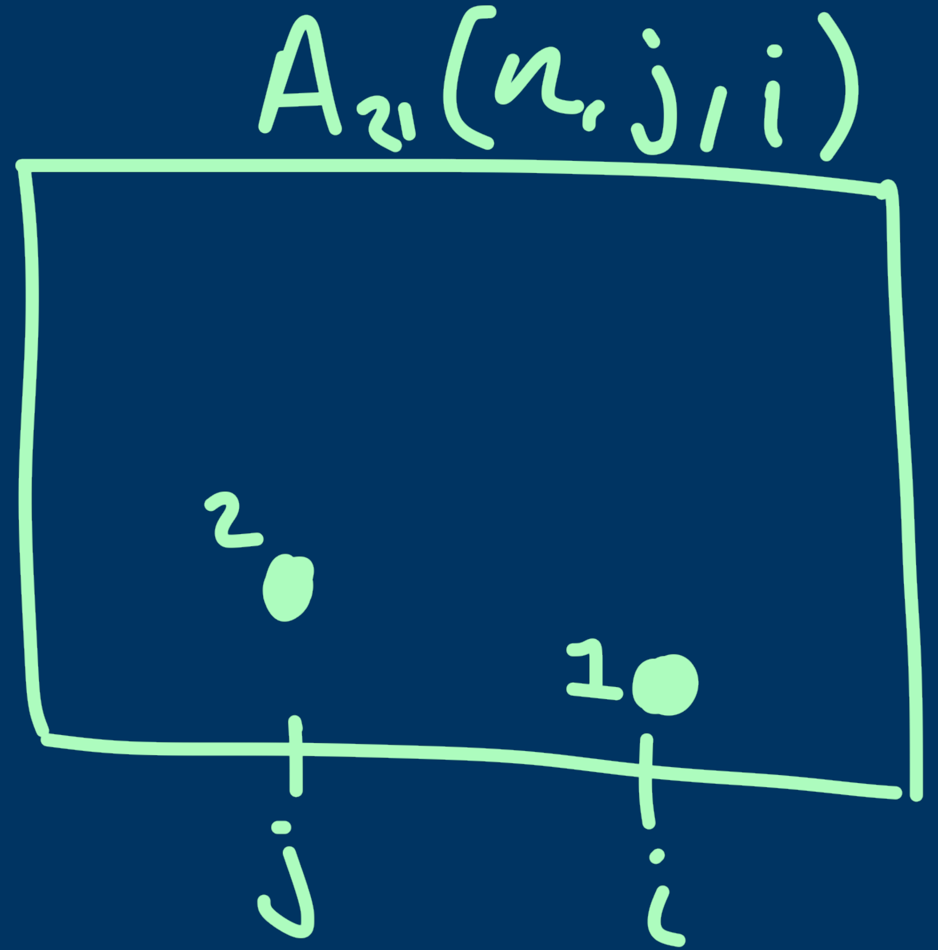

Obviously $$A_1(n, i) = \left(\bigcup_{j=1}^{i-1} A_{21}(n, j, i) \right) \cup \left(\bigcup_{j=i+1}^{n} A_{12}(n, i, j) \right).$$

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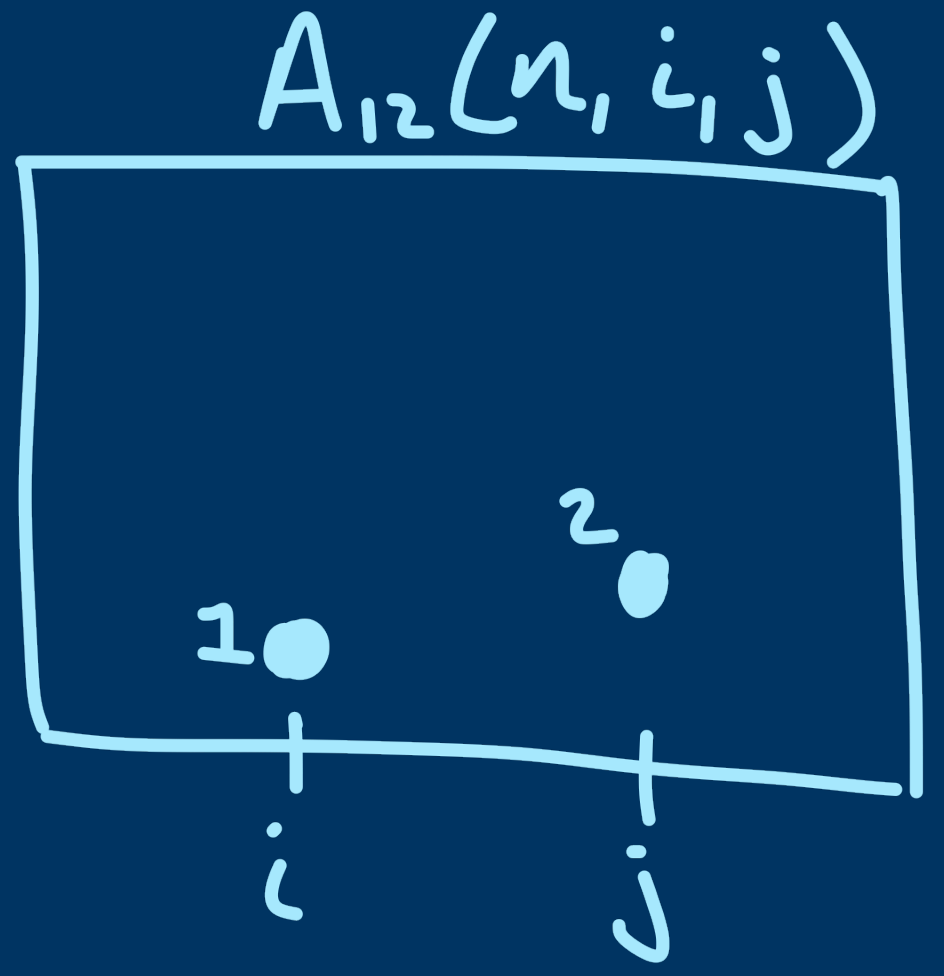

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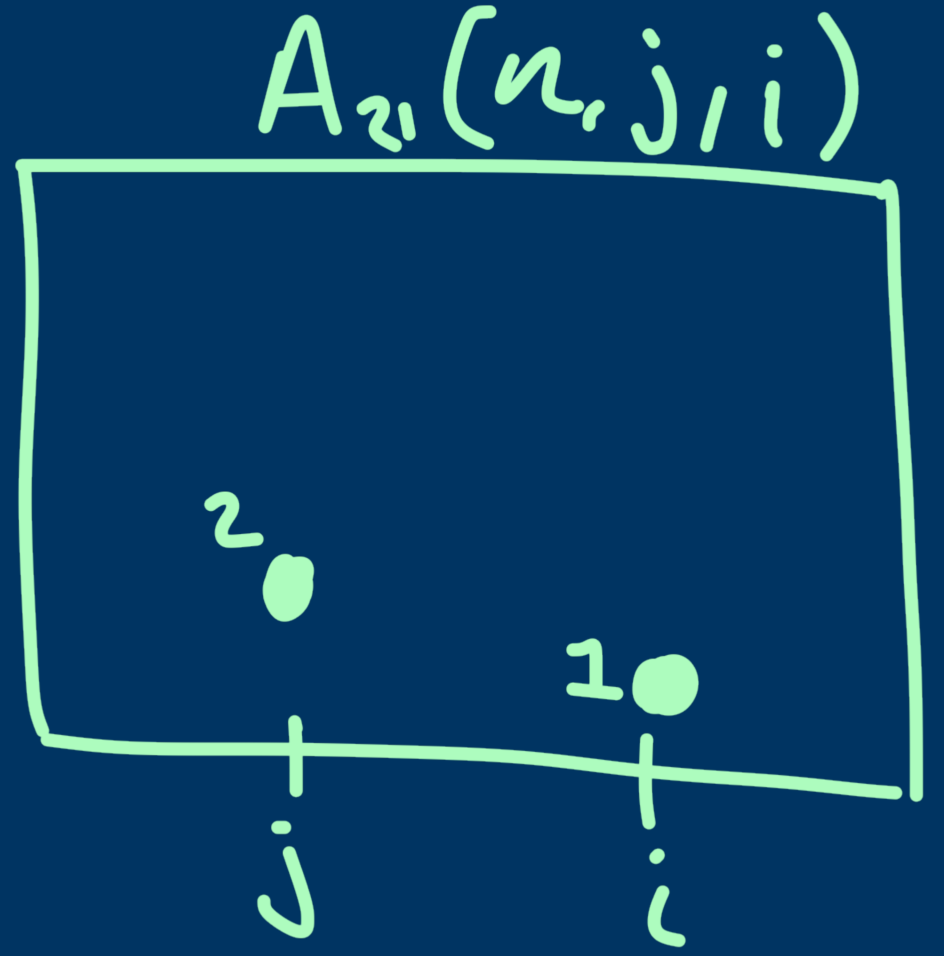

We could do this forever...

Enumeration Schemes

"The Most Trivial Non-Trivial Example" – Av(123)

Claim 1: $|A_{21}(n, j, i)| = |A_1(n-1, j)|$

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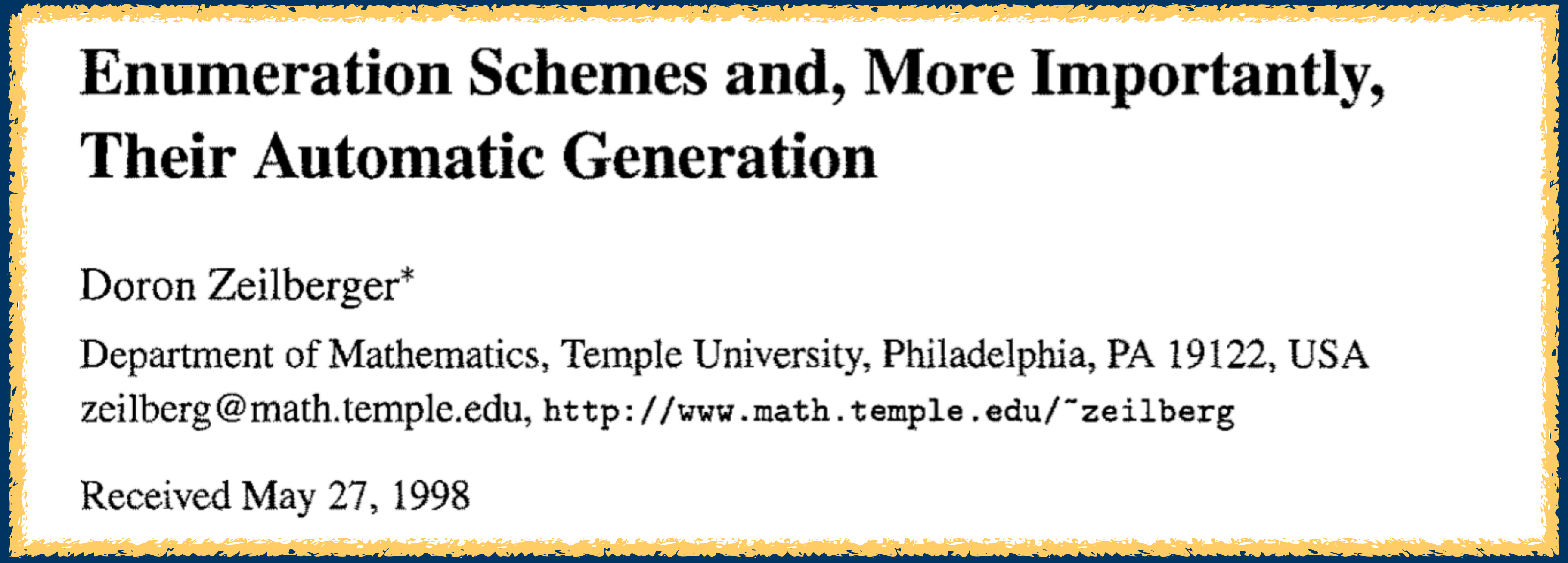
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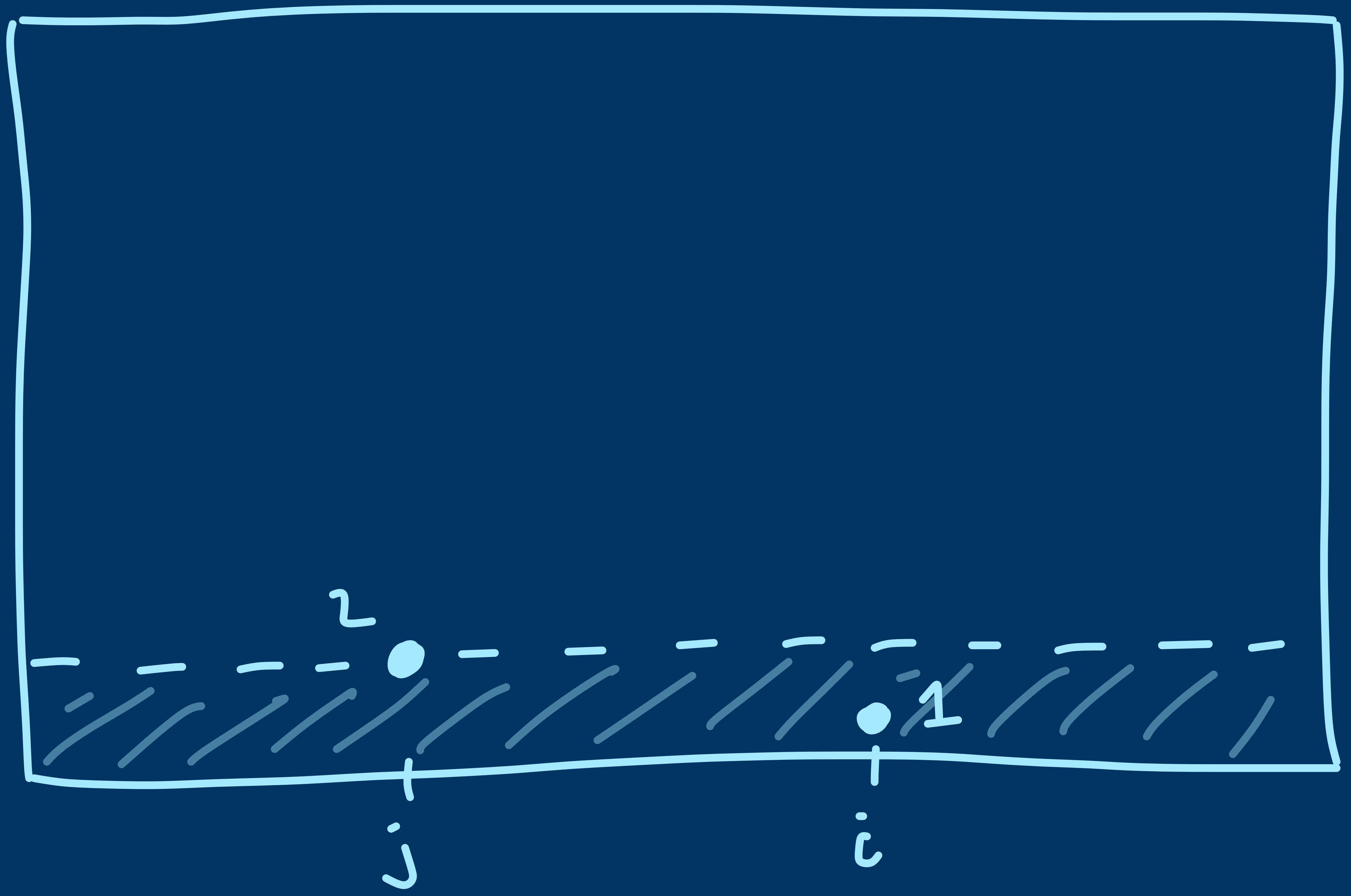

Let π be any permutation with $\pi(j)=2$, $\pi(i)=1$, $j<i$.

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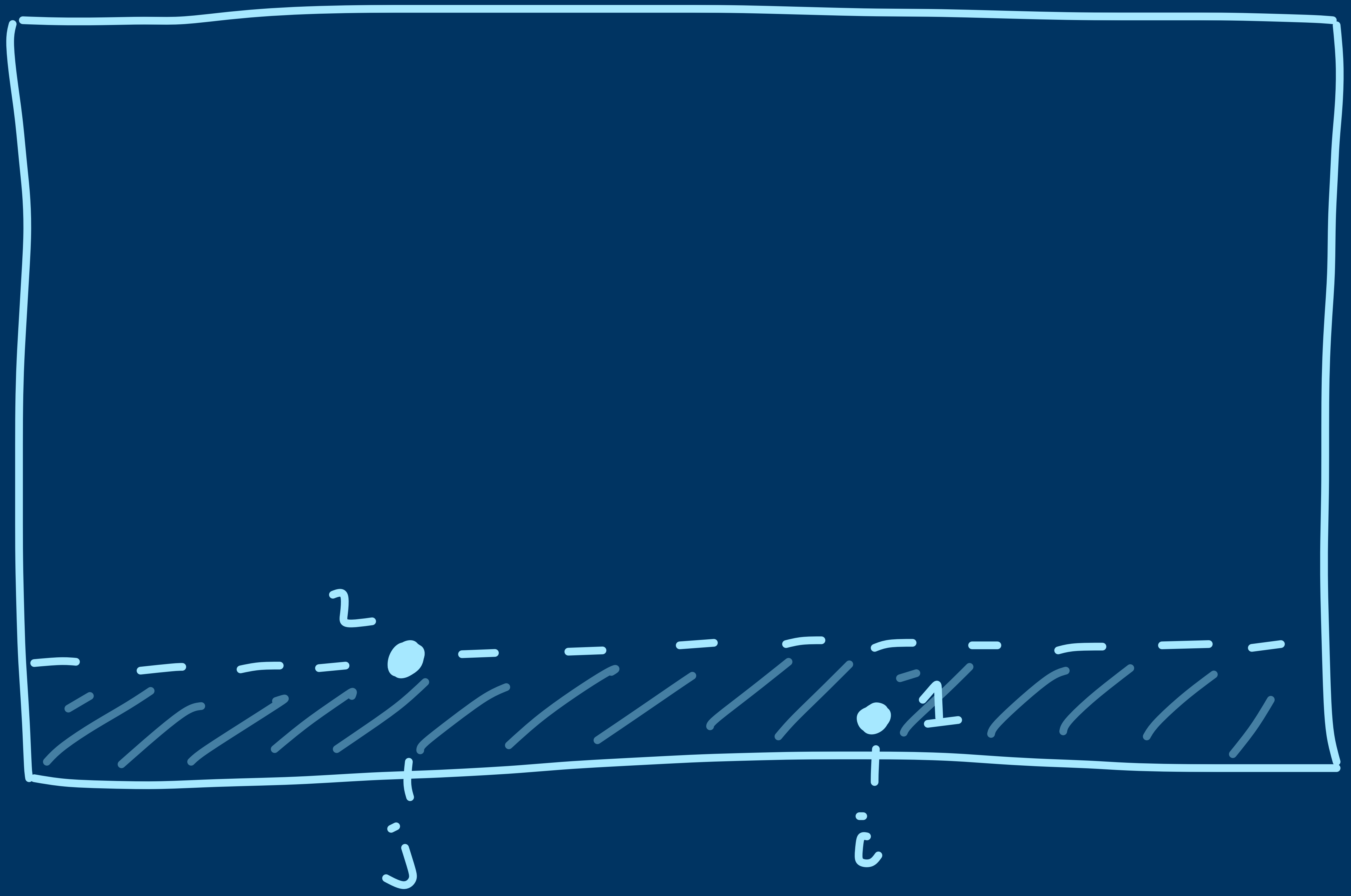

Let π be any permutation with $\pi(j) = 2$, $\pi(i) = 1$, $j < i$.

π contains 123
iff
$\pi - \pi(i)$ contains 123

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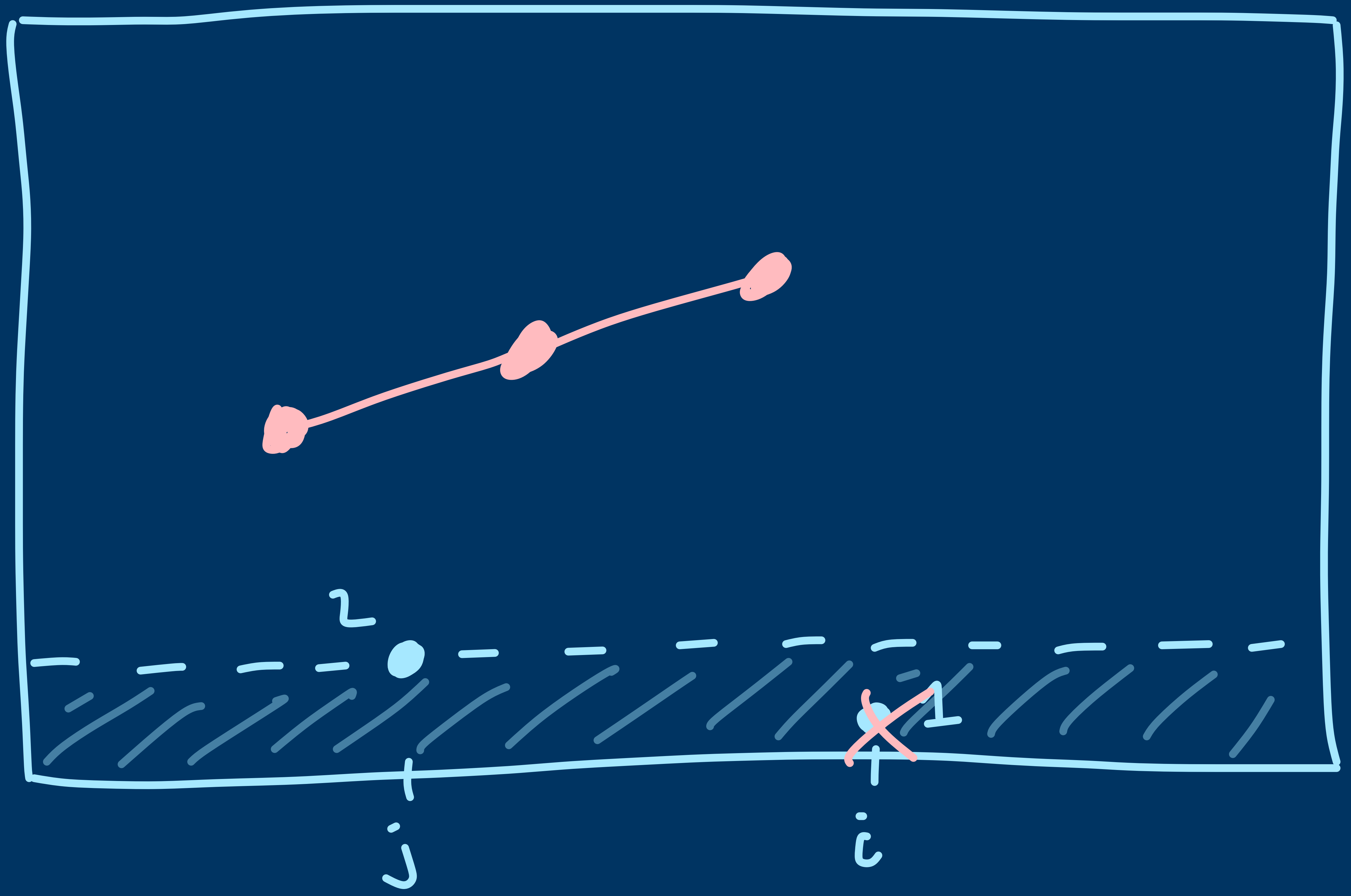

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Any 123 that doesn't involve $\pi(i)$ is obviously still in $\pi - \pi(i)$.

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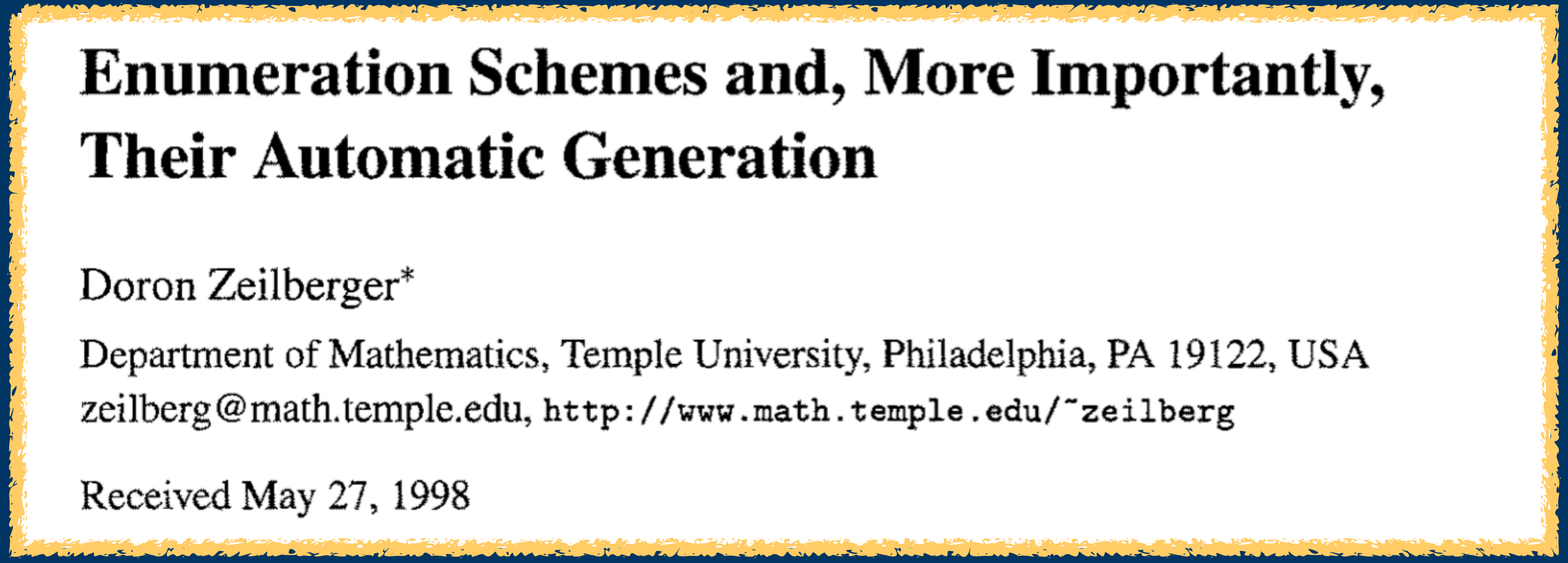
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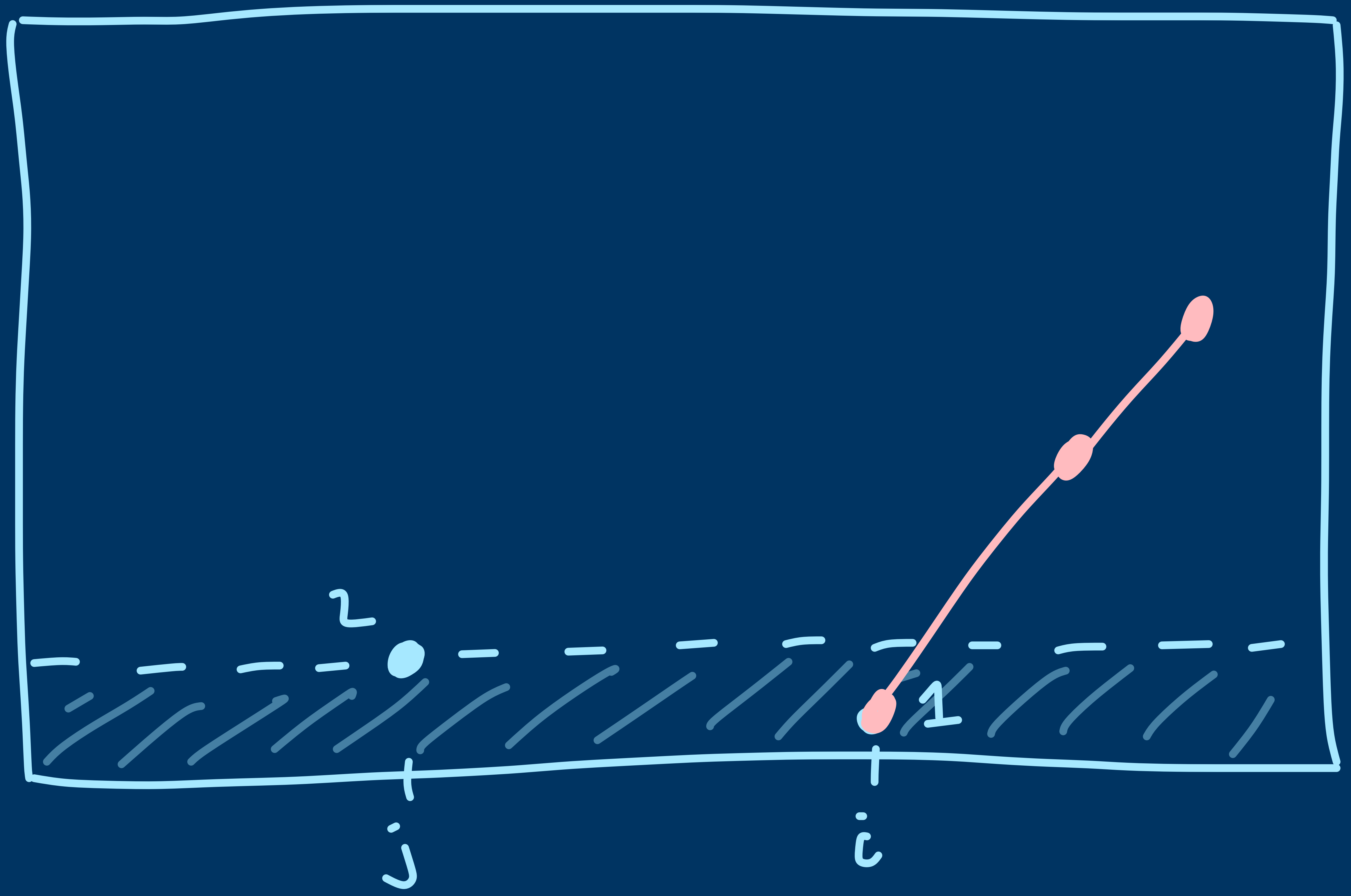

π contains 123
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If an occurrence of 123 involves $\pi(i)$, it must be the 1.

$\pi - \pi(i)$ has a 123 with $\pi(j)$ substituting for $\pi(i)$.

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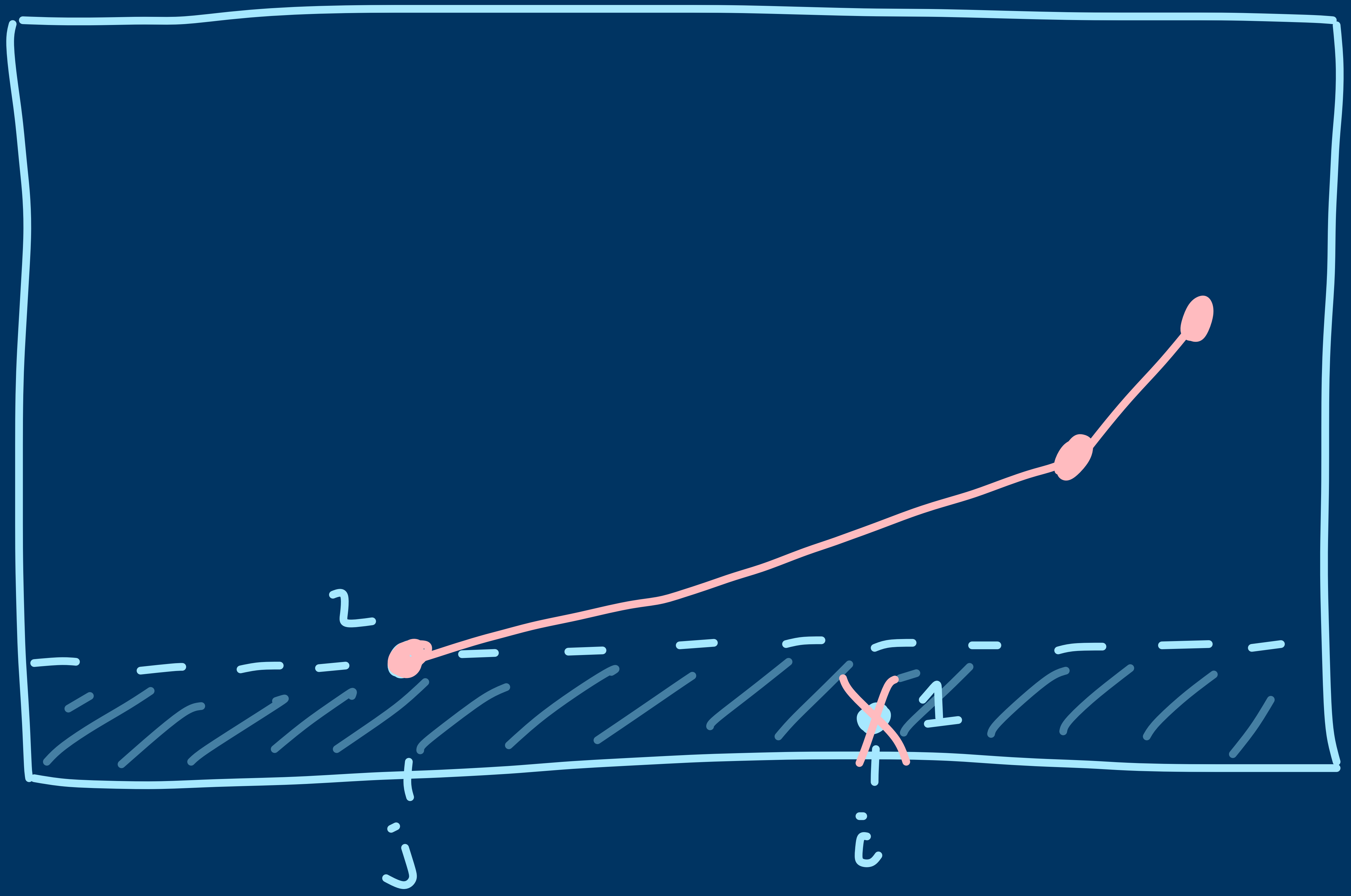

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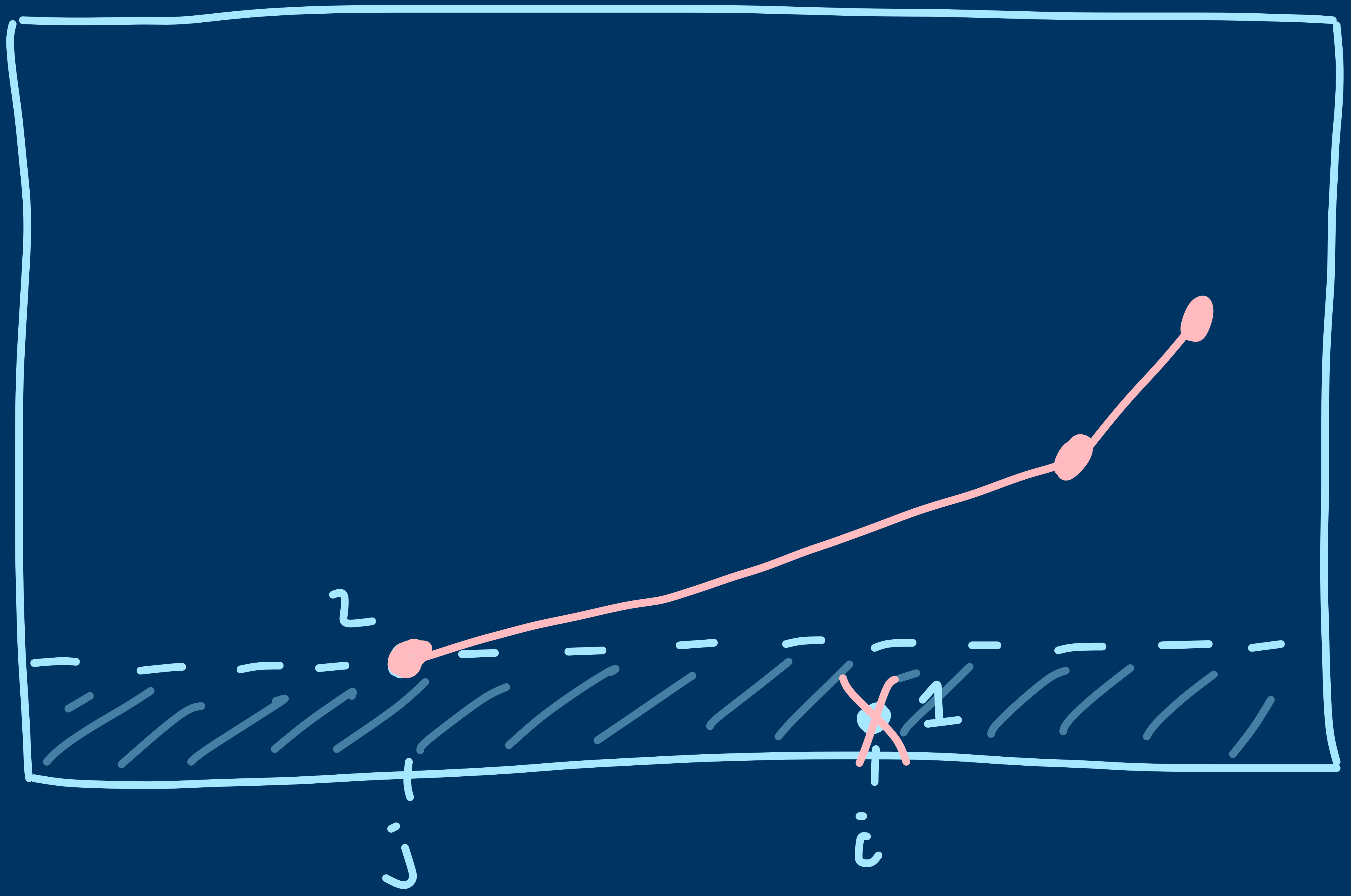

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So there is a bijection between $A_{21}(n,j,i)$ and $A_1(n-1,j)$.

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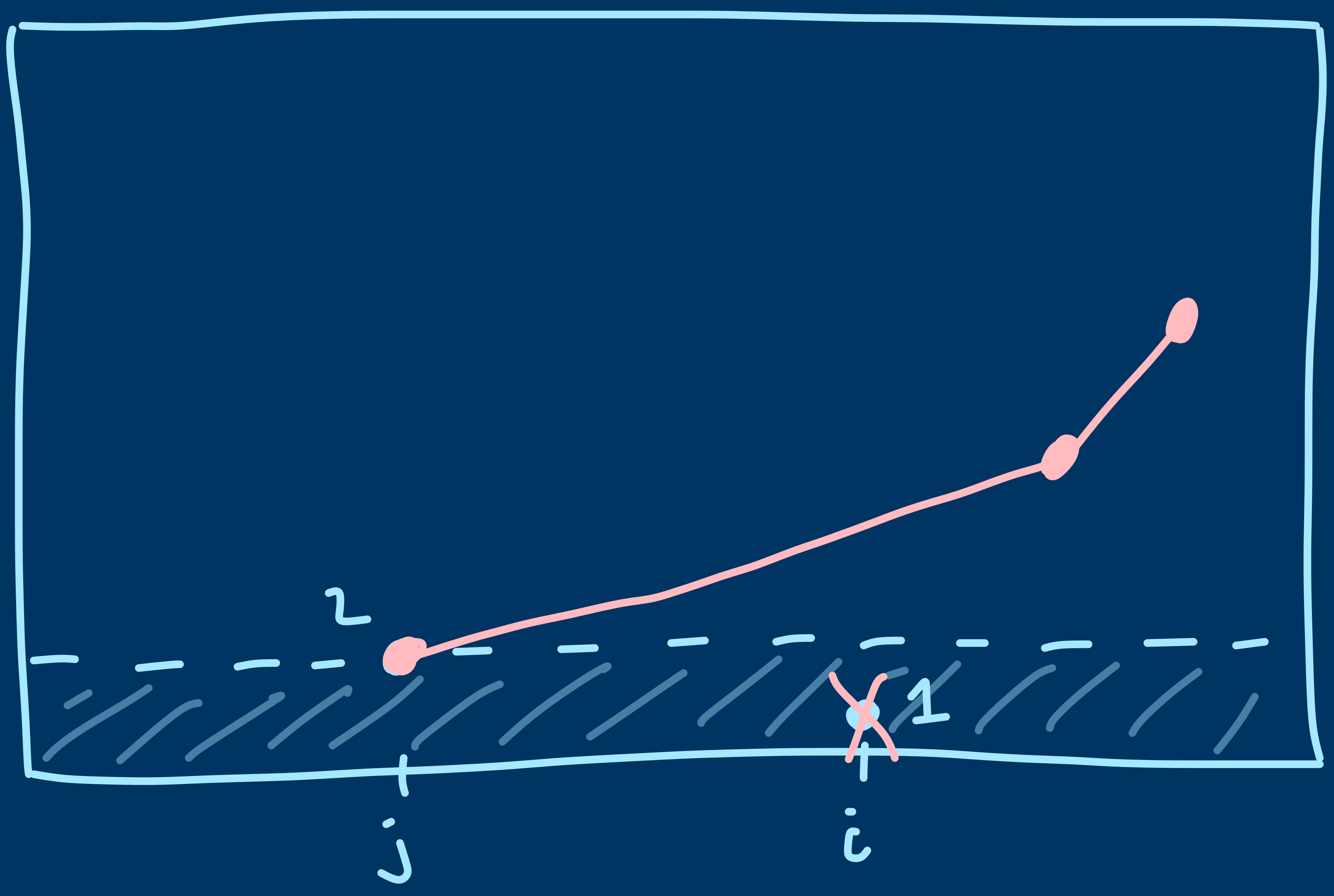

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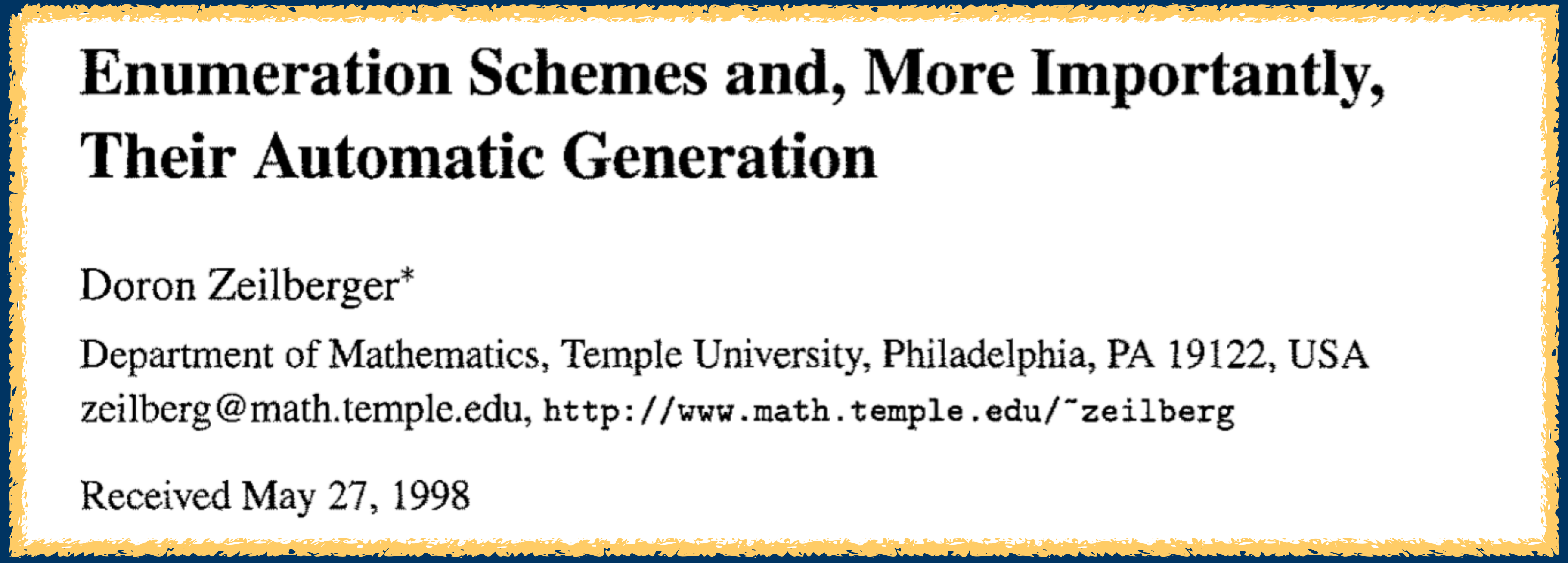

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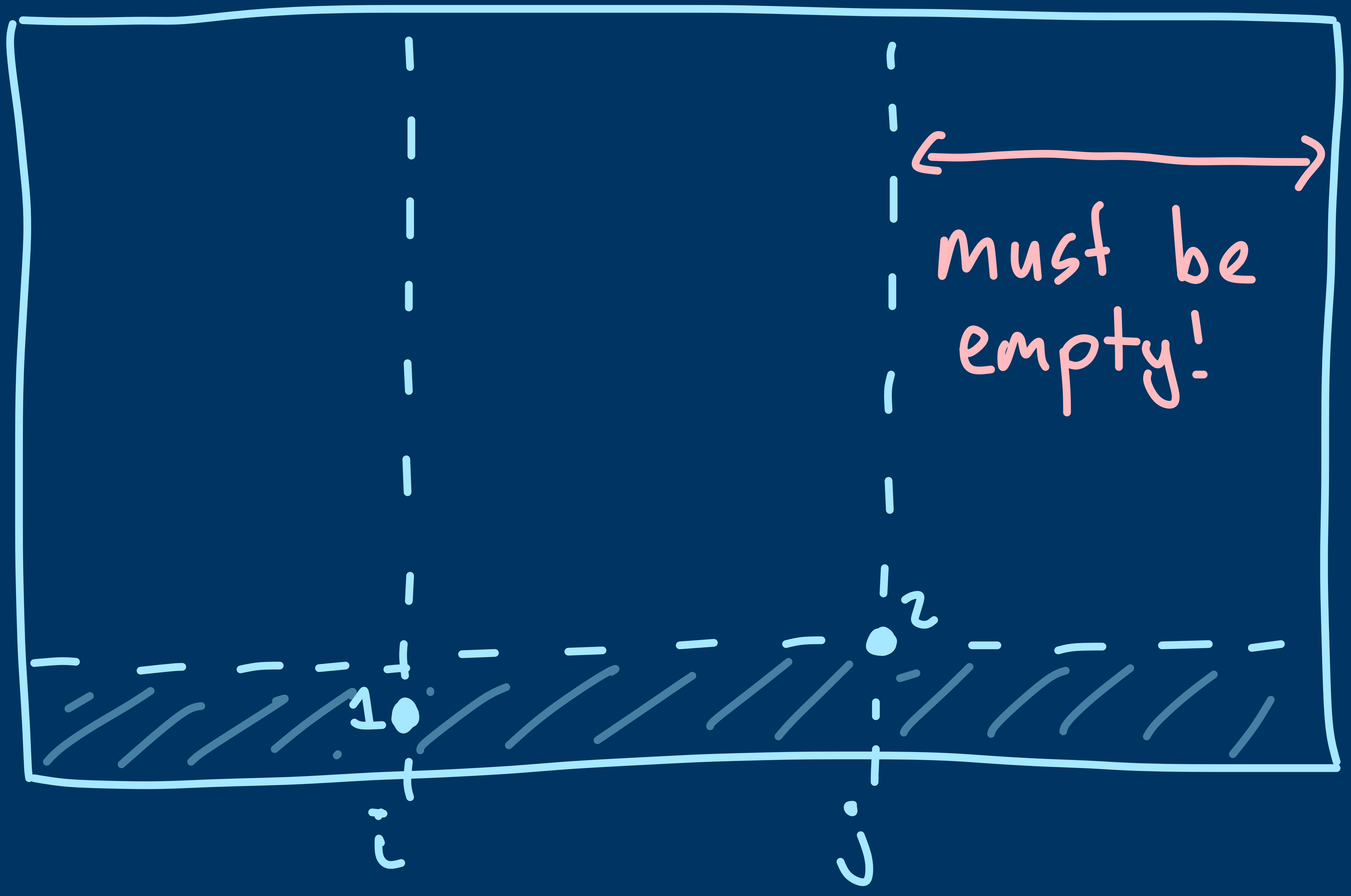
Claim 2: $|A_{12}(n,i,j)| = \begin{cases} 0, & j < n \\ |A_1(n-1,i)|, & j = n \end{cases}$



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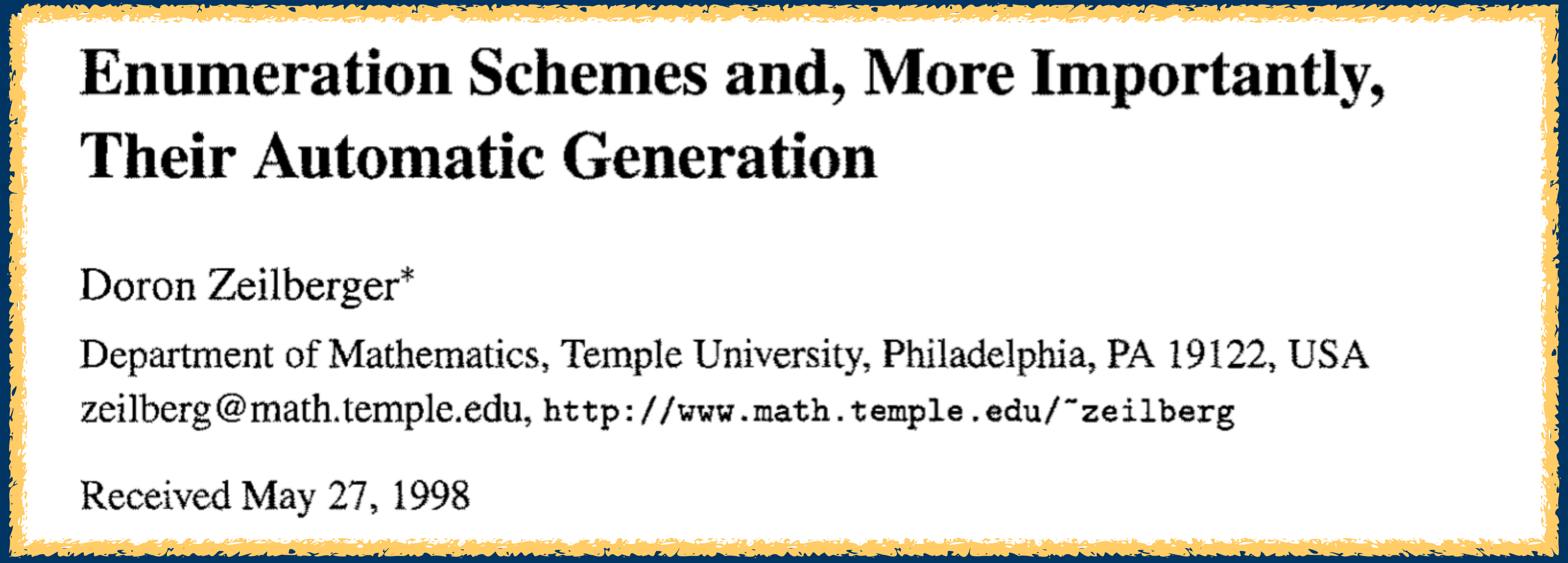
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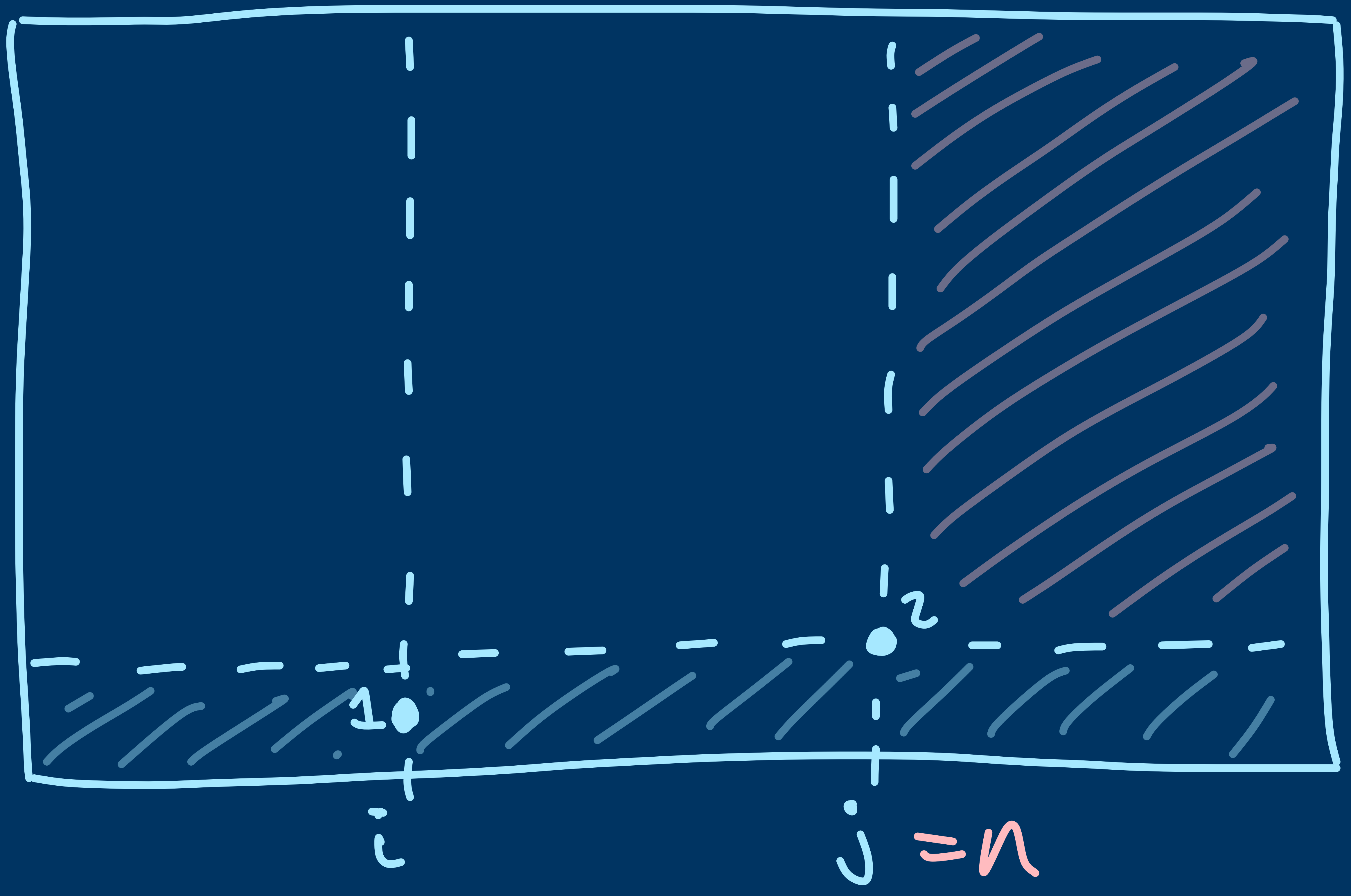

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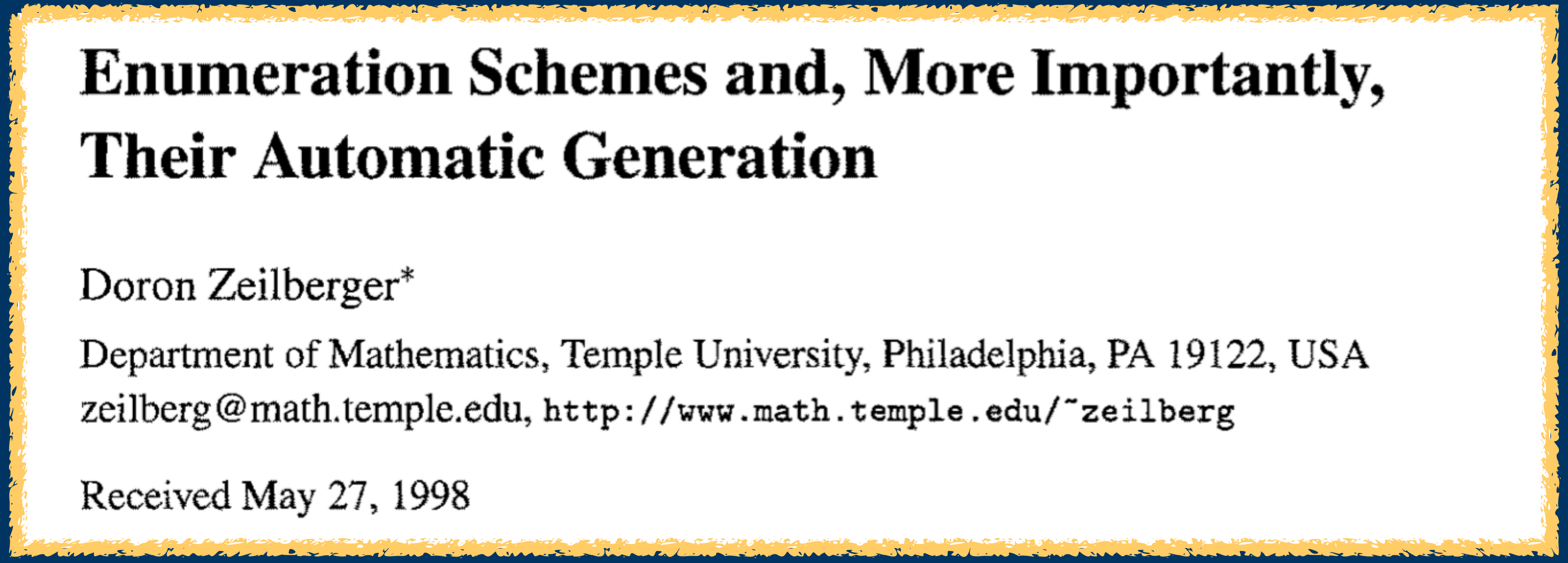
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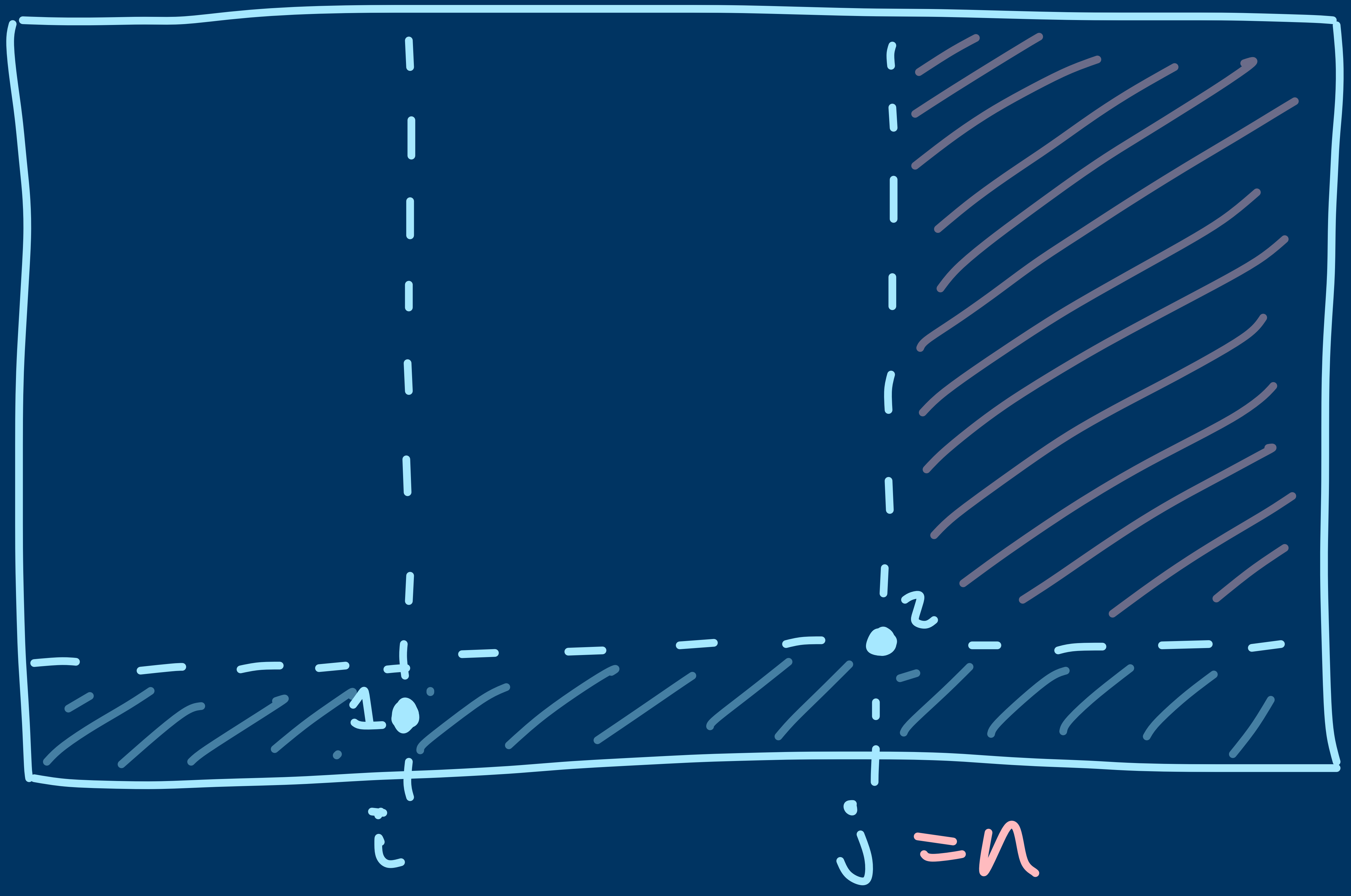

$\Rightarrow A_{12}(n,i,j) = \emptyset$ when $j < n$.

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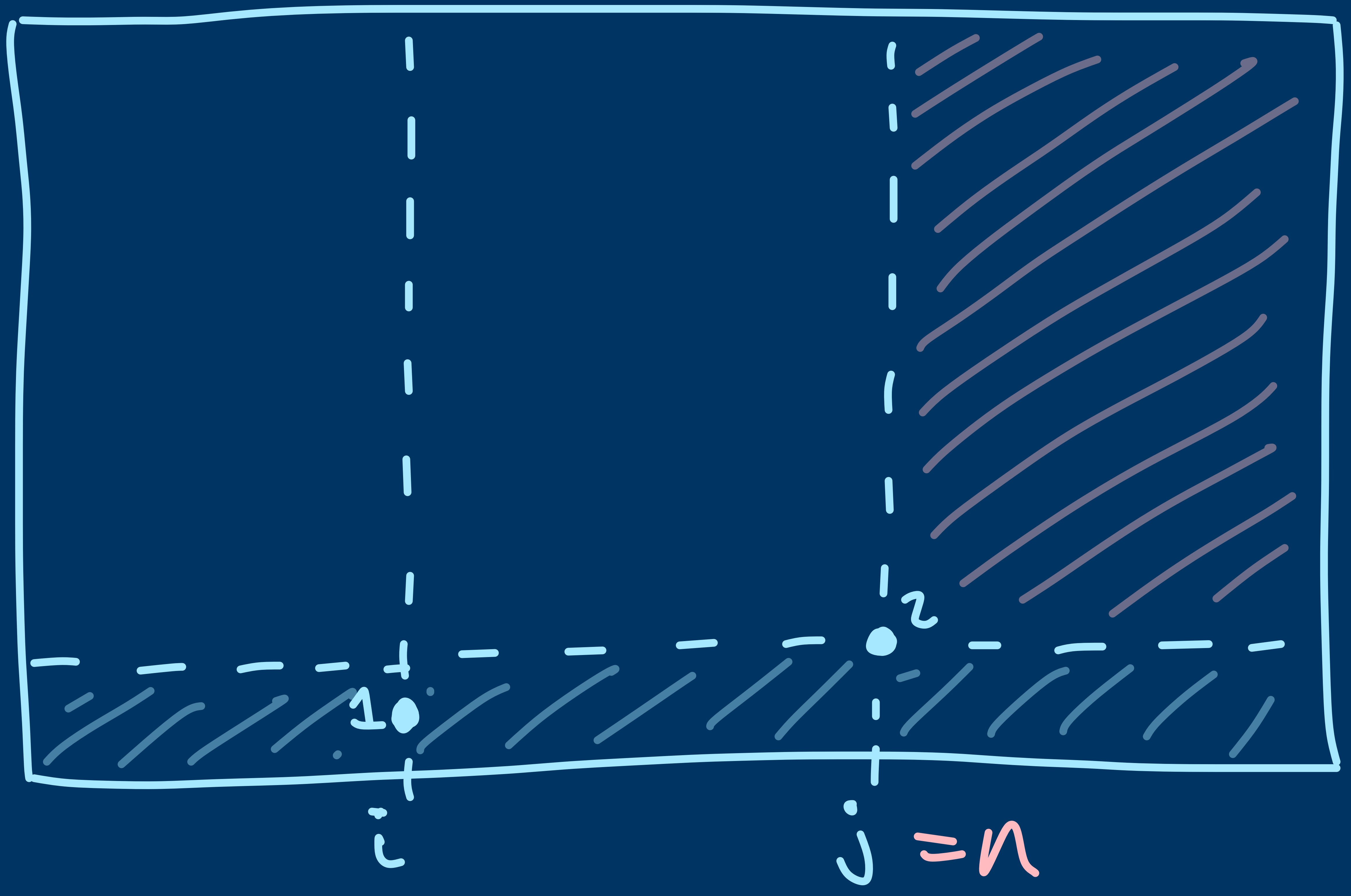

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When $j=n$, $\pi(j)$ can be deleted without destroying any 123 patterns.

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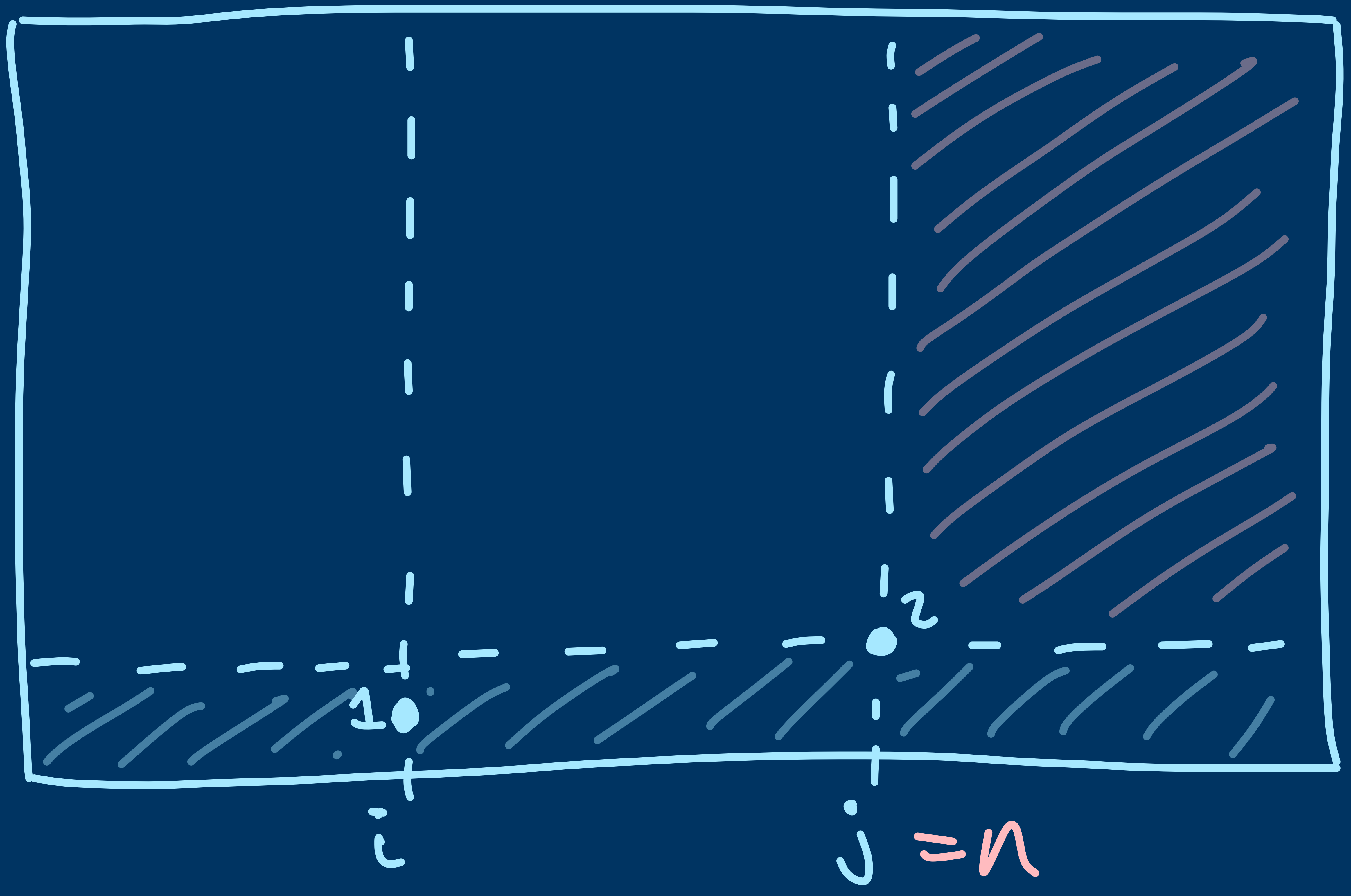

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$\Rightarrow A_{12}(n,i,n) \cong A_1(n-1,i)$

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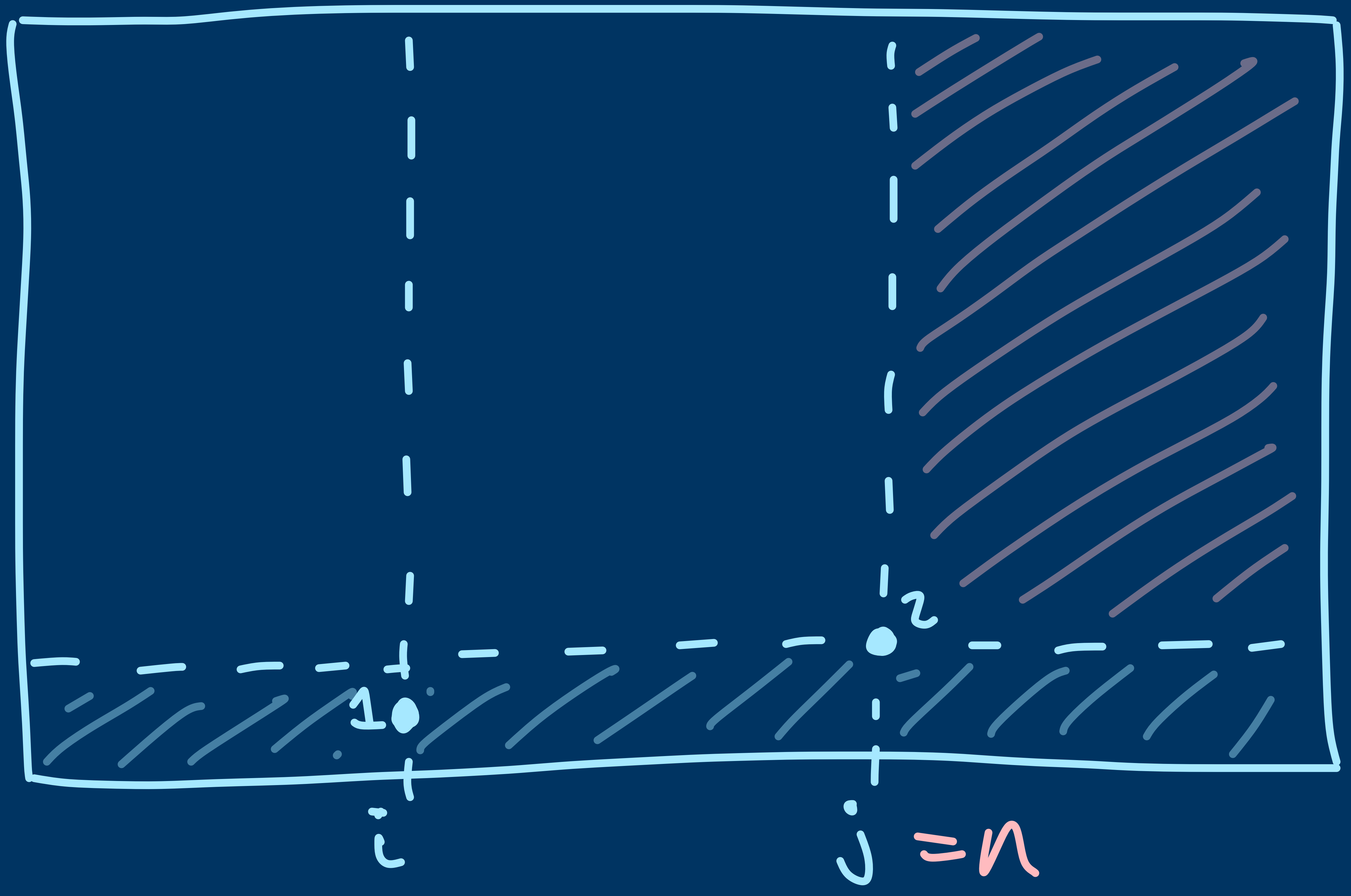
Claim 2: $|A_{12}(n,i,j)| = \begin{cases} 0, & j < n \\ |A_1(n-1,i)|, & j = n \end{cases}$ ✓



$\Rightarrow A_{12}(n,i,j) = \emptyset$ when $j < n$.

When $j = n$, $\pi(j)$ can be deleted without destroying any 123 patterns.

$\Rightarrow A_{12}(n,i,n) \cong A_1(n-1,i)$

Enumeration Schemes

"The Most Trivial Non-Trivial Example" – Av(123)

Enumeration Schemes and, More Importantly, Their Automatic Generation

Doron Zeilberger*
Department of Mathematics, Temple University, Philadelphia, PA 19122, USA
zeilberg@math.temple.edu, http://www.math.temple.edu/~zeilberg

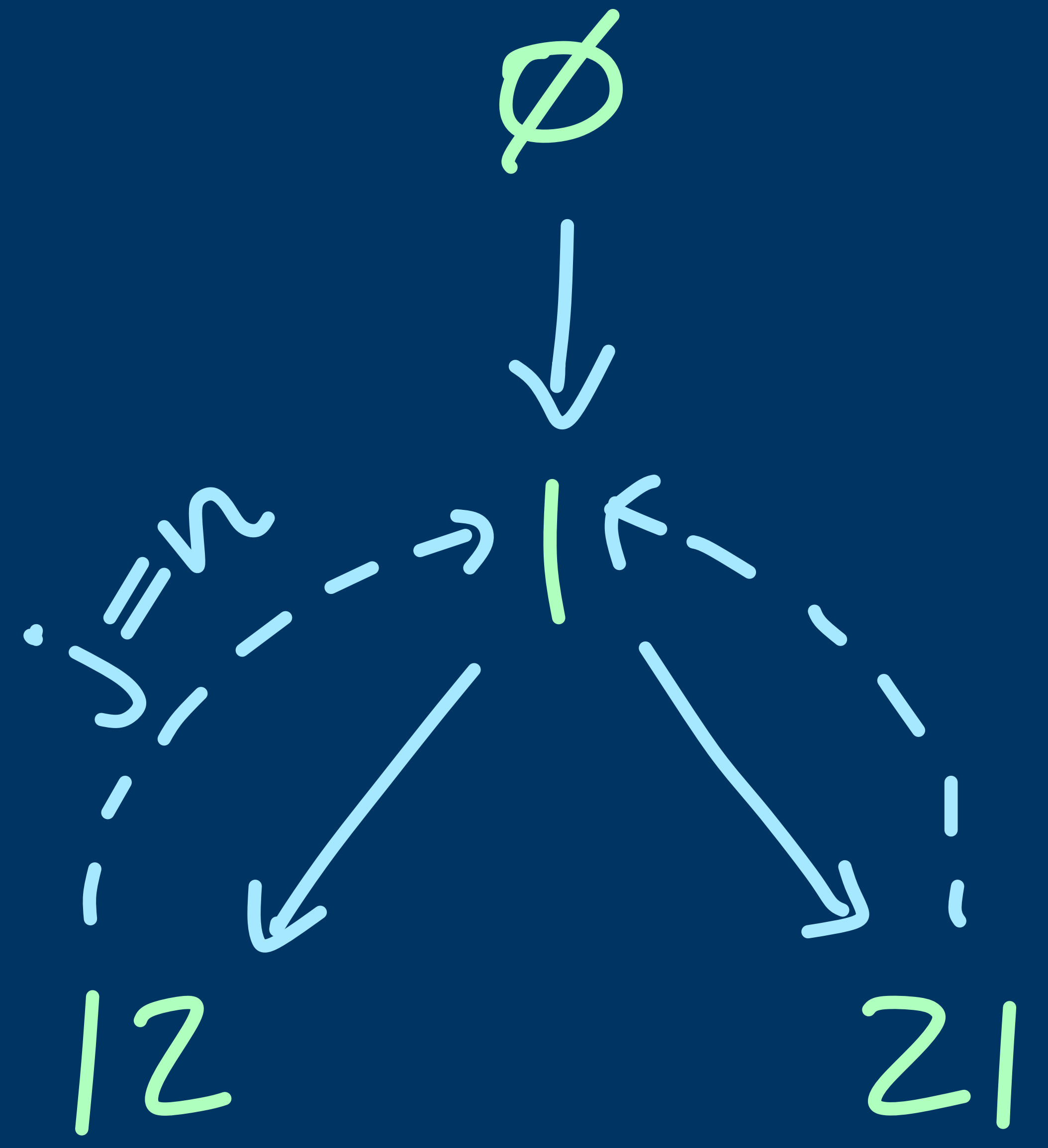
Received May 27, 1998

$$A(n) = \bigcup_{i=1}^{n} A_1(n, i)$$

$$A_1(n, i) = \left(\bigcup_{j=1}^{i-1} A_{21}(n, j, i) \right) \cup \left(\bigcup_{j=i+1}^{n} A_{12}(n, i, j) \right)$$

$$A_{21}(n, j, i) \cong A_1(n-1, j)$$

$$A_{12}(n, i, j) \cong \begin{cases} \varnothing & , j < n \\ A_1(n-1, i), & j = n \end{cases}$$



Enumeration Schemes

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Big Picture:

- The computer splits the whole set $A(n)$ further and further based on the pattern formed by the bottom entries.

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- At each step it checks if any of the entries are "reversibly deletable". If so, this branch of the search tree doesn't need to be split further.

Enumeration Schemes

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- The computer splits the whole set $A(n)$ further and further based on the pattern formed by the bottom entries.
- At each step it checks if any of the entries are “reversibly deletable”. If so, this branch of the search tree doesn’t need to be split further.
- If all branches finish, we get an enumeration scheme, which gives us a *polynomial-time algorithm* to count the number of permutations of length n, but does not give us the *generating function*.

Enumeration Schemes

Zeilberger's method is:

Enumeration Schemes and, More Importantly, Their Automatic Generation

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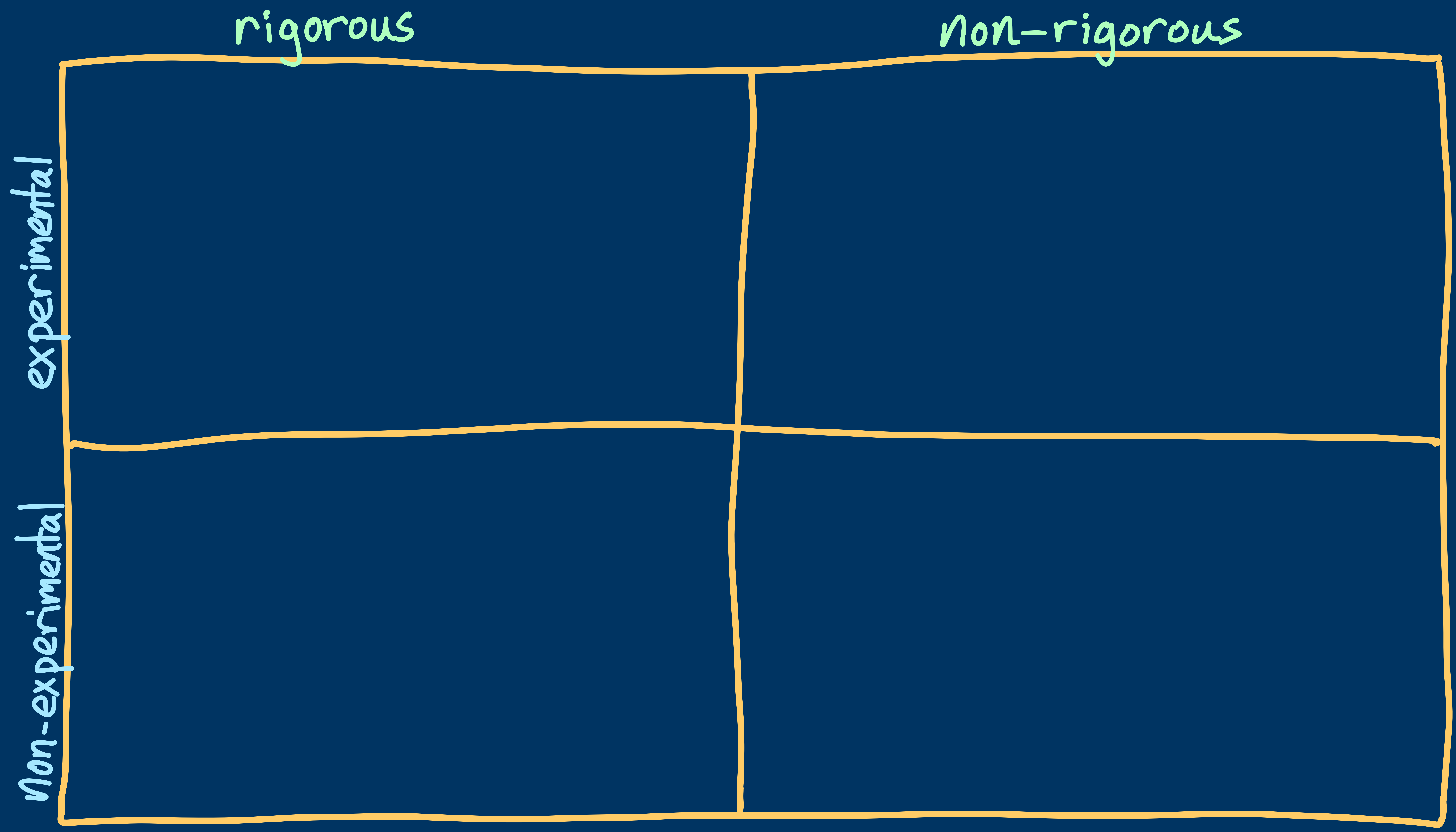
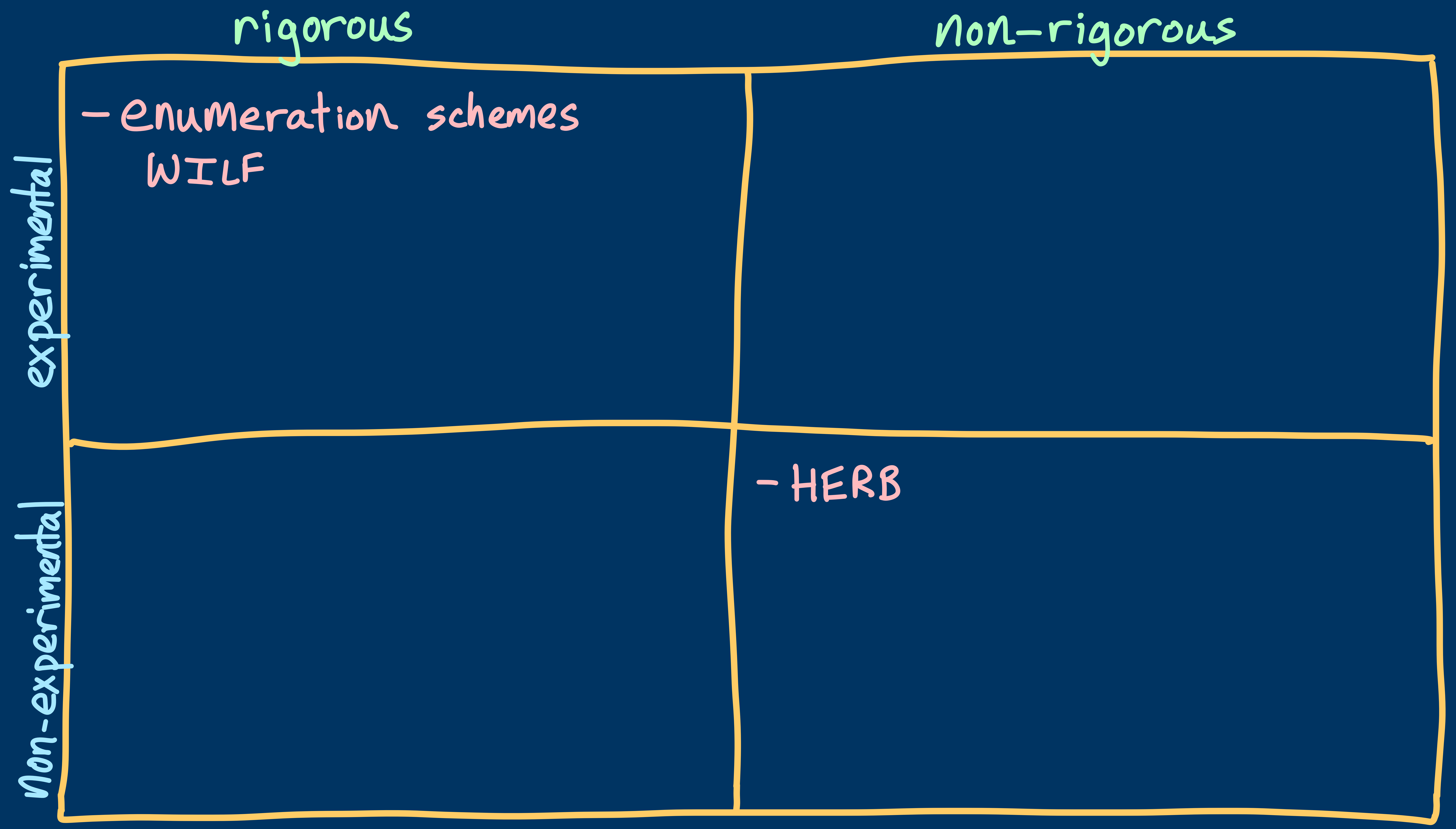
Experimental: when you "hit go", you don't know whether or not it will return an answer

Rigorous: if it does give an answer, it's guaranteed to be correct


rigorous
non-rigorous
experimental
non-experimental

	rigorous	non-rigorous
experimental	- enumeration schemes WILF	
non-experimental		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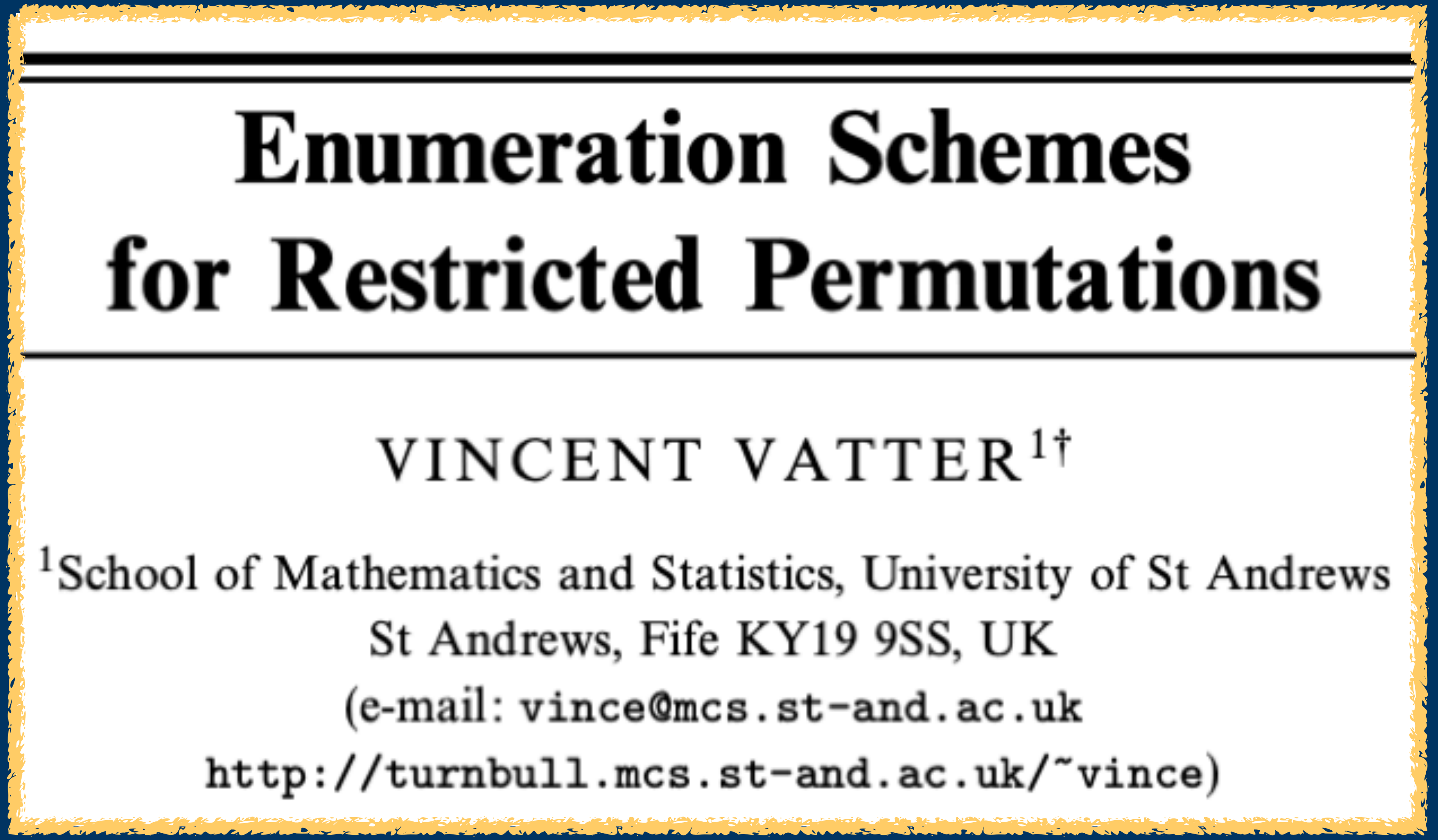
Enumeration Schemes — WILFPLUS

In 2007, Vince Vatter made the method more powerful by increasing the number of situations in which a point can be declared reversibly deletable.

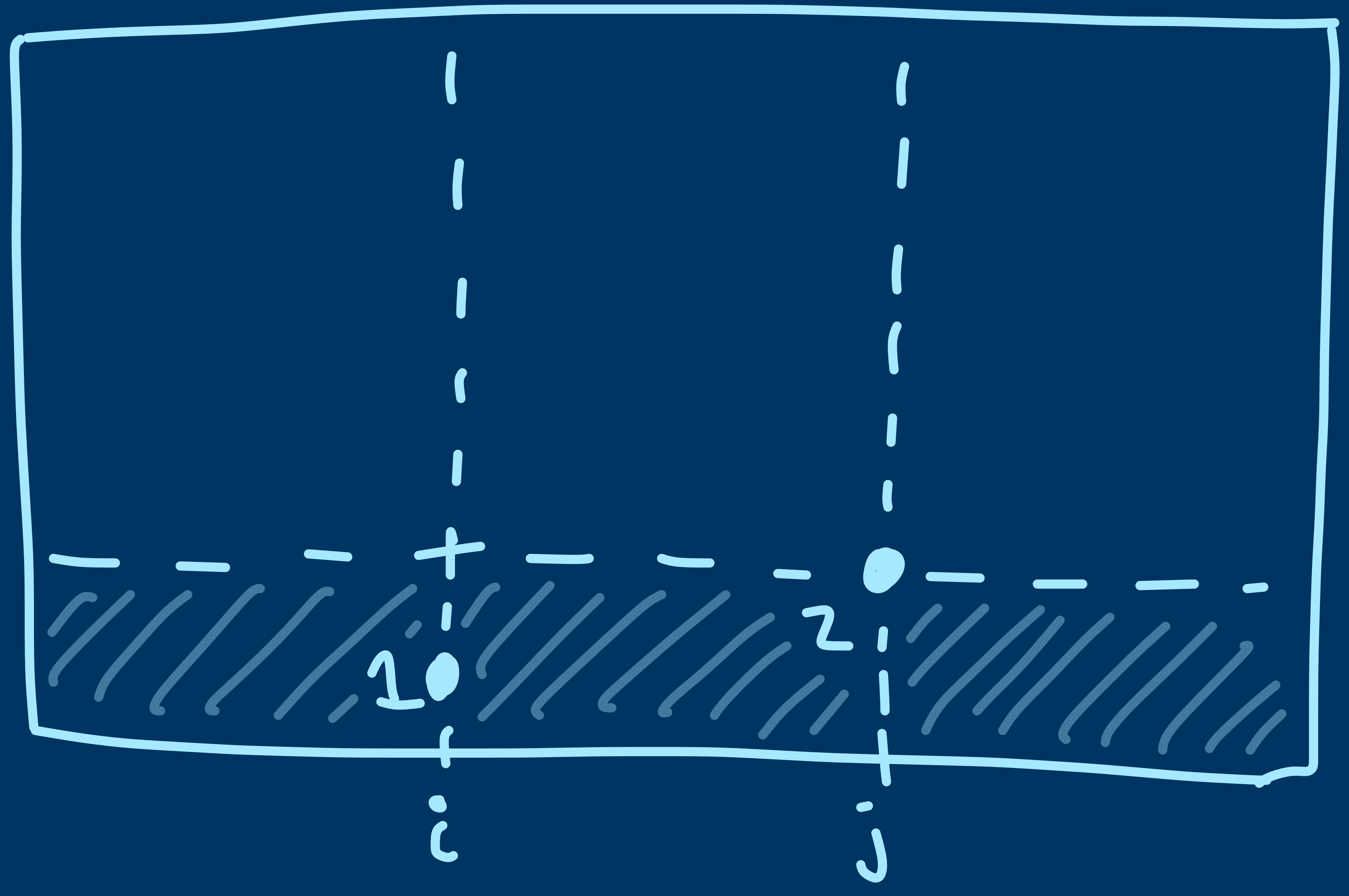


Enumeration Schemes for Restricted Permutations

VINCENT VATTER[1†]

[1]School of Mathematics and Statistics, University of St Andrews
St Andrews, Fife KY19 9SS, UK
(e-mail: vince@mcs.st-and.ac.uk
http://turnbull.mcs.st-and.ac.uk/~vince)

Av(1342,1432)



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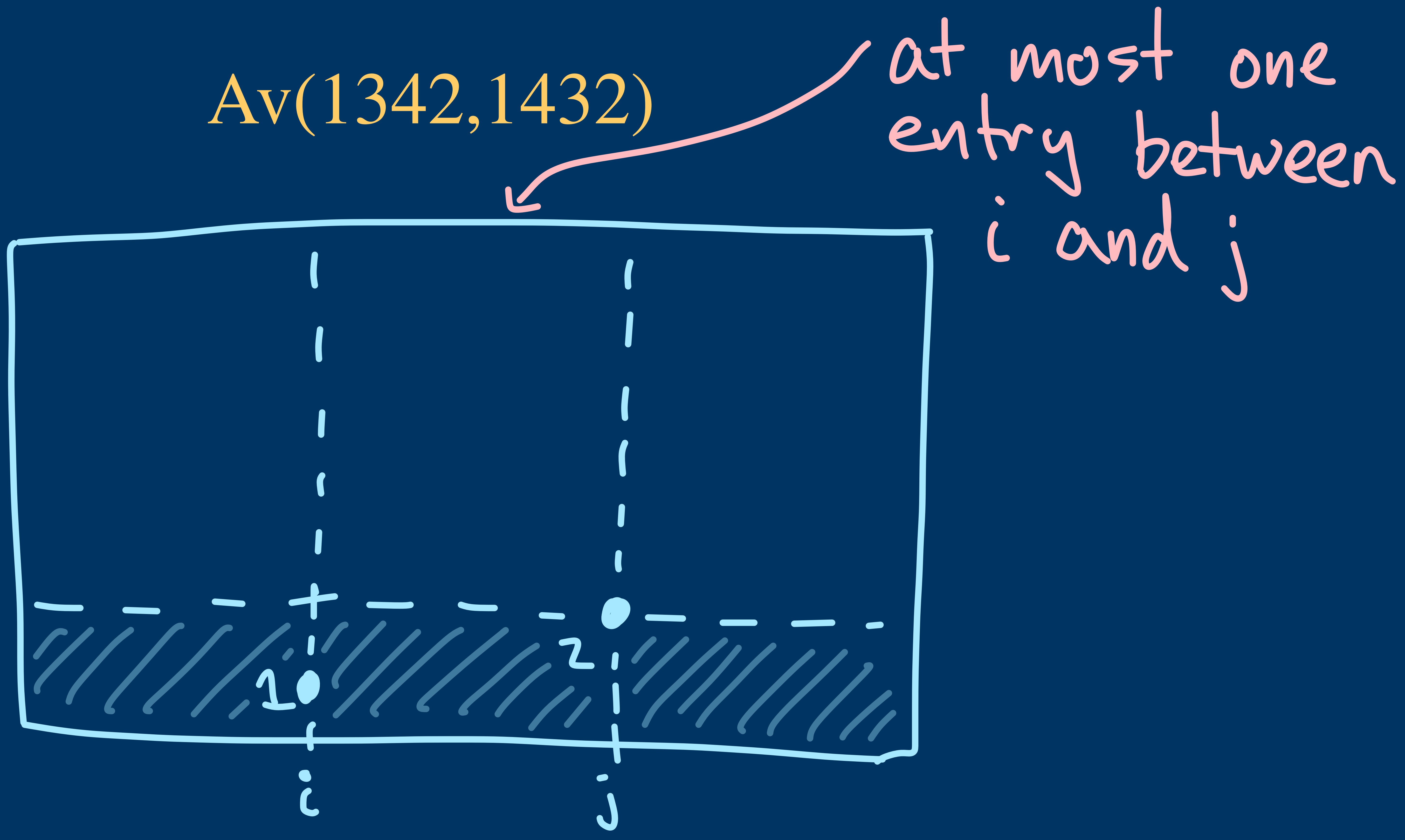

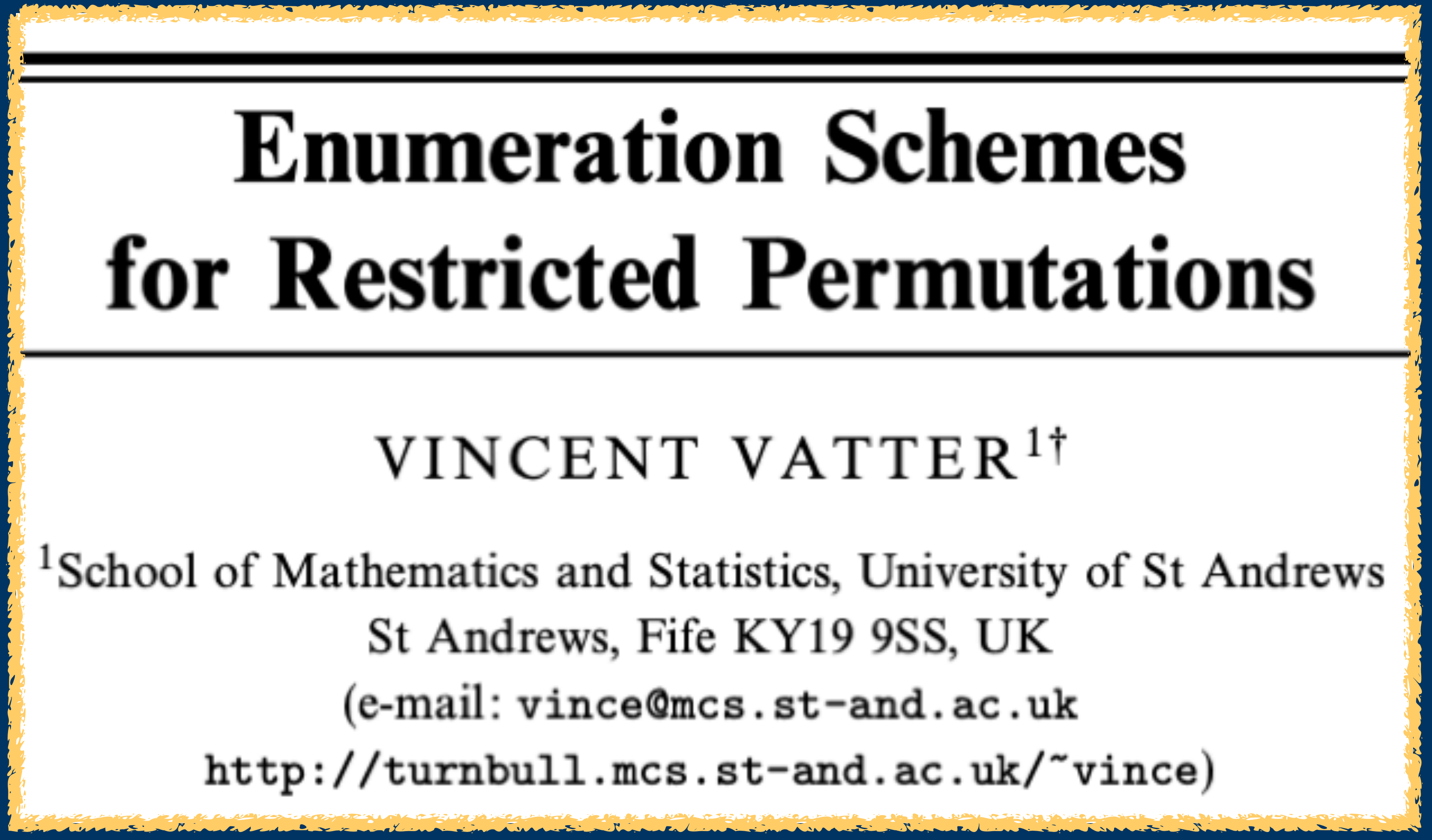

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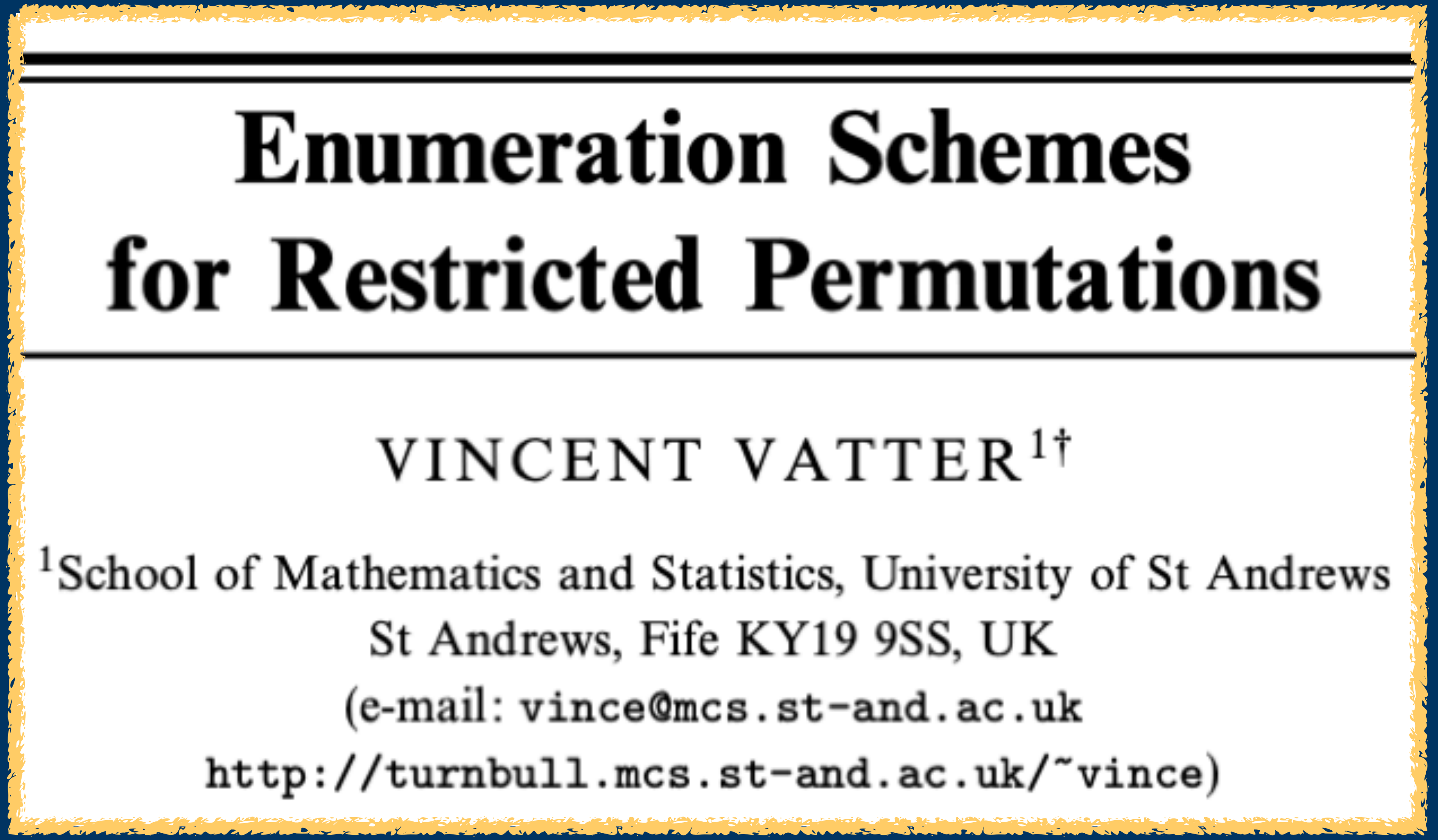
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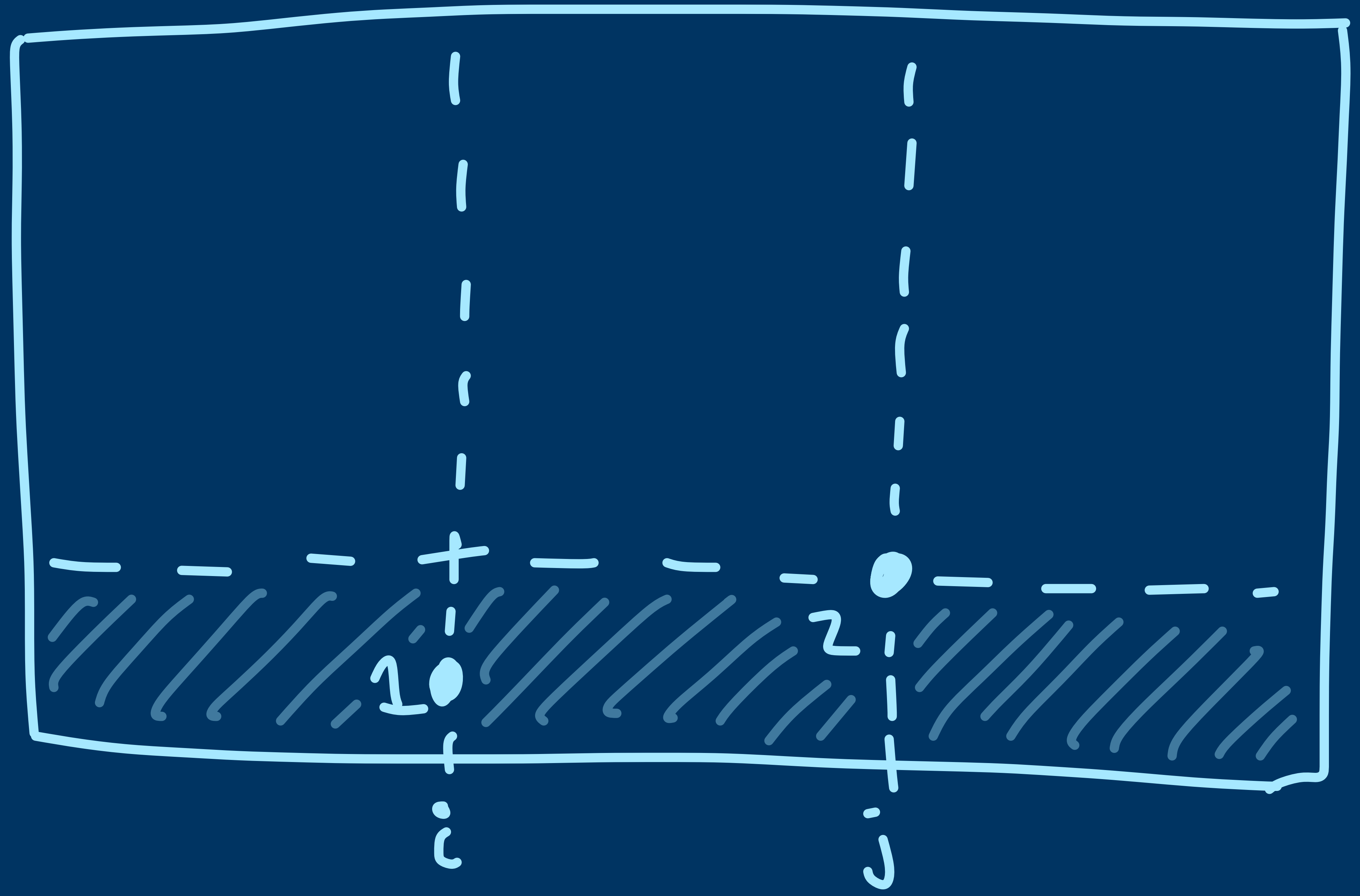
Av(1342,1432)

at most one entry between i and j



Knowing this: the entry $\pi(j)$ can be deleted.

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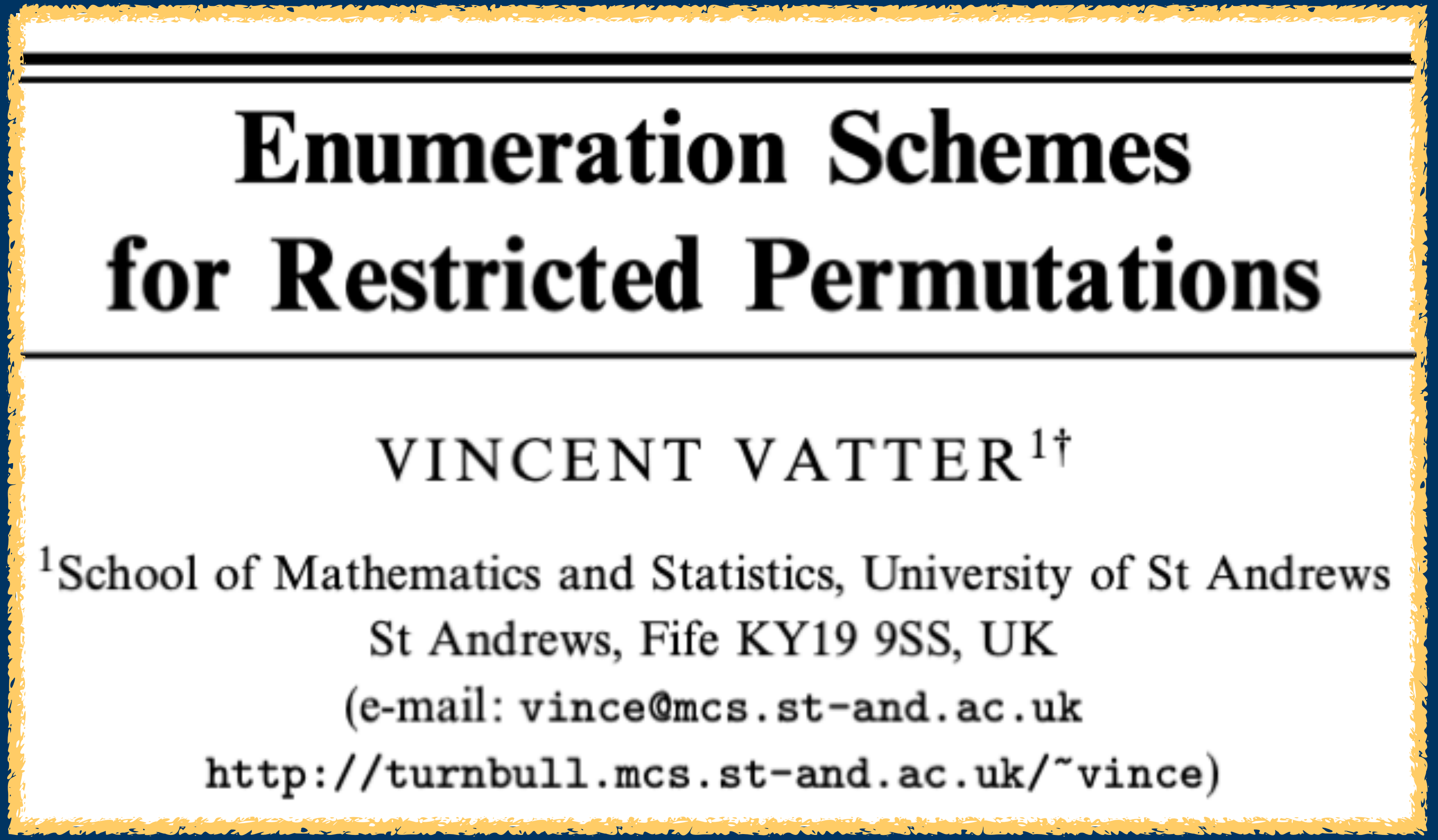
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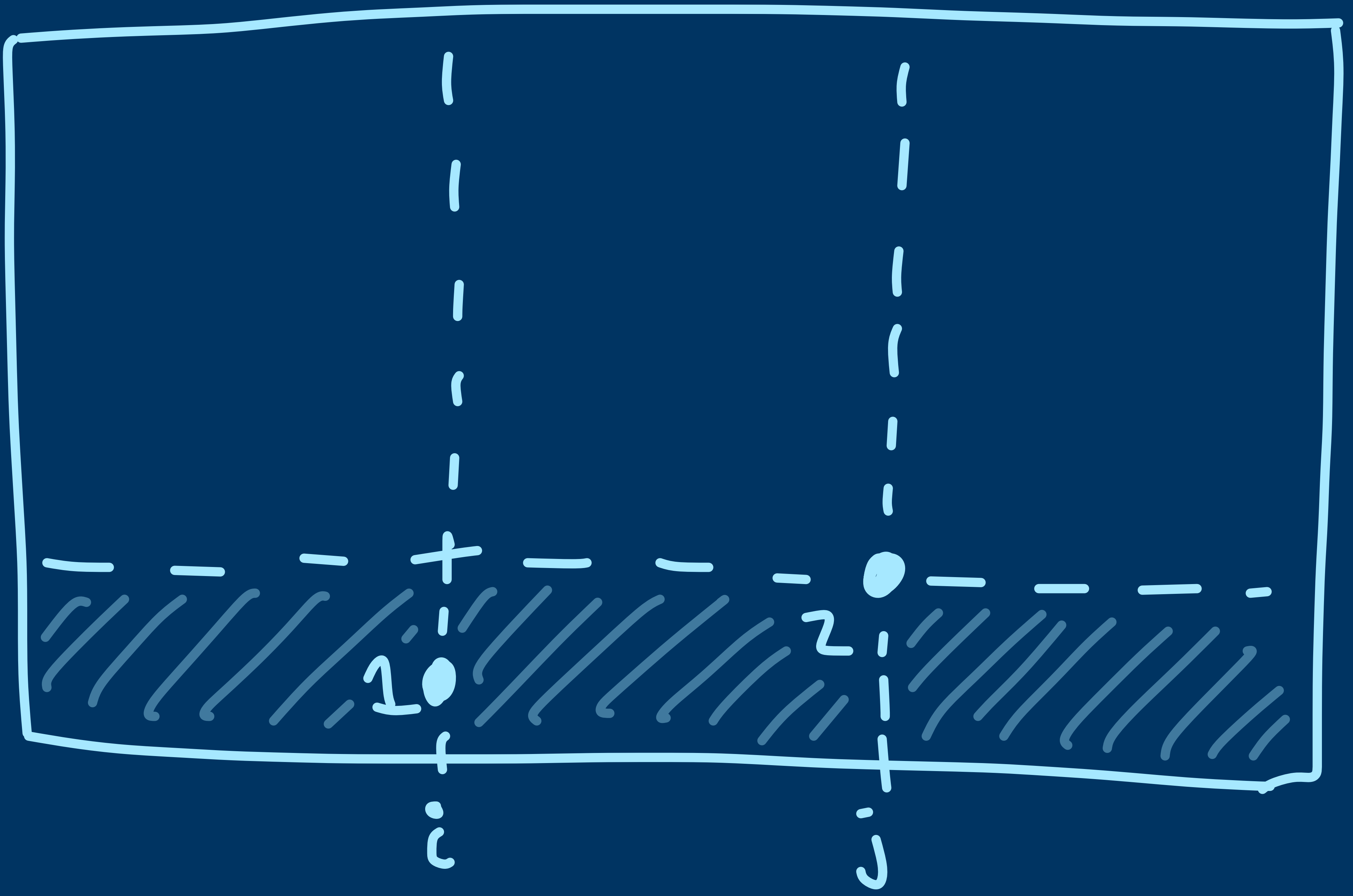

Knowing this: the entry $\pi(j)$ can be deleted.

Zeilberger's "logical reasoning" won't notice this.

Enumeration Schemes — WILFPLUS

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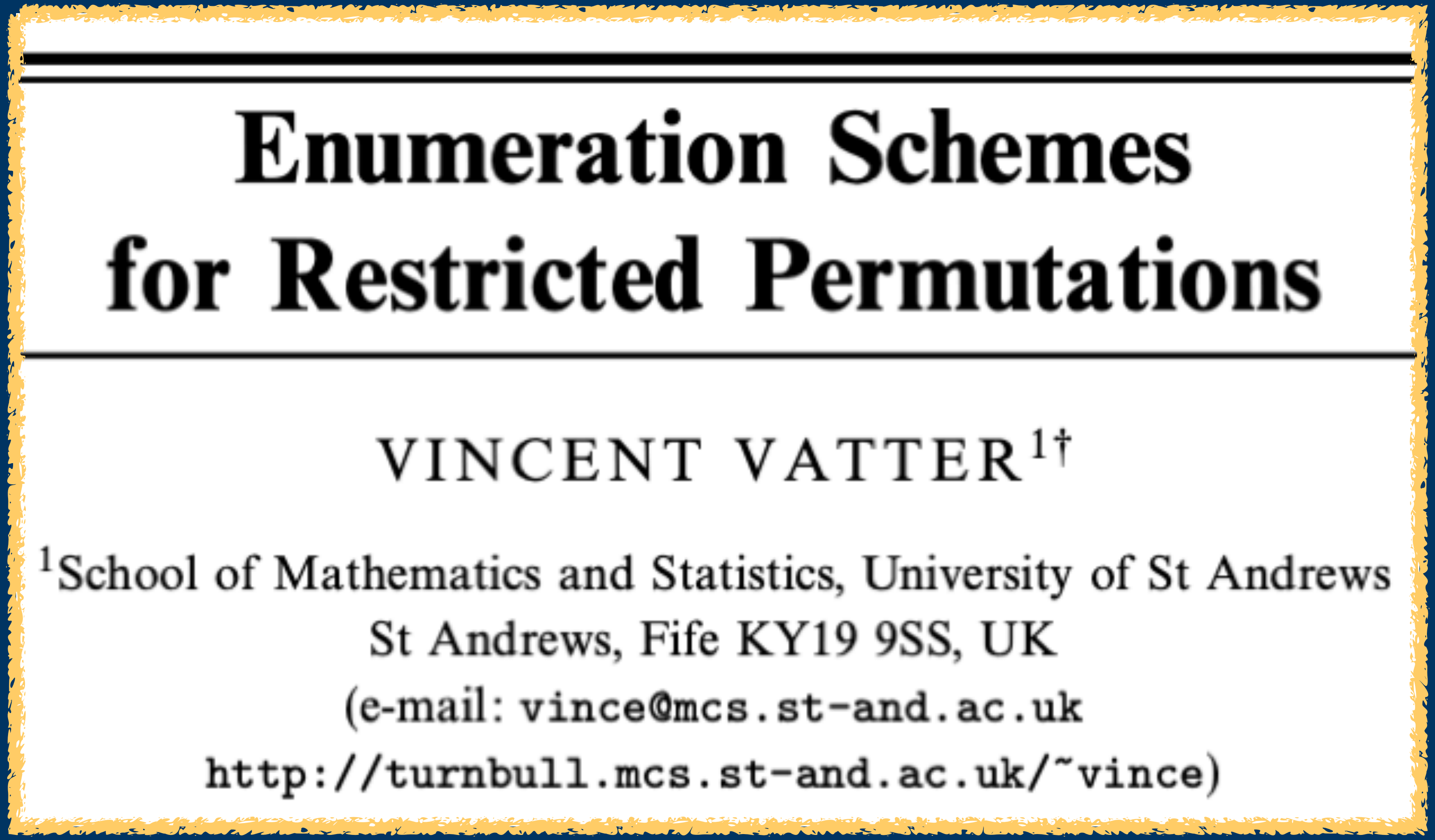

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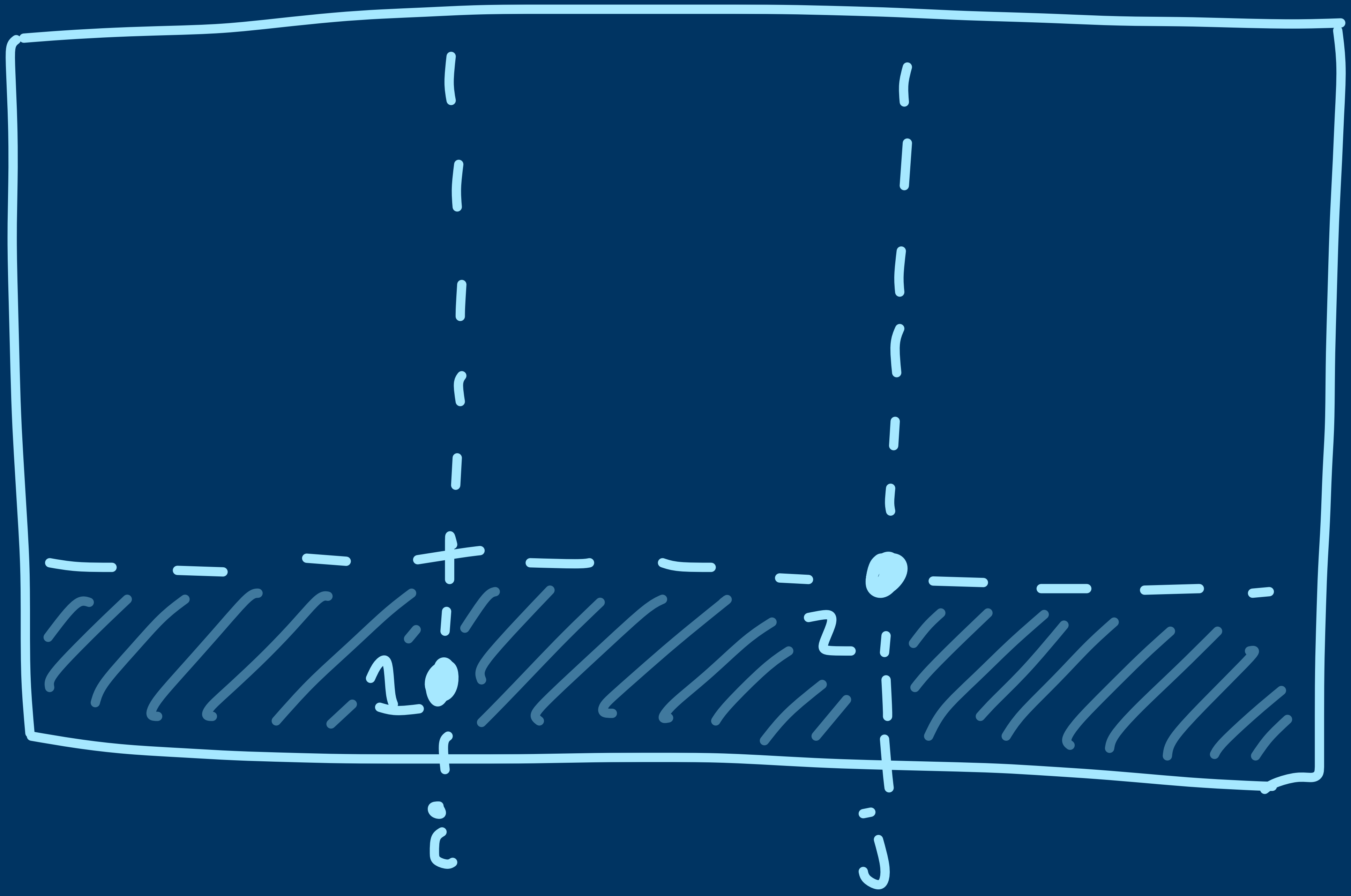

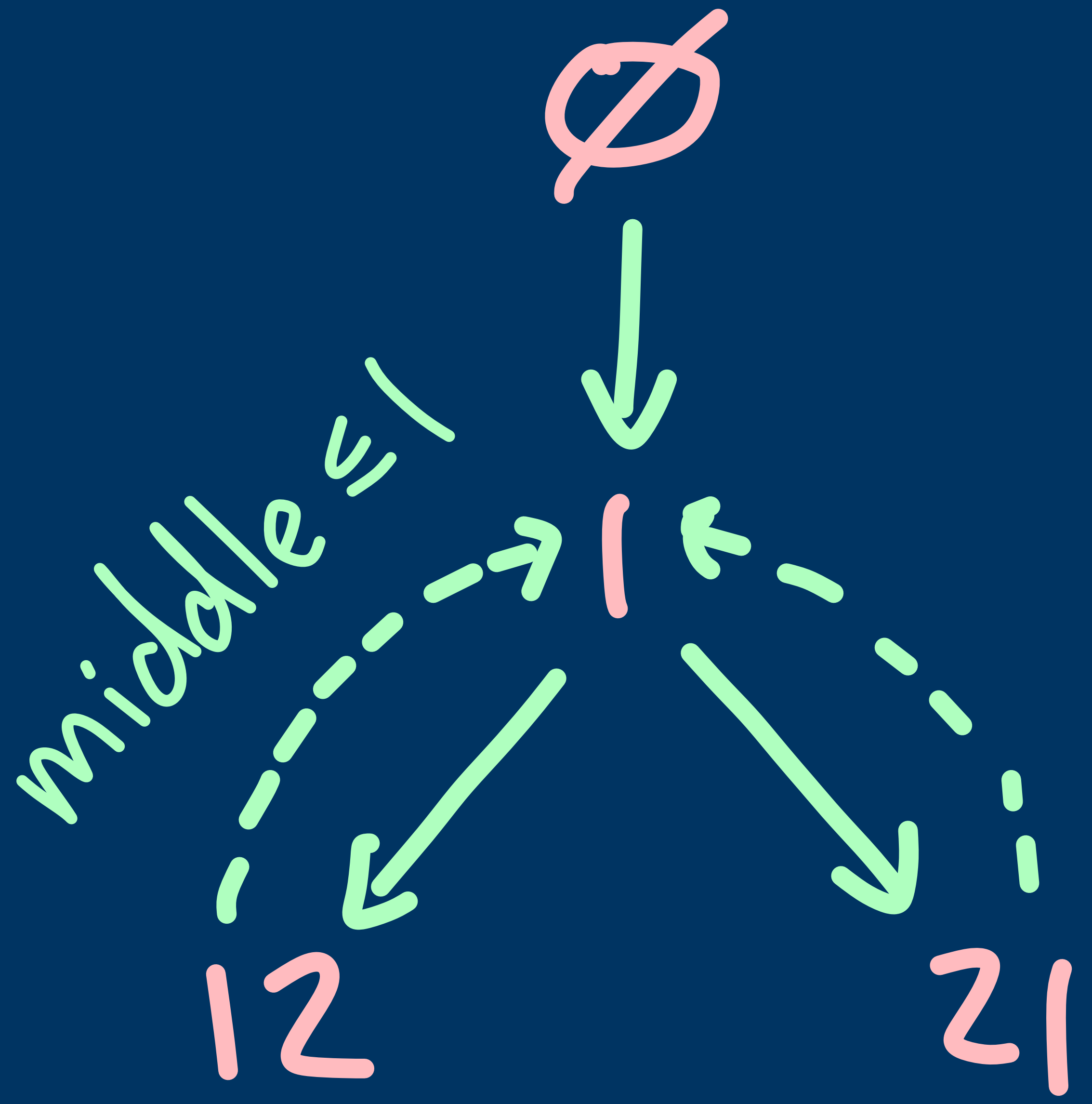

Enumeration Schemes — WILFPLUS

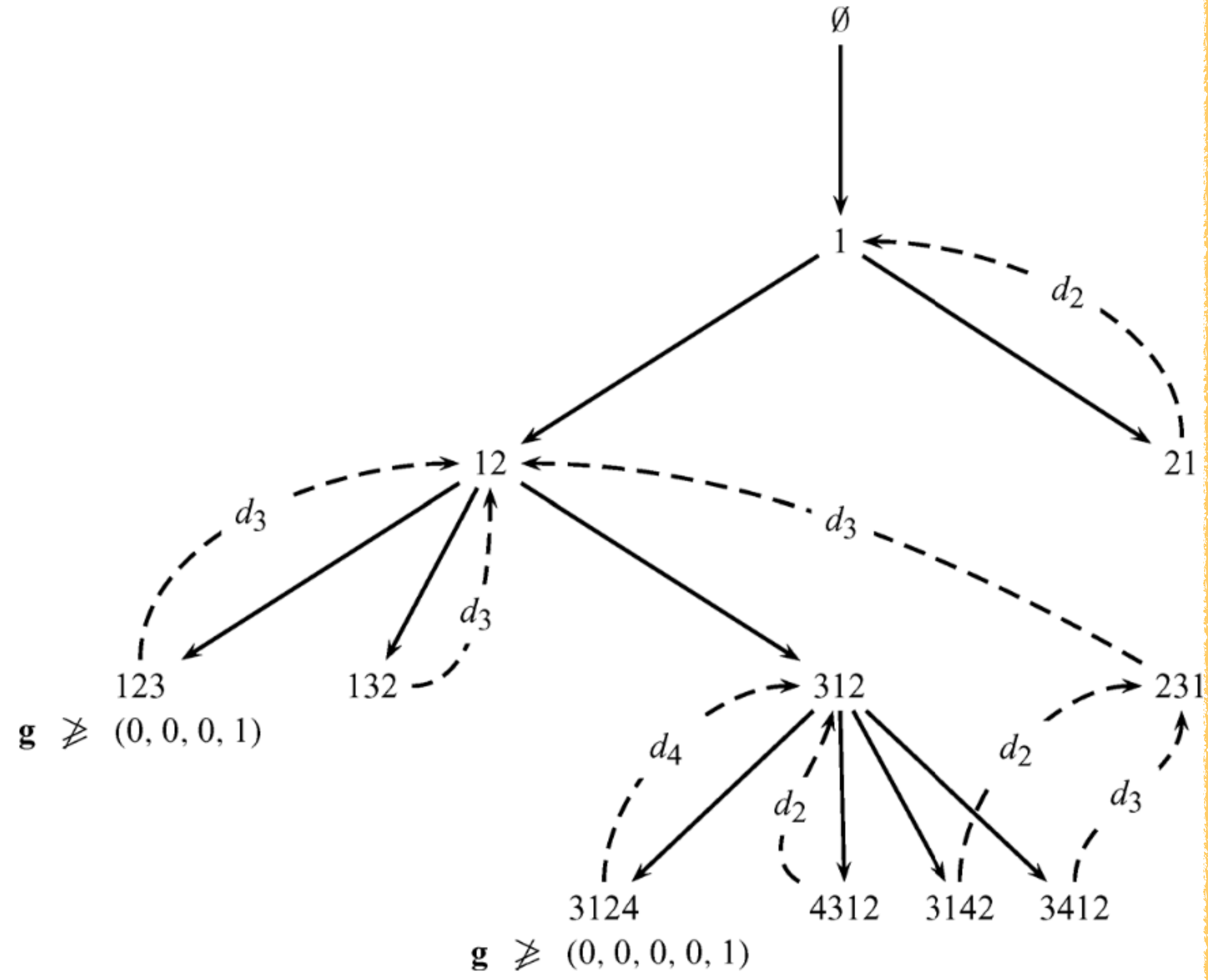

Figure 5. The enumeration scheme for Av(1234).

Enumeration Schemes for Restricted Permutations

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Enumeration Schemes — Flexible Schemes

Z: When checking if a point is reversibly deletable, can take into account whether a gap between two entries <u>must be empty</u>.

can do only a few simple classes

FLEXIBLE SCHEMES AND BEYOND: EXPERIMENTAL ENUMERATION OF PATTERN AVOIDANCE CLASSES

By

YONAH BIERS-ARIEL

Enumeration Schemes — Flexible Schemes

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V: Can take into account when a gap is constrained to have a finite number of entries (and more complicated similar constraints)

can do more classes

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B-A: Sometimes if a gap is constrained to a finite number of possibilities, there could be one entry deletable for some of these possibilities, and a *different entry* deletable for the other possibilities.

can do even more classes

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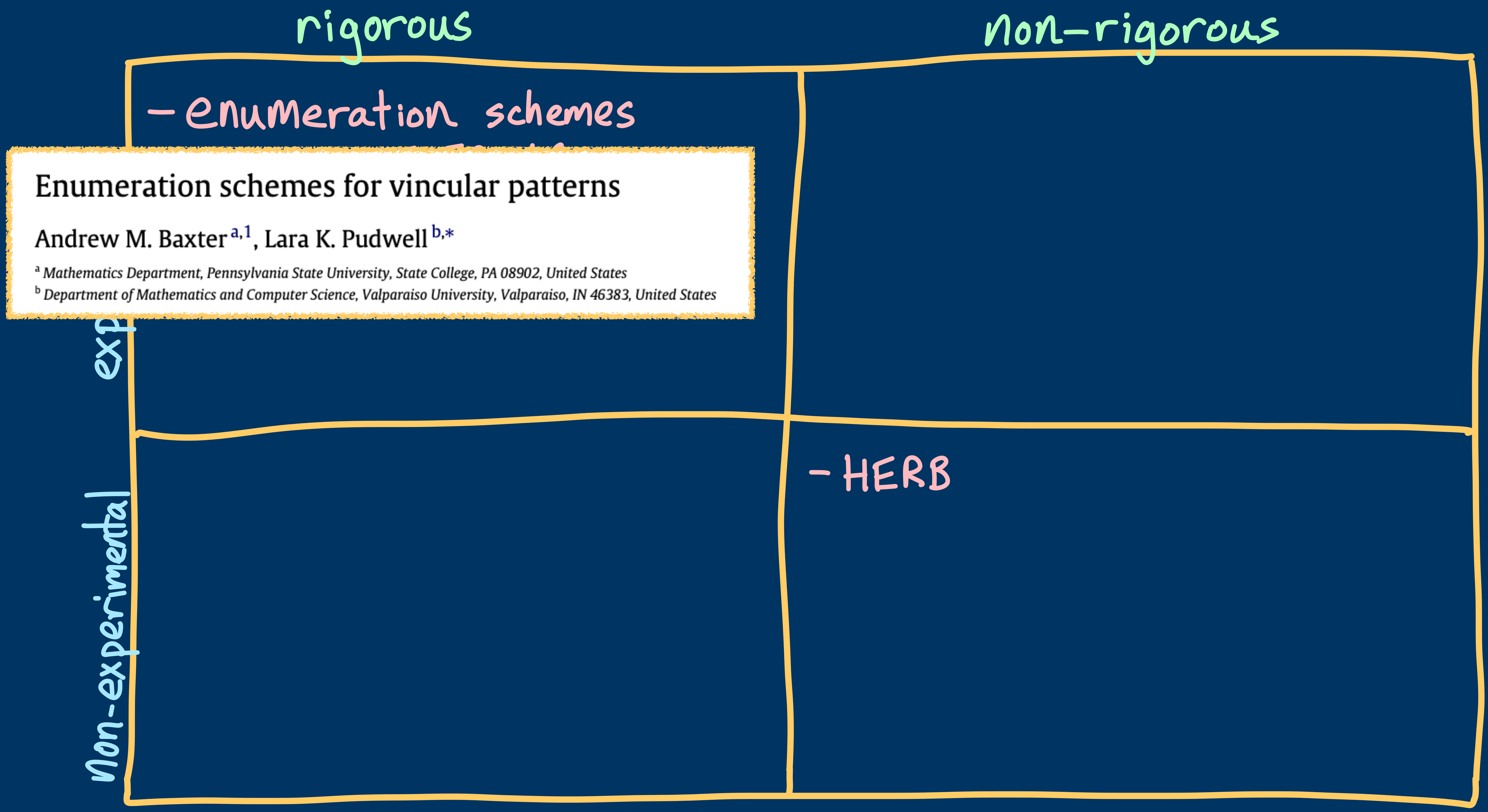
Pat length[1]	Sym Classes[2]	Ins. Enc.	ES	FS	New with FS
[3]	2	0	2	2	0
[4]	7	0	2	2	0
[5]	23	0	2	2	0
[3], [3]	5	5	5	5	0
[4], [4]	56	13	33	44	9
[4], [5]	434	30	112	173	59

V: Ca … ned to have a finite number of entries (and …

B-A: Sometimes if a gap is constrained to a finite number of possibilities, there could be one entry deletable for some of these possibilities, and a *different entry* deletable for the other possibilities.

can do even more classes

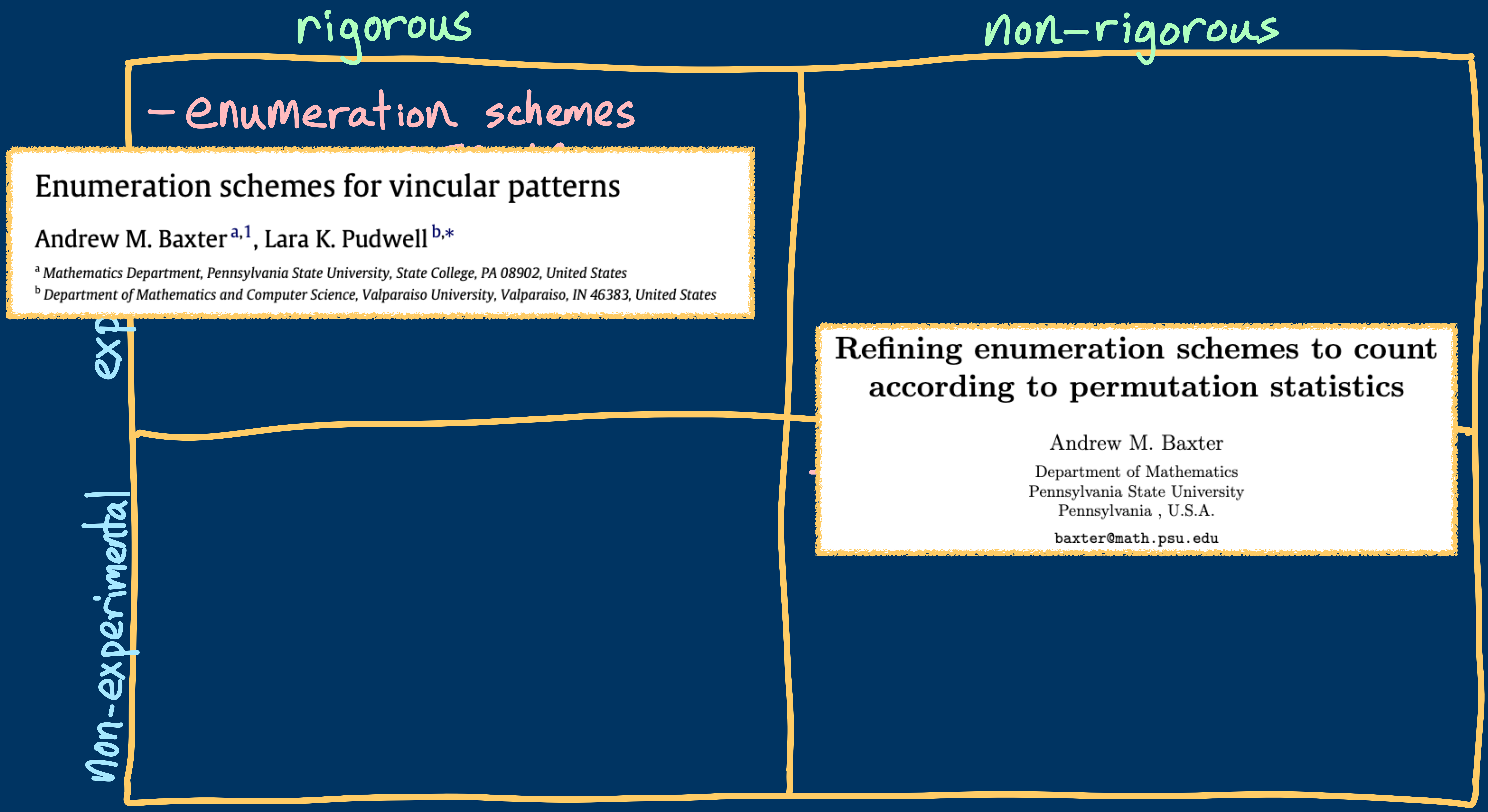
	rigorous	non-rigorous
experimental	- enumeration schemes WILF, WILFPLUS, Flexible Schemes (E)	
non-experimental		- HERB



Enumeration schemes for vincular patterns

Andrew M. Baxter[a,1], Lara K. Pudwell[b,*]

[a] Mathematics Department, Pennsylvania State University, State College, PA 08902, United States
[b] Department of Mathematics and Computer Science, Valparaiso University, Valparaiso, IN 46383, United States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exp
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Refining enumeration schemes to count according to permutation statistics
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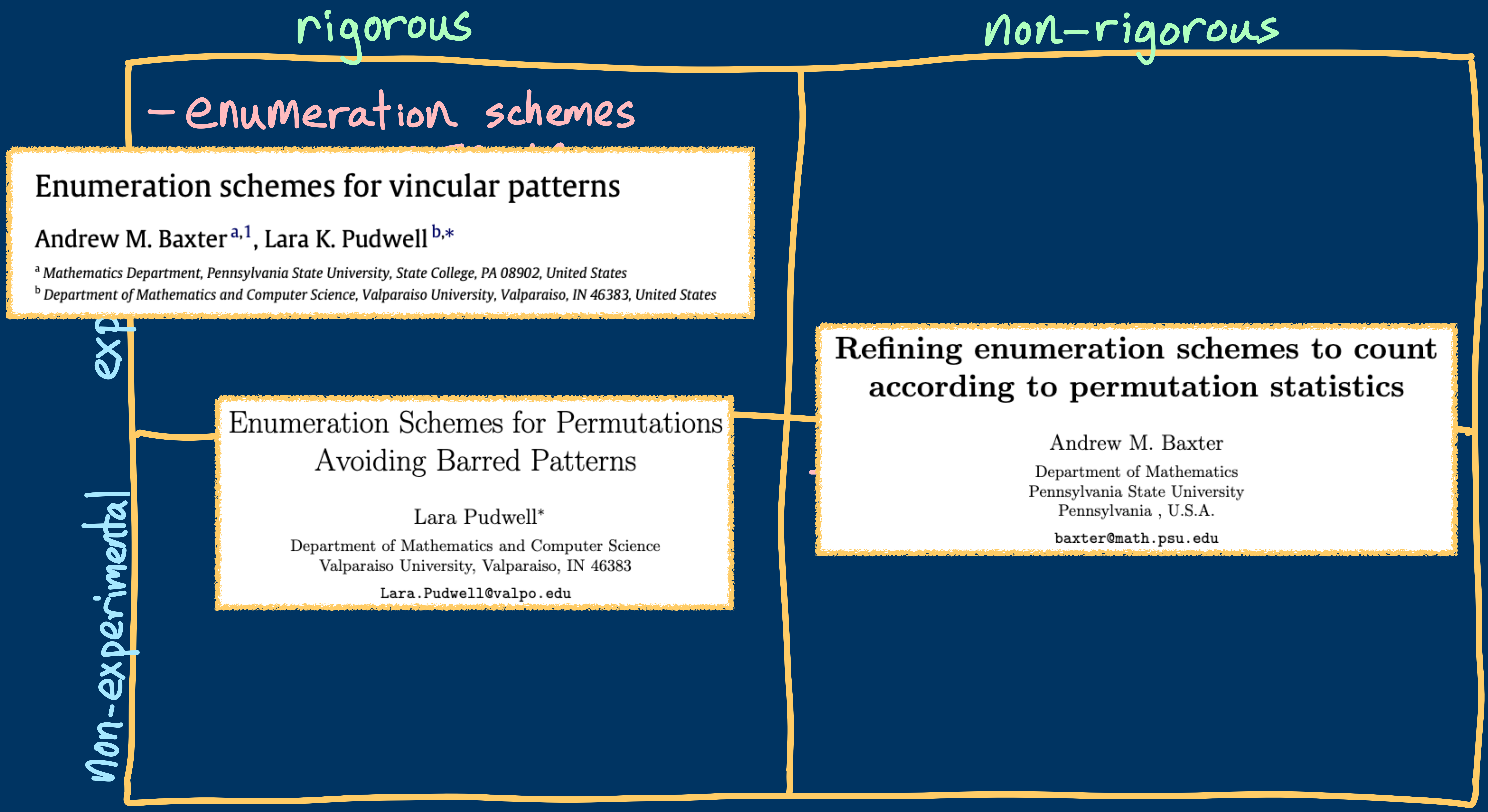
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b Department of Mathematics and Computer Science, Valparaiso University, Valparaiso, IN 46383, United States
Enumeration Schemes for Permutations Avoiding Barred Patterns
Lara Pudwell*
Department of Mathematics and Computer Science
Valparaiso University, Valparaiso, IN 46383
Lara.Pudwell@valpo.edu
Refining enumeration schemes to count according to permutation statistics
Andrew M. Baxter
Department of Mathematics
Pennsylvania State University
Pennsylvania , U.S.A.
baxter@math.psu.edu

Generating Trees

- A "generating tree" for a set of permutations is a way of rigorously representing its structure. It describes where new maximum entries can be inserted into permutations so that they remain in the set.

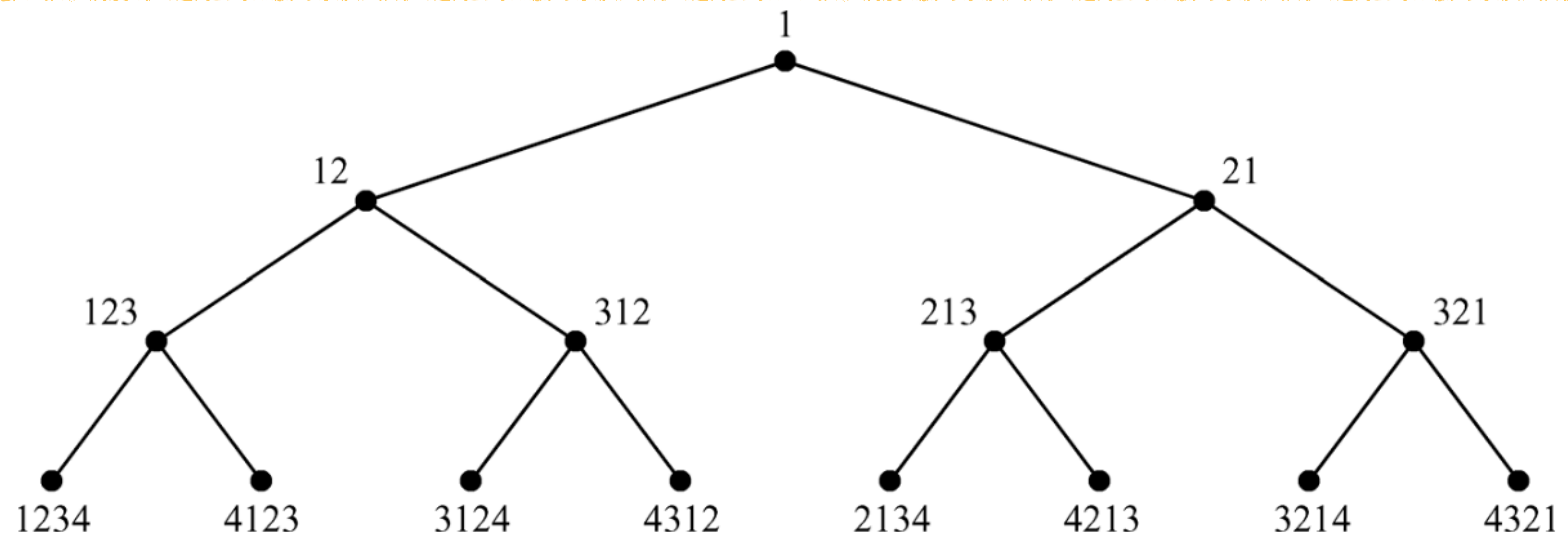

Figure 2. The first four levels of the pattern-avoidance tree $T(132, 231)$.

(Vatter 2007)

Generating Trees

- 1978: Chung, Graham, Hoggatt Jr., and Kleiman invented generating trees to enumerate the *Baxter permutations*.

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- 1995/1996: West uses generating trees to enumerate several permutation classes.

- 2006: Vatter categorizes the permutation classes that have finitely labeled generating trees and writes the Maple package FINLABEL to enumerate them automatically.

Generating Trees

- 1998 – present: ECO Method

  Exports the idea of generating trees to other combinatorial objects and uses them to do many things: enumeration, generating functions, exhaustively generating all objects in a fast way, …

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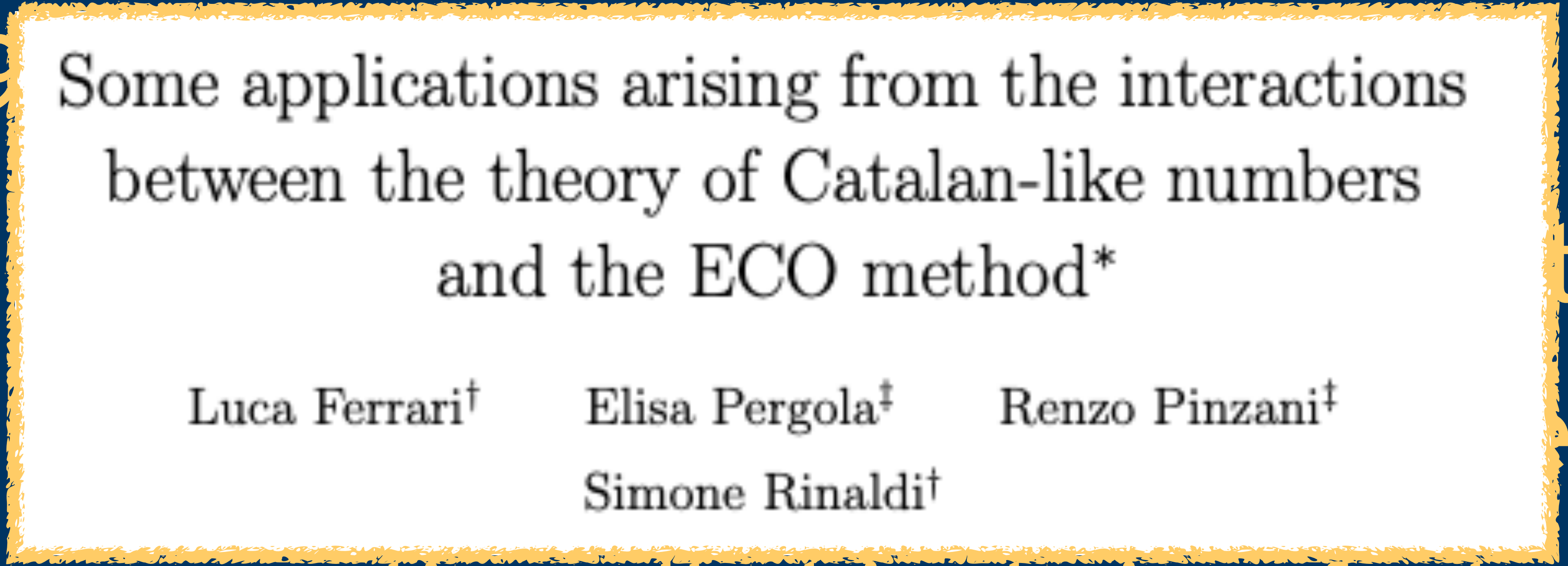
ECO: A Methodology for the Enumeration of Combinatorial Objects

ELENA BARCUCCI, ALBERTO DEL LUNGO, ELISA PERGOLA and RENZO PINZANI*

Dipartimento di Sistemi e Informatica, Via Lombroso 6/17, 50134 Firenze, Italy

Generating Trees

- 199

Some applications arising from the interactions between the theory of Catalan-like numbers and the ECO method*

Luca Ferrari[†] Elisa Pergola[‡] Renzo Pinzani[‡]
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Integer Partitions in Discrete Dynamical Models and ECO Method*

Le Manh Ha[1] and Phan Thi Ha Duong[2]

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[2] *Institute of Mathematics, 18 Hoang Quoc Viet Road, 10307 Hanoi, Vietnam*

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ECO Method and the Exhaustive Generation of Convex Polyominoes

Alberto Del Lungo, Andrea Frosini, and Simone Rinaldi

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ECO-generation for some restricted classes of compositions

Jean-Luc Baril, Phan-Thuan Do

and RENZO PINZ…

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Integer Partitions in Discrete Dynamical Models and ECO Method*

…n Thi Ha Duong[2]
…ity, 34 Le Loi, Hue, Vietnam
…tnam

Generating involutions, derangements, and relatives by ECO

Vincent Vajnovszki
LE2I – UMR CNRS, Université de Bourgogne, B.P. 47 870, 21078 DIJON-Cedex France.
Email: vvajnov@u-bourgogne.fr

…eneration of

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- On Thursday: More algorithms for exhaustive generation!

	rigorous	non-rigorous
experimental	- enumeration schemes WILF, WILFPLUS, Flexible Schemes (E)	
non-experimental	- generating trees (E) - FINLABEL - ECO Method - Combinatorial Generation	- HERB

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Struct

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Volume 88, Number 318, July 2019, Pages 1967–1990
https://doi.org/10.1090/mcom/3386
Article electronically published on December 11, 2018

AUTOMATIC DISCOVERY OF STRUCTURAL RULES OF PERMUTATION CLASSES

CHRISTIAN BEAN, BJARKI GUDMUNDSSON, AND HENNING ULFARSSON

Most of the methods described so far:

“expand a particular structure tree and hope it ends up being finite”

Struct is a software package that takes a permutation class as input and searches for a *set cover* that decomposes it into simpler disjoint parts.

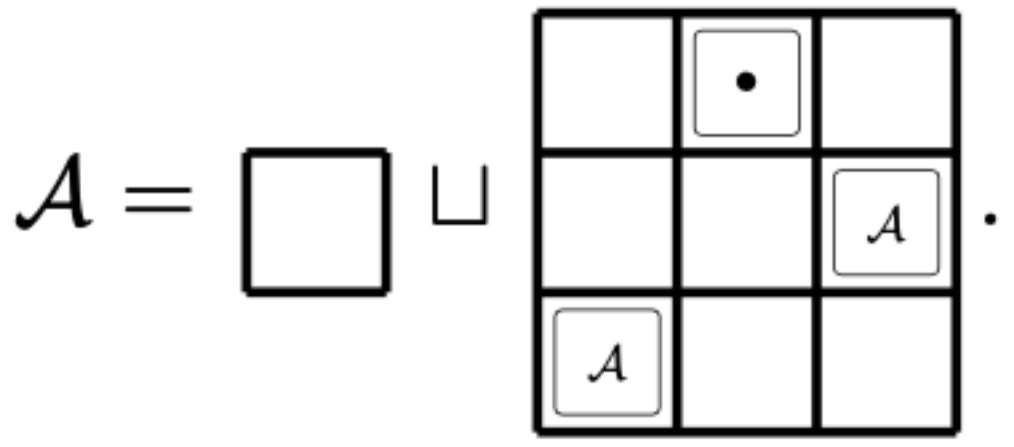
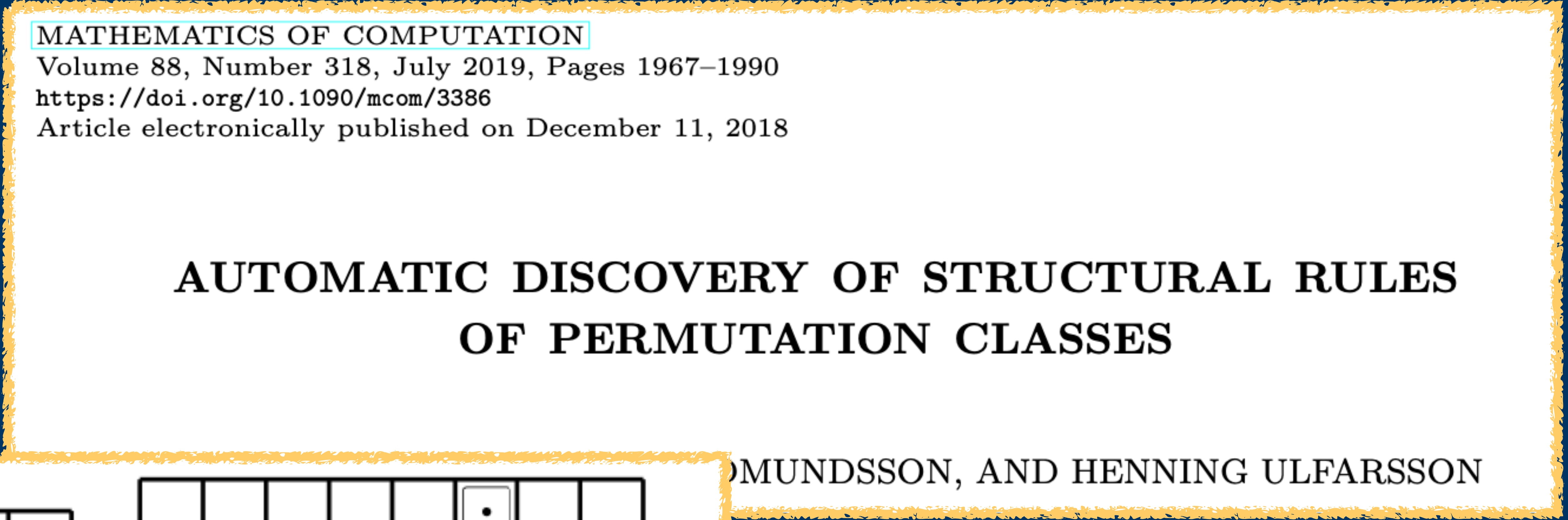


FIGURE 1. The structure of Av(231)

Struct

$$\mathcal{G}_4 = \mathrm{Av}(321,1324)$$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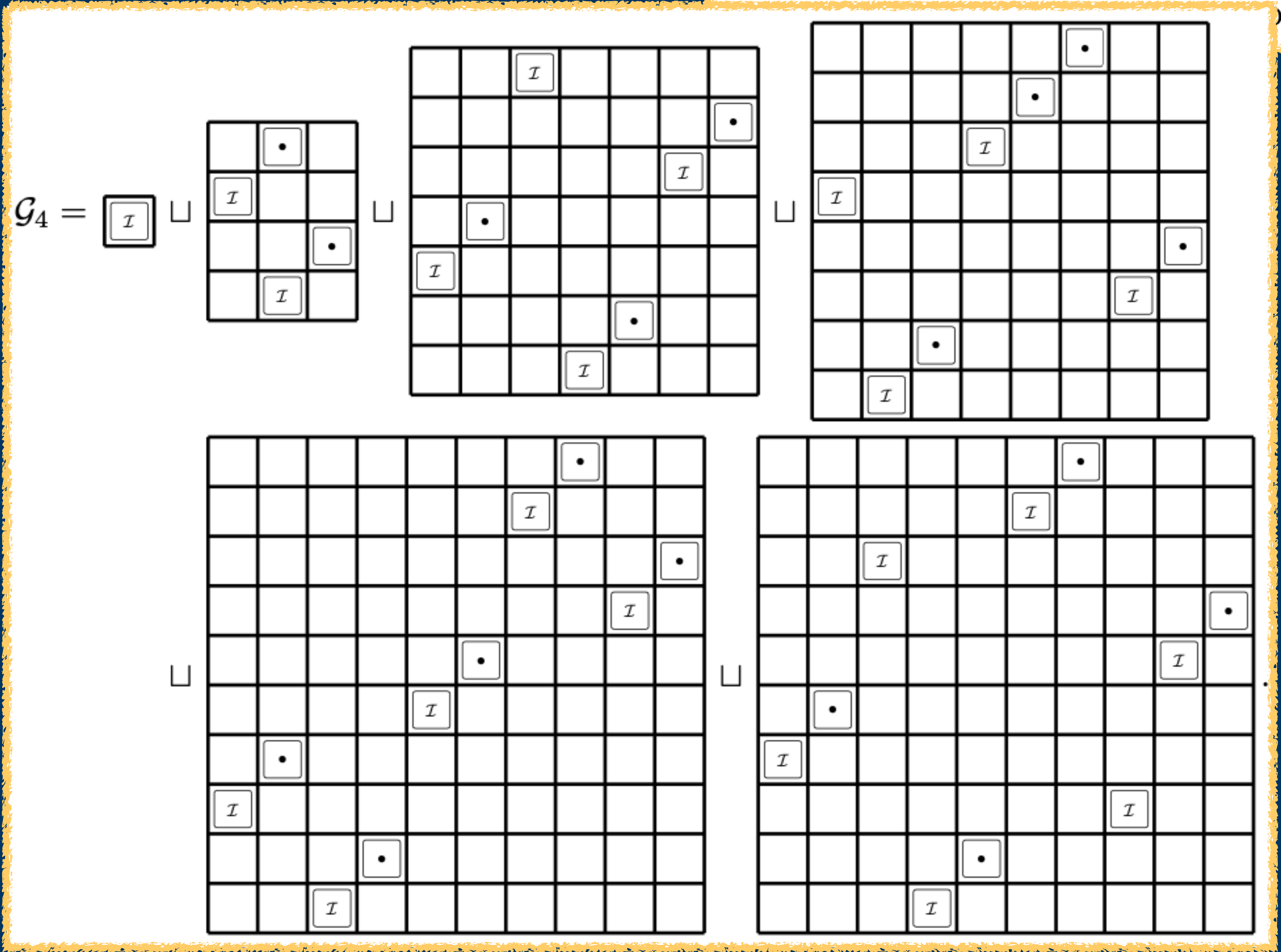

Struct

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AUTOMATIC DISCOVERY OF STRUCTURAL RULES OF PERMUTATION CLASSES

CHRISTIAN BEAN, BJARKI GUDMUNDSSON, AND HENNING ULFARSSON

Method:

- Construct a big list of grids that make subsets of the input class.
- Set up an integer linear programming problem to pick a subset of grids that forms a set cover (each permutation in the class gives one constraint).
- Feed it into an ILP solver like Gurobi and wait patiently for a solution.

	rigorous	non-rigorous
experimental	- enumeration schemes WILF, WILFPLUS, Flexible Schemes (E)	- Struct - BiSC
non-experimental	- generating trees (E) - FINLABEL - ECO Method - Combinatorial Generation - Regular Insertion Enc. - Finite Simples - Poly Classes	- HERB

Combinatorial Exploration

At the end of the Struct paper, the authors discuss some classes that Struct can't do along with a possible future approach.

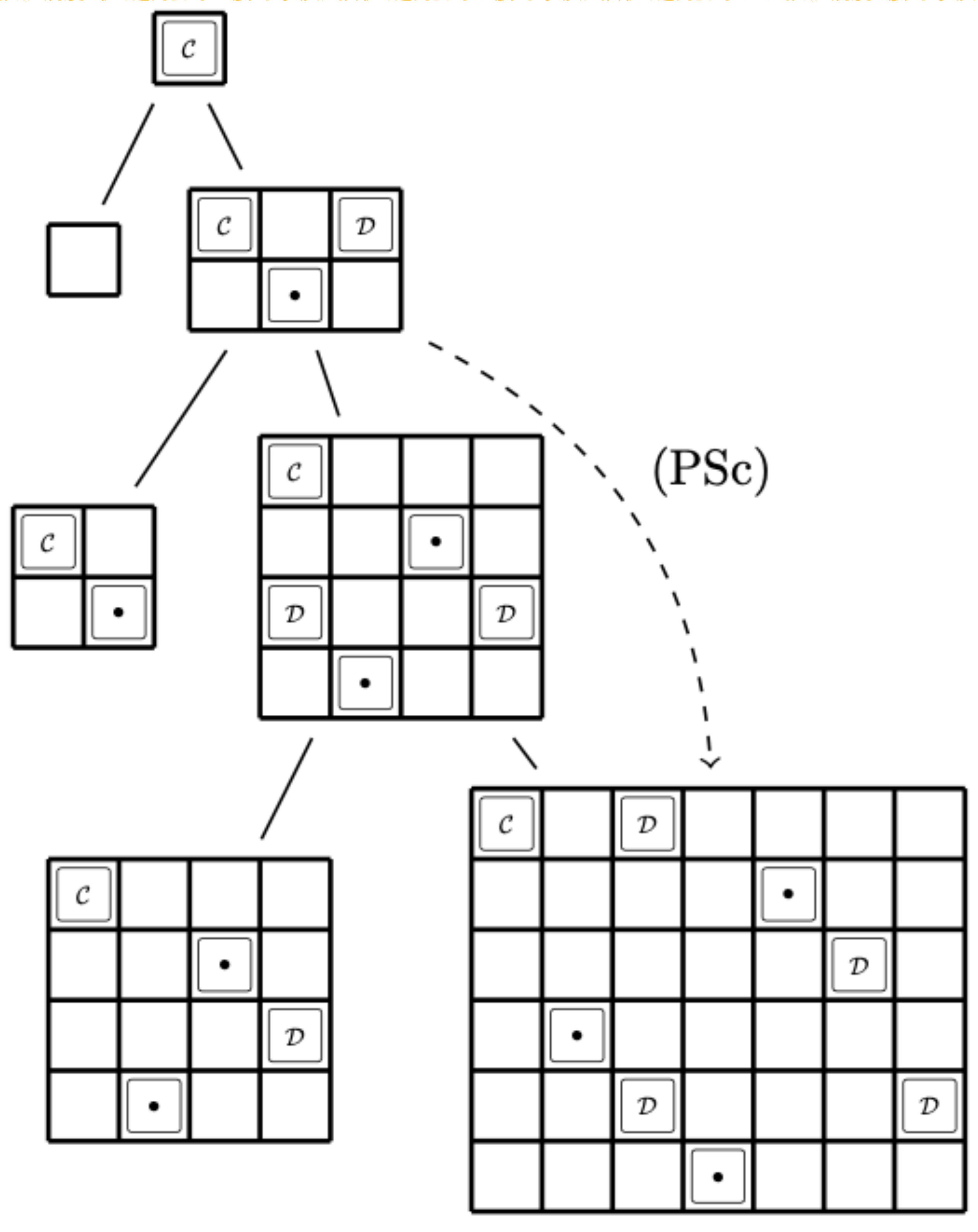

FIGURE 5. The structure of $\mathcal{C} = \mathrm{Av}(123)$

"proof tree"

Combinatorial Exploration

I started talking with Henning Ulfarsson and Christian Bean at PP 2016.

6 years later…

[Submitted on 15 Feb 2022 (v1), last revised 8 Aug 2022 (this version, v2)]

Combinatorial Exploration: An algorithmic framework for enumeration

Michael H. Albert, Christian Bean, Anders Claesson, Émile Nadeau, Jay Pantone, Henning Ulfarsson

Combinatorial Exploration is a new domain-agnostic algorithmic framework to automatically and rigorously study the structure of combinatorial objects and derive their counting sequences and generating functions. We describe how it works and provide an open-source Python implementation. As a prerequisite, we build up a new theoretical foundation for combinatorial decomposition strategies and combinatorial specifications.
We then apply Combinatorial Exploration to the domain of permutation patterns, to great effect. We rederive hundreds of results in the literature in a uniform manner and prove many new ones. These results can be found in a new public database, the Permutation Pattern Avoidance Library (PermPAL) at this https URL. Finally, we give three additional proofs-of-concept, showing examples of how Combinatorial Exploration can prove results in the domains of alternating sign matrices, polyominoes, and set partitions.

Combinatorial Exploration

Key insights:

1. Instead of expanding one particular structure tree and hoping it ends up being finite: produce a bunch of independent "rules" that relate a parent set to child sets, and hope that some subset of these rules can be assembled into a tree

Combinatorial Exploration

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1. Instead of expanding one particular structure tree and hoping it ends up being finite: produce a bunch of independent "rules" that relate a parent set to child sets, and hope that some subset of these rules can be assembled into a tree
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Combinatorial Exploration

Key insights:

1. Instead of expanding one particular structure tree and hoping it ends up being finite: produce a bunch of independent "rules" that relate a parent set to child sets, and hope that some subset of these rules can be assembled into a tree
2. We need a much more efficient way to represent sets of permutations.
3. If (1) and (2) are done correctly, then the result can still be fully rigorous.

$C = Av(132)$

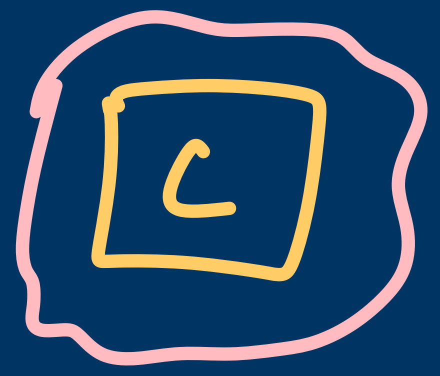
$C^+ =$ nonempty perms

empty or not

point placement

row/col separation

factor



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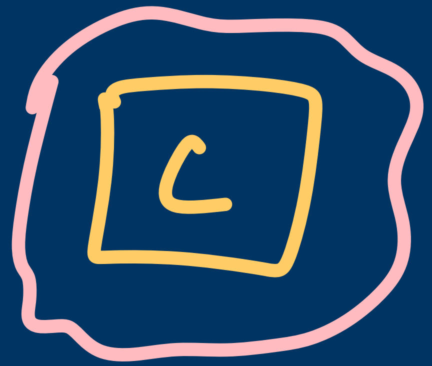

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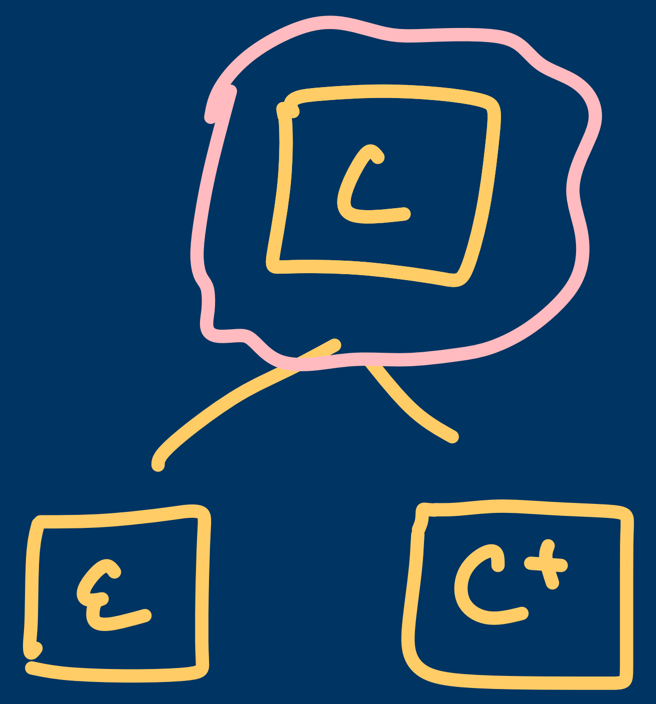

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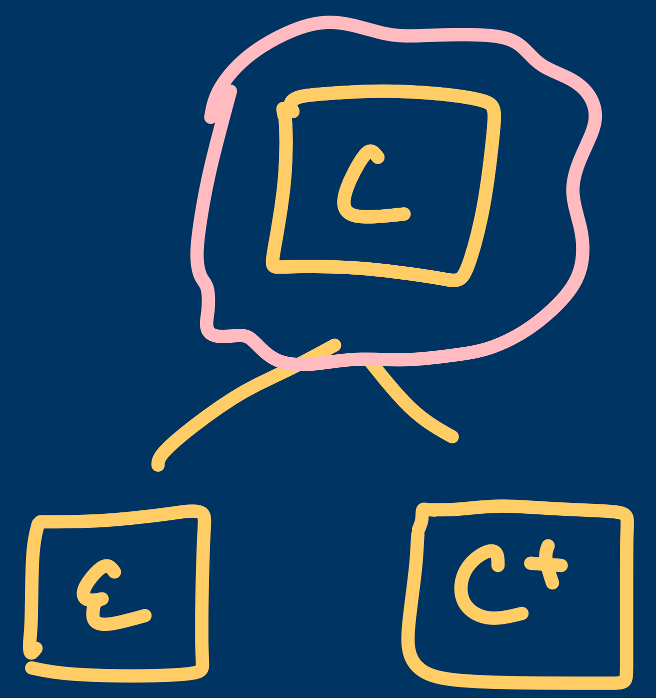

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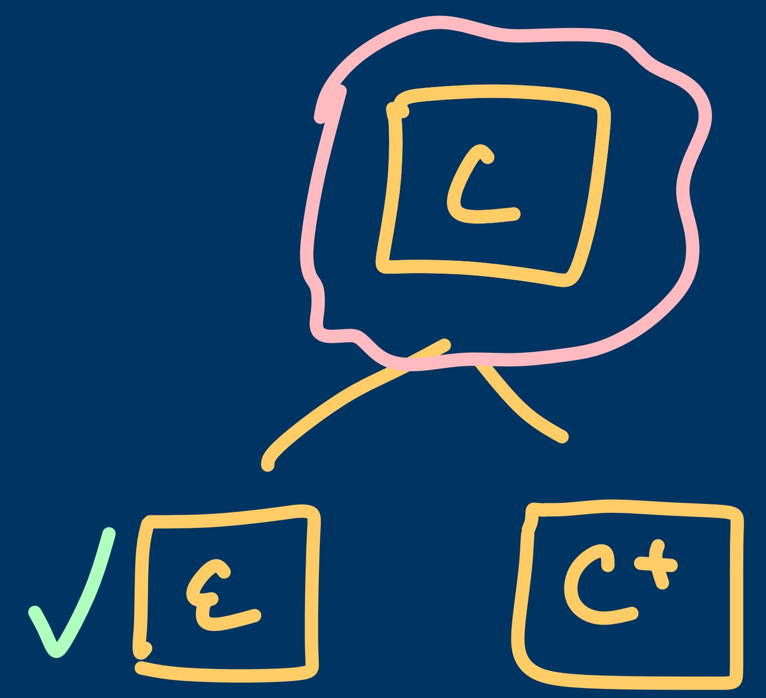

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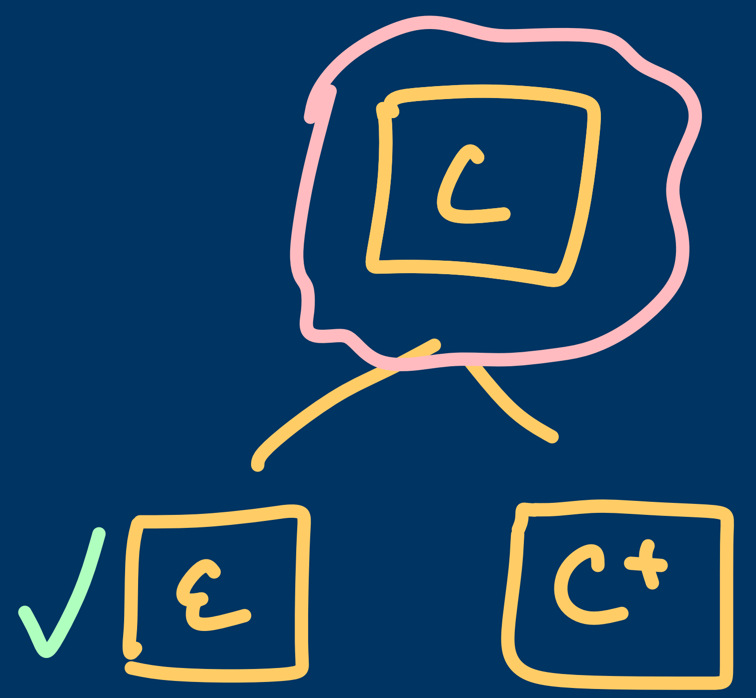

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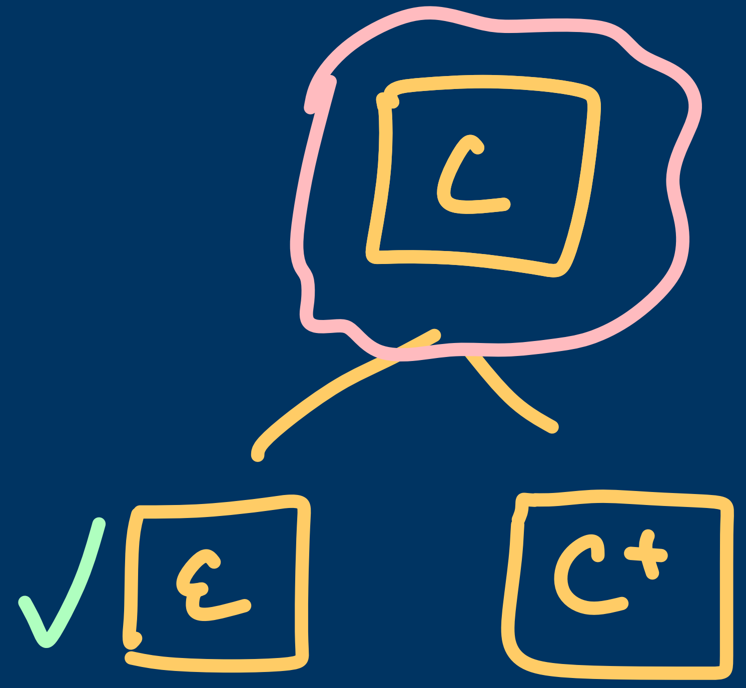

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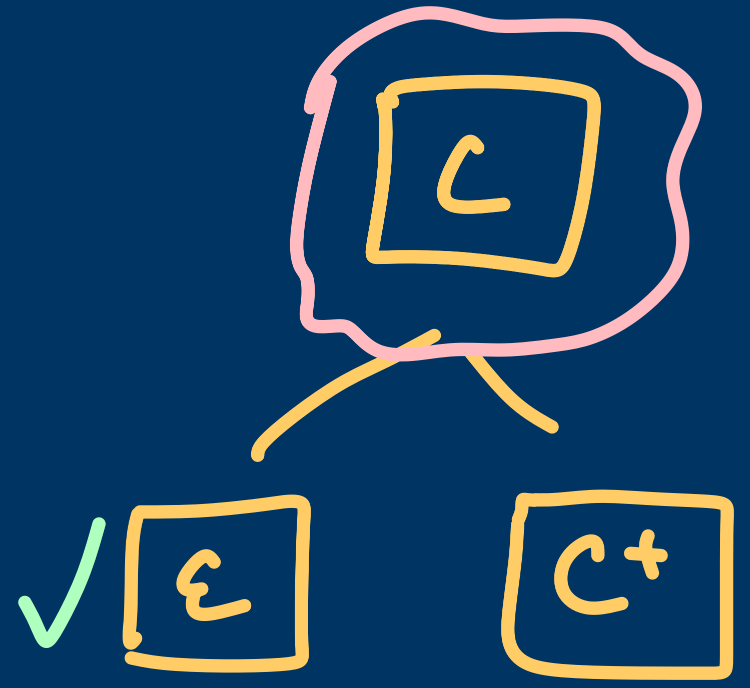

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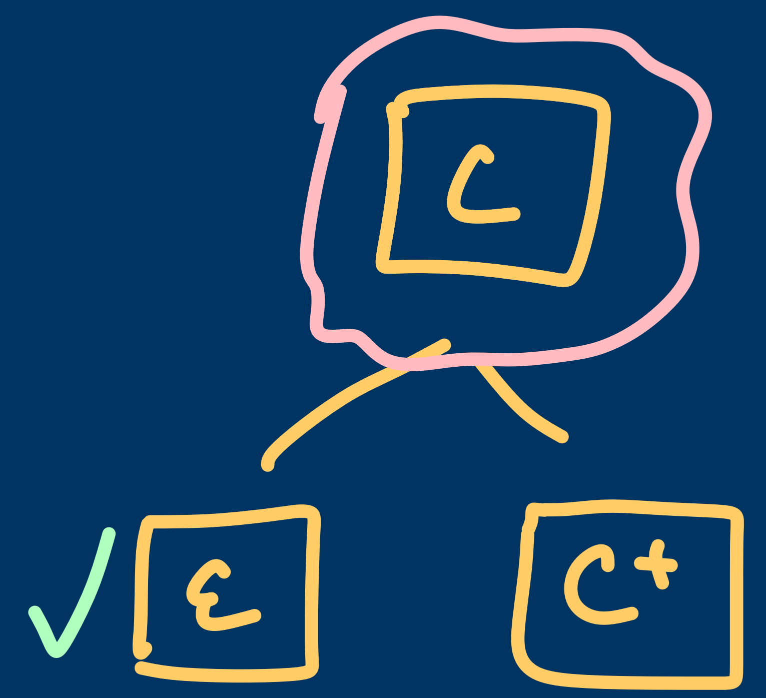

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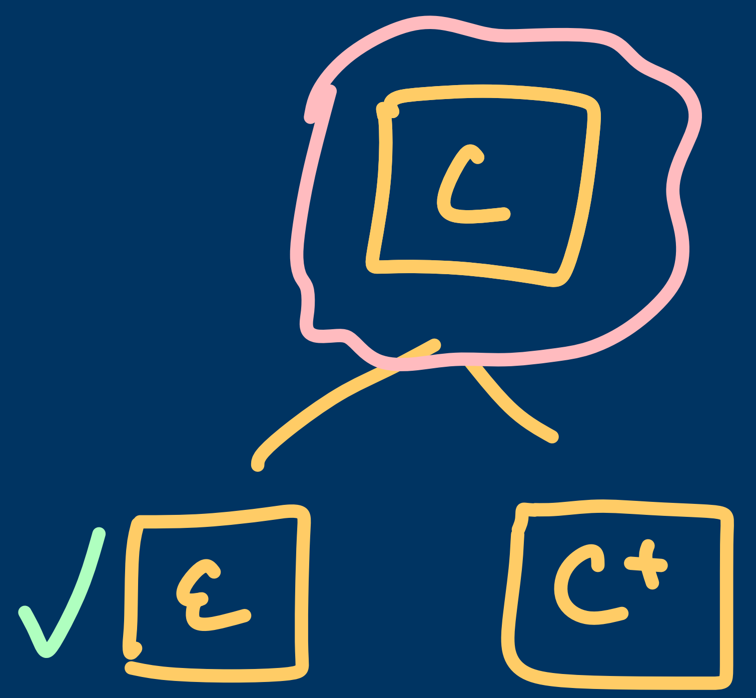

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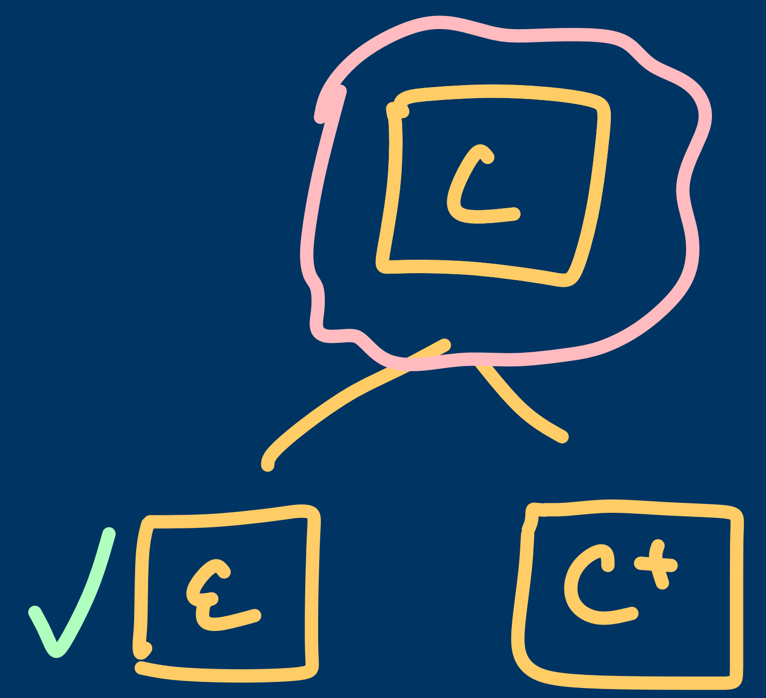

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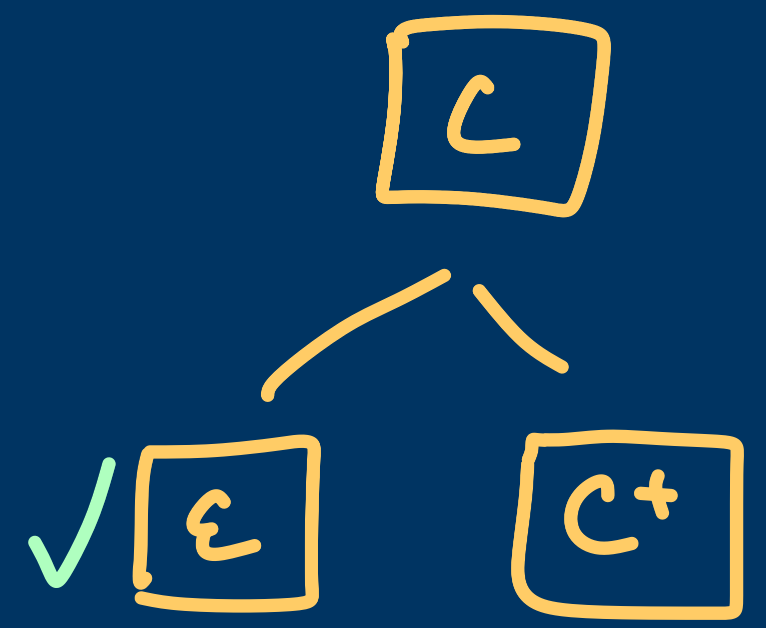

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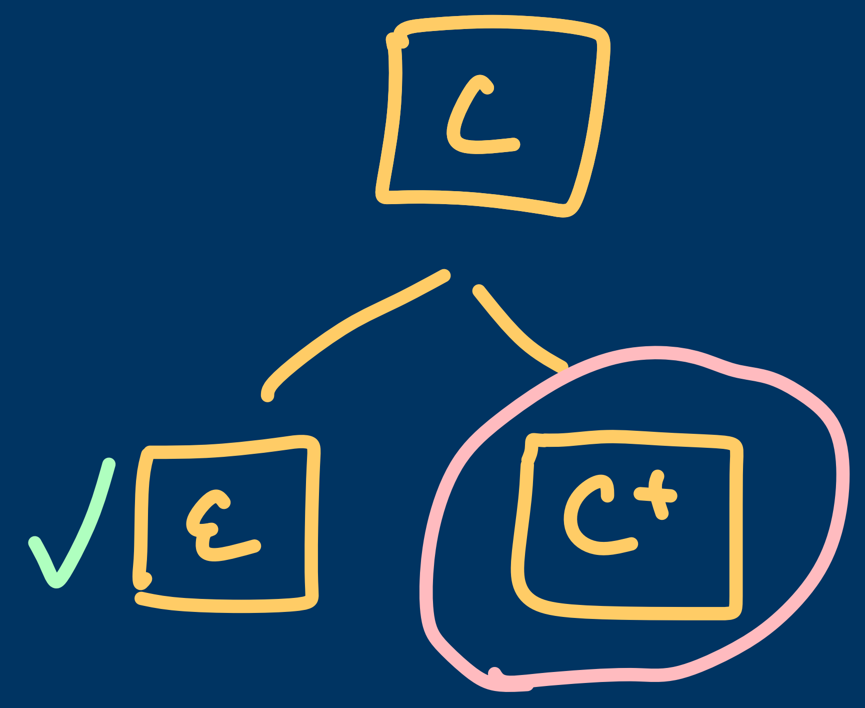

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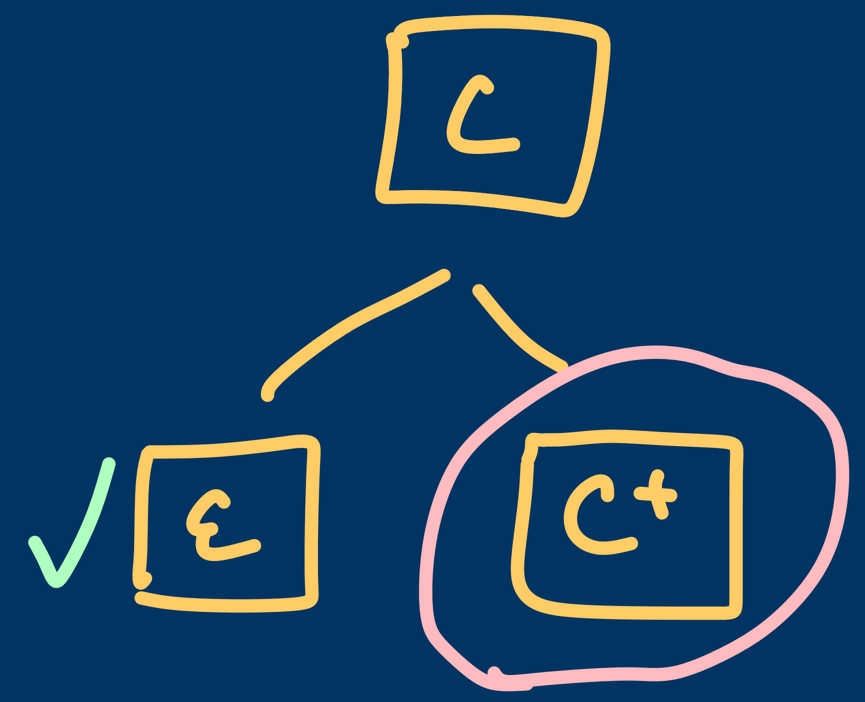

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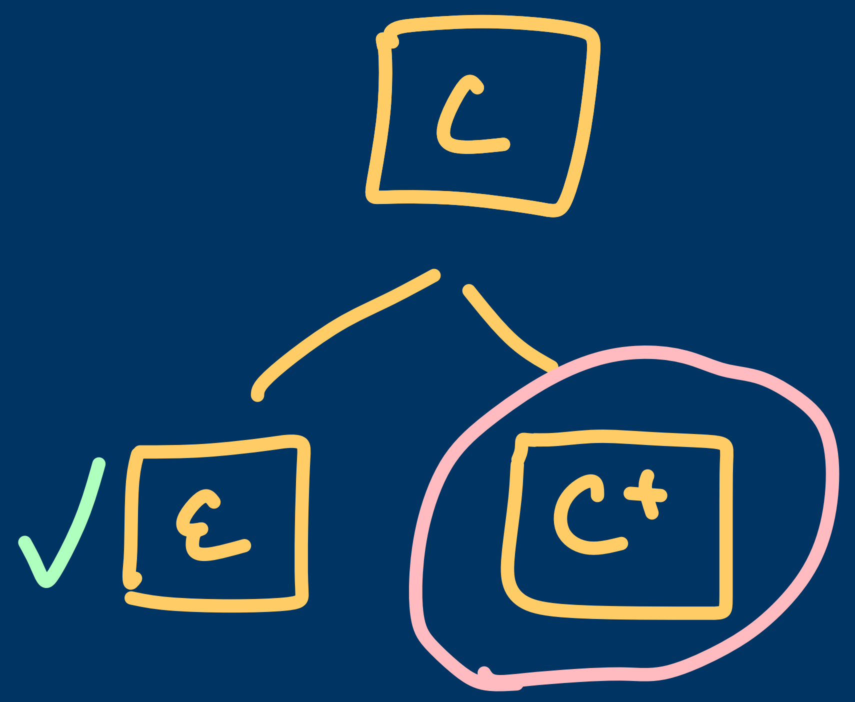

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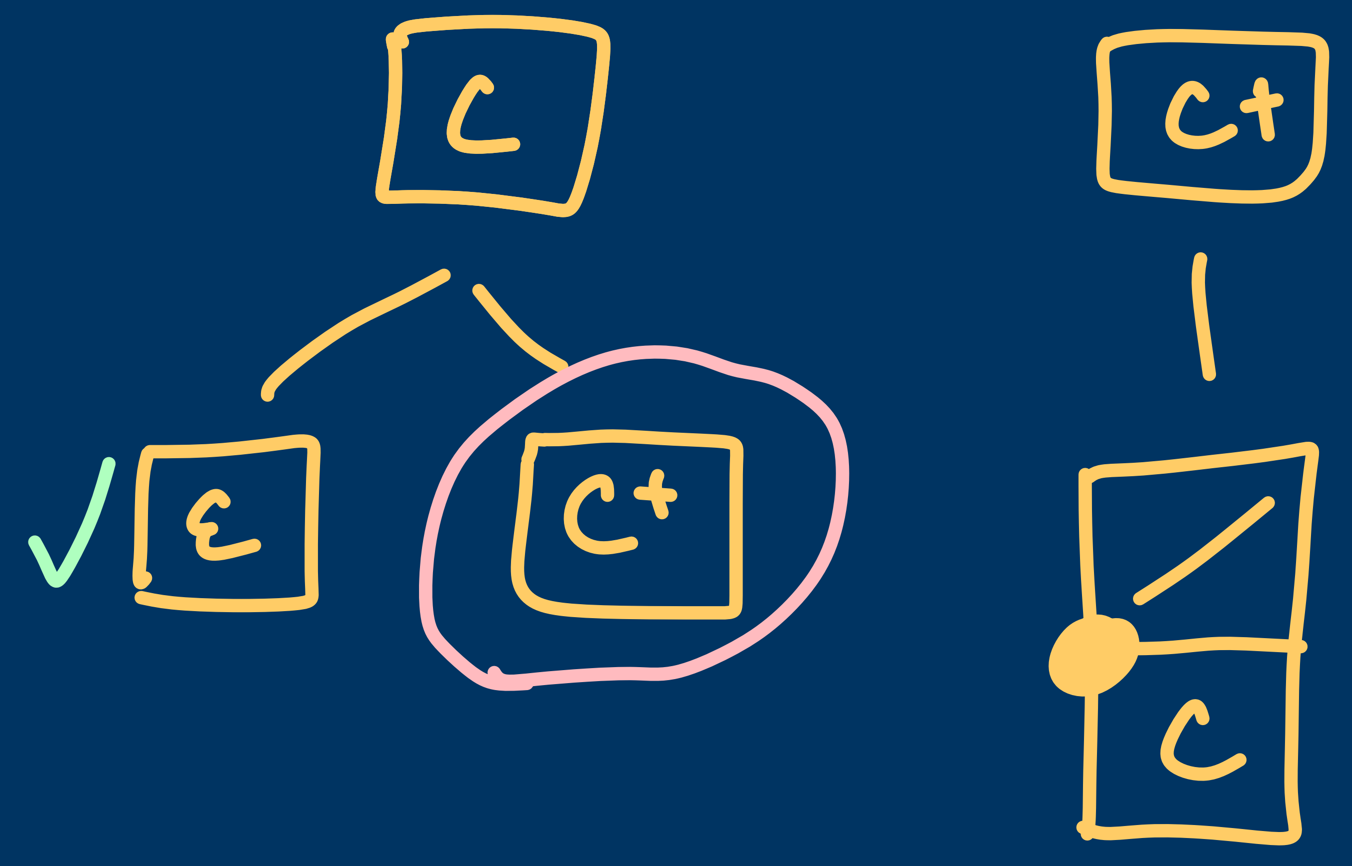

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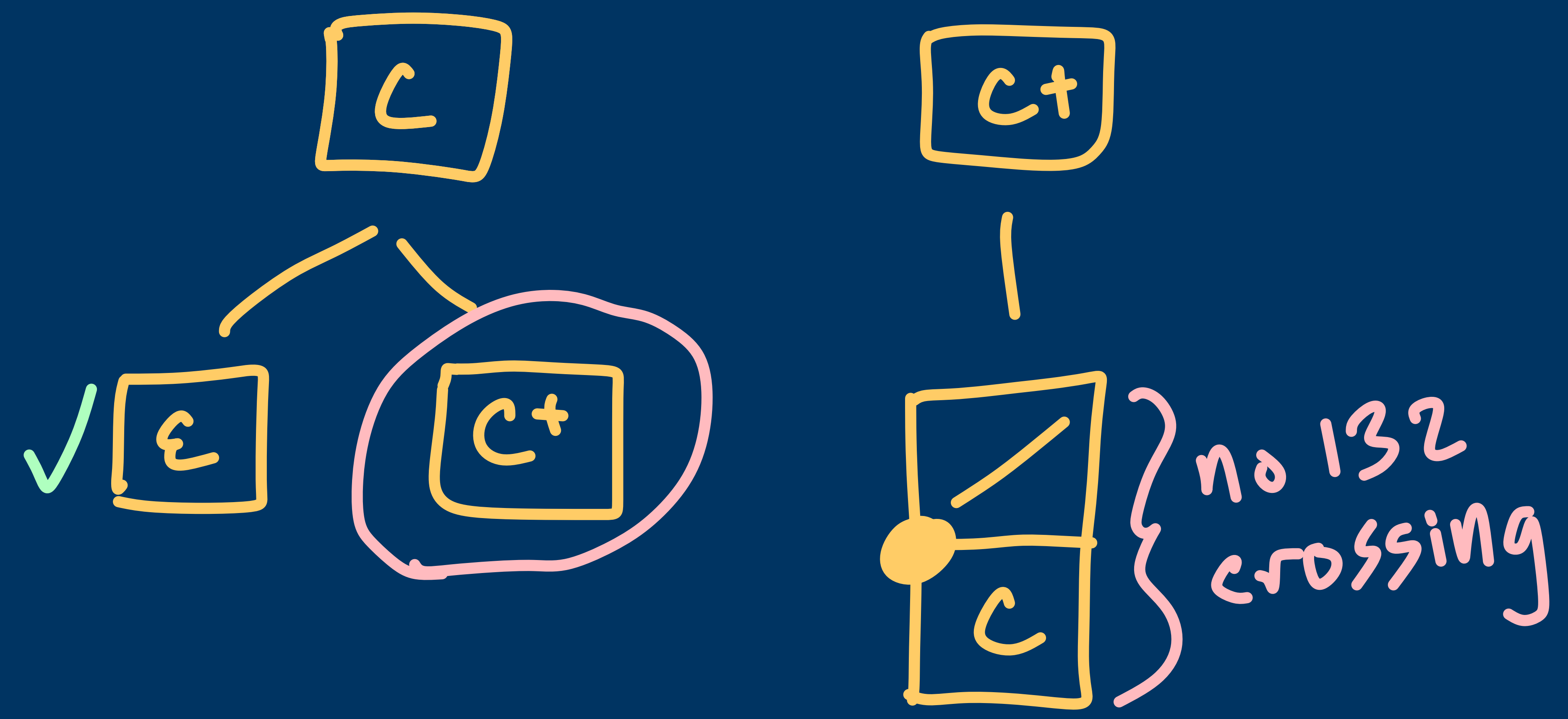

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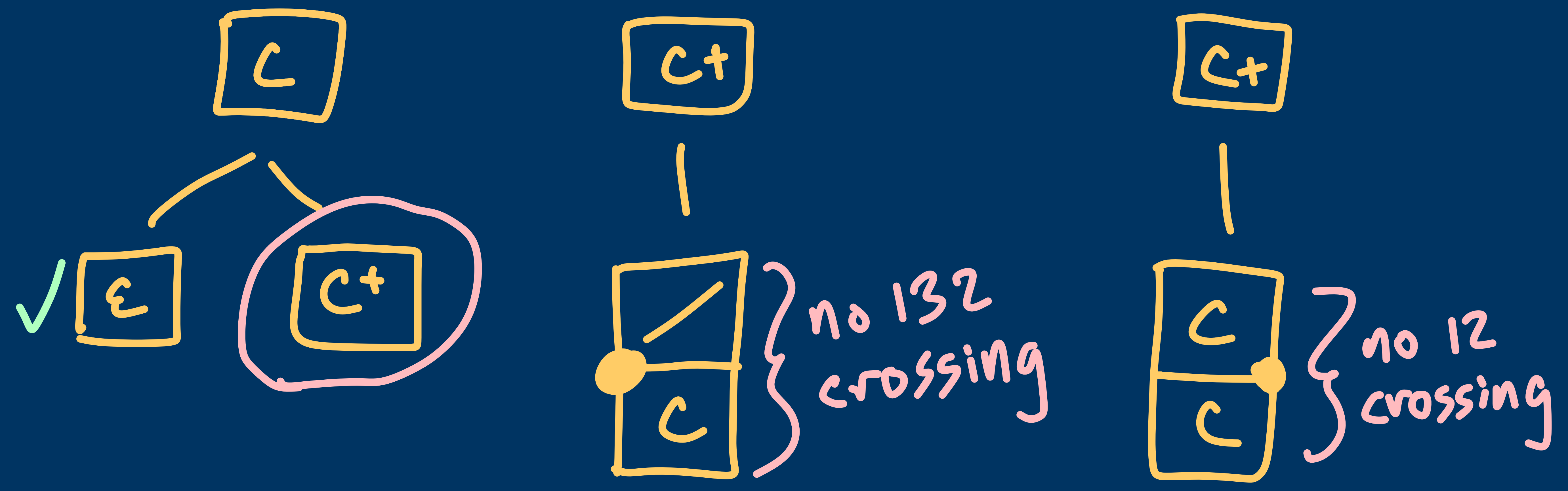

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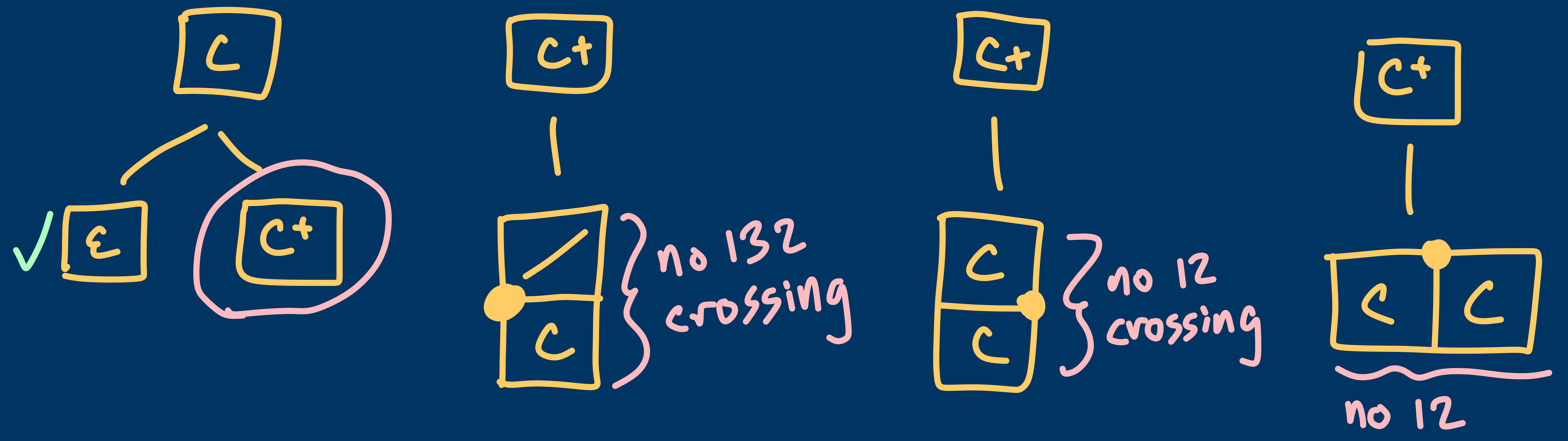

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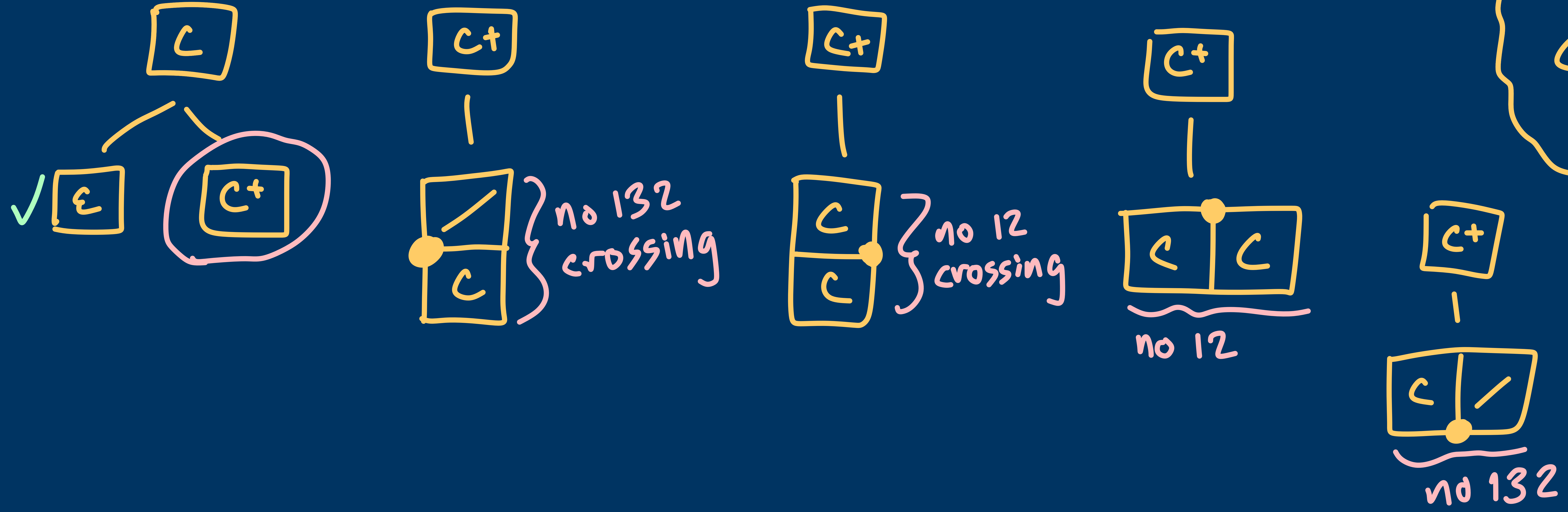

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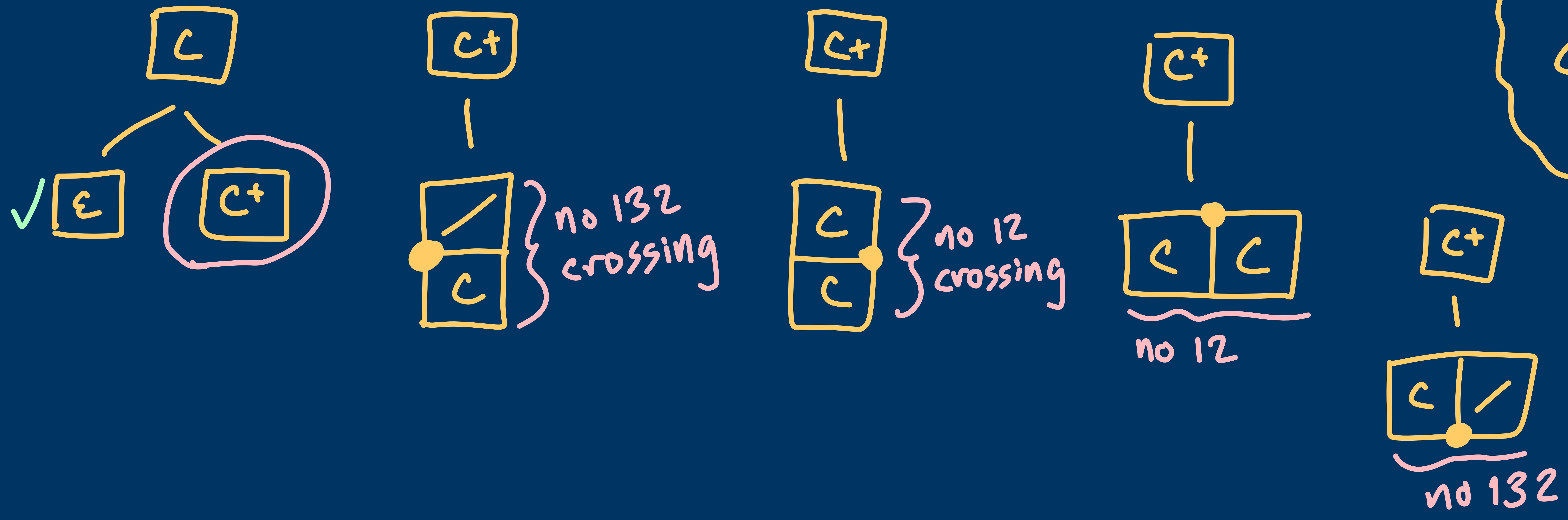

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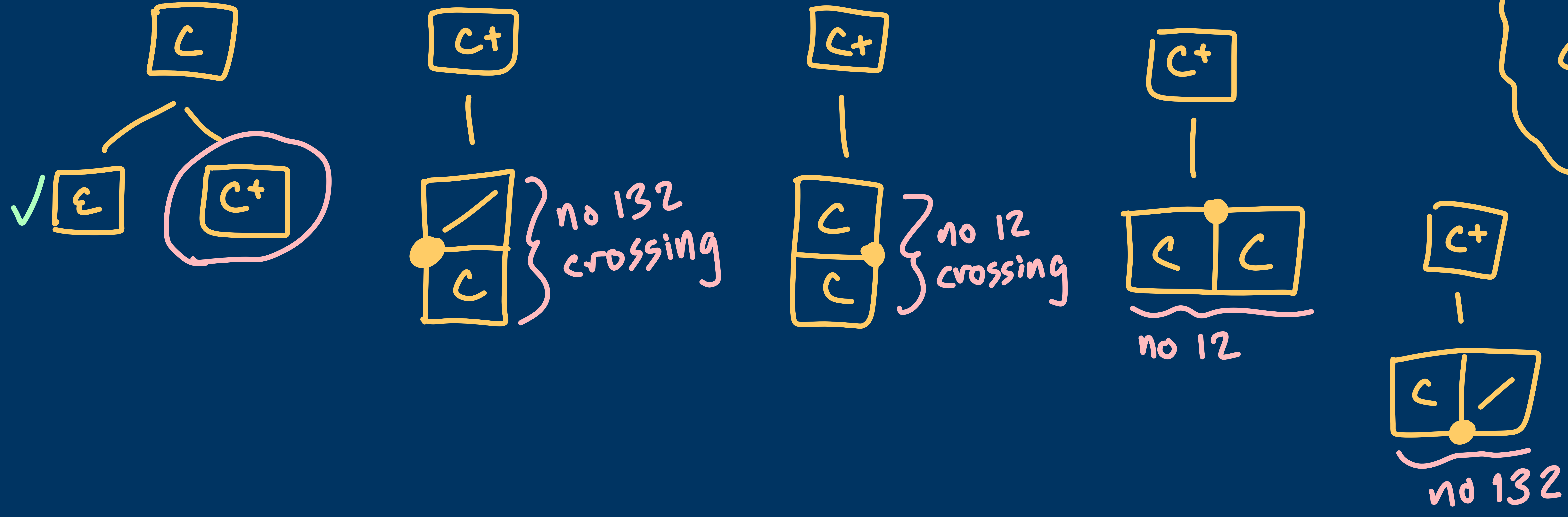

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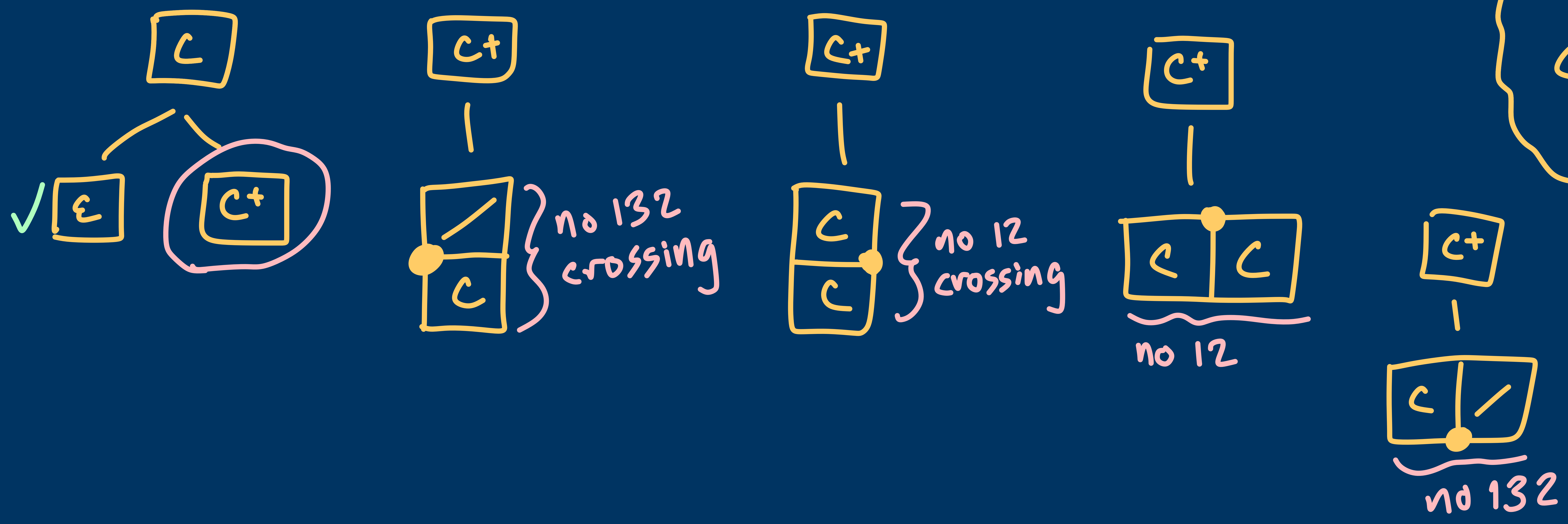

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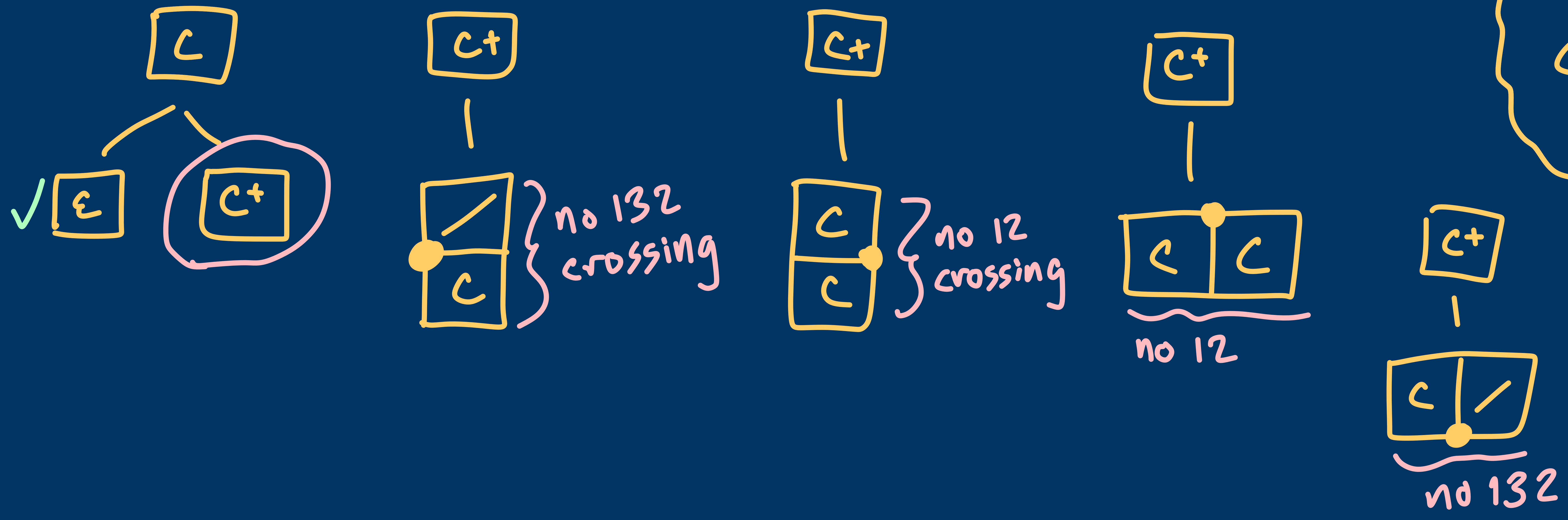

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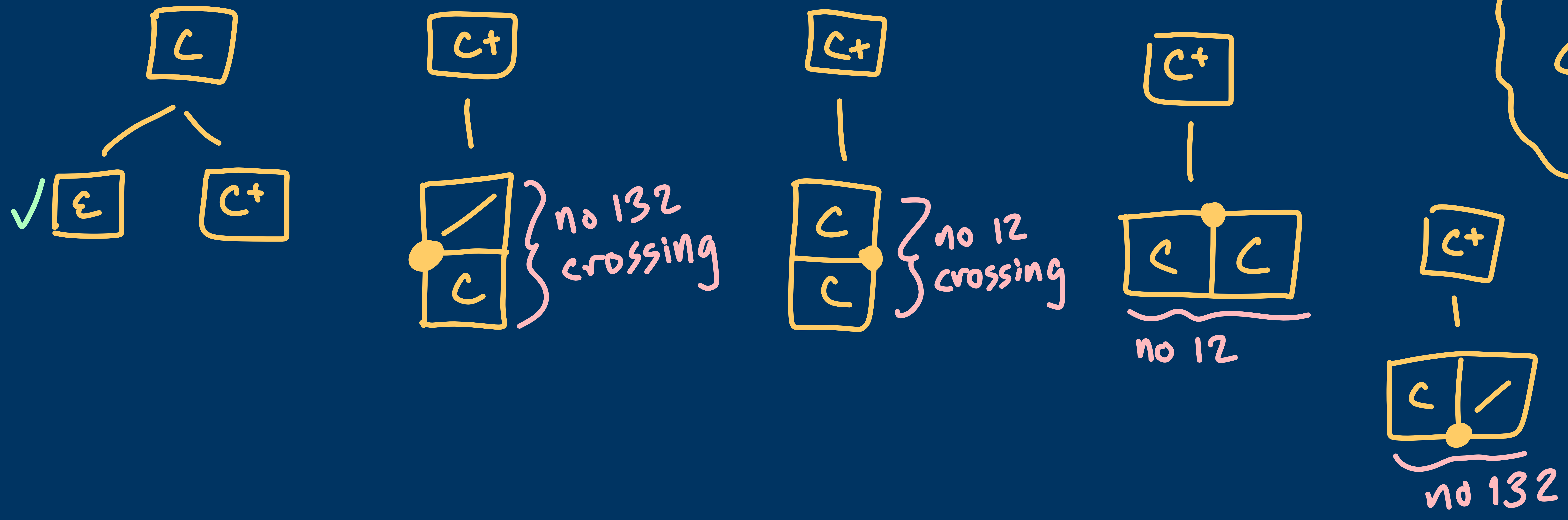

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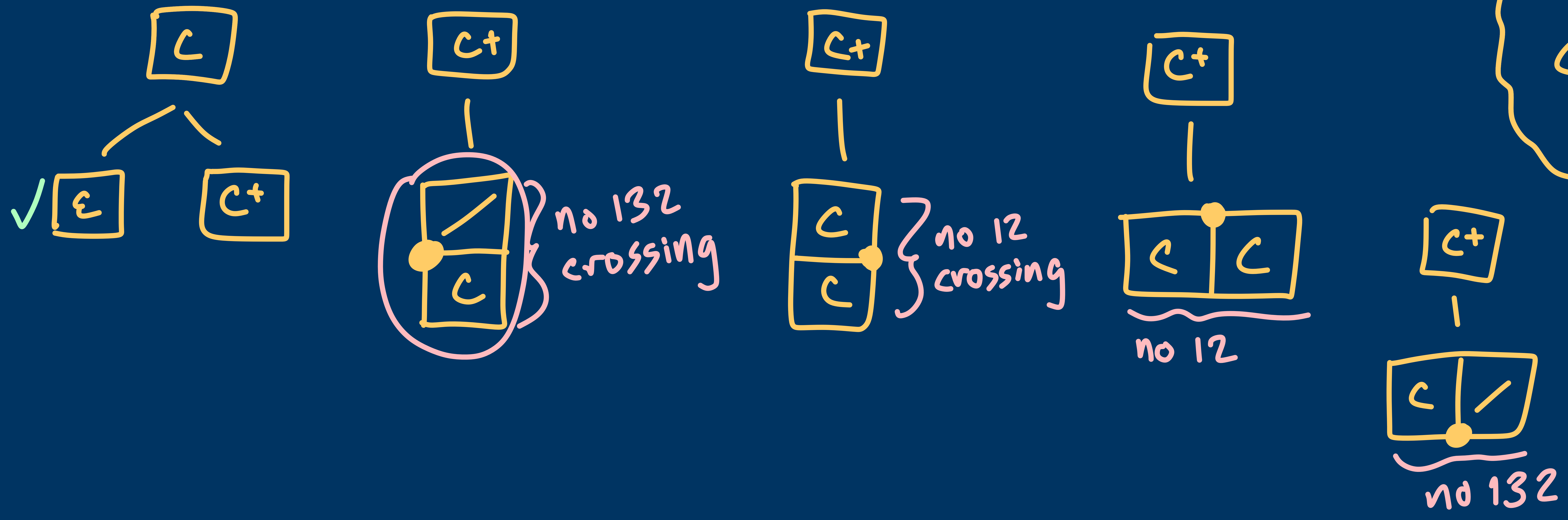

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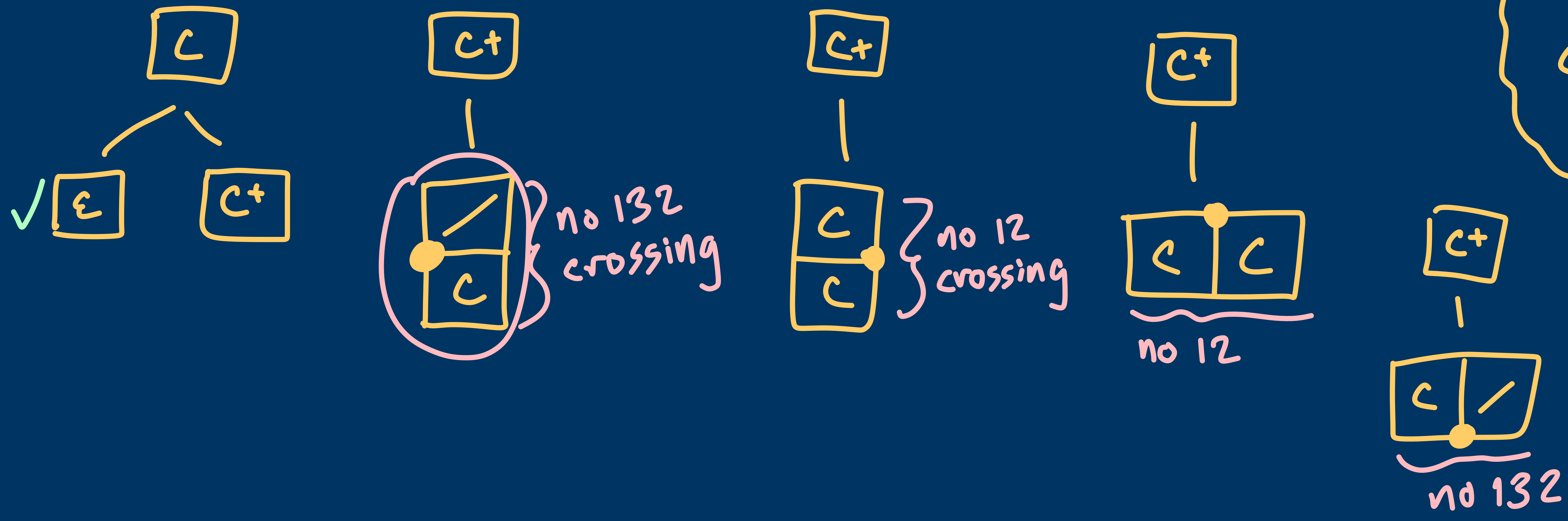

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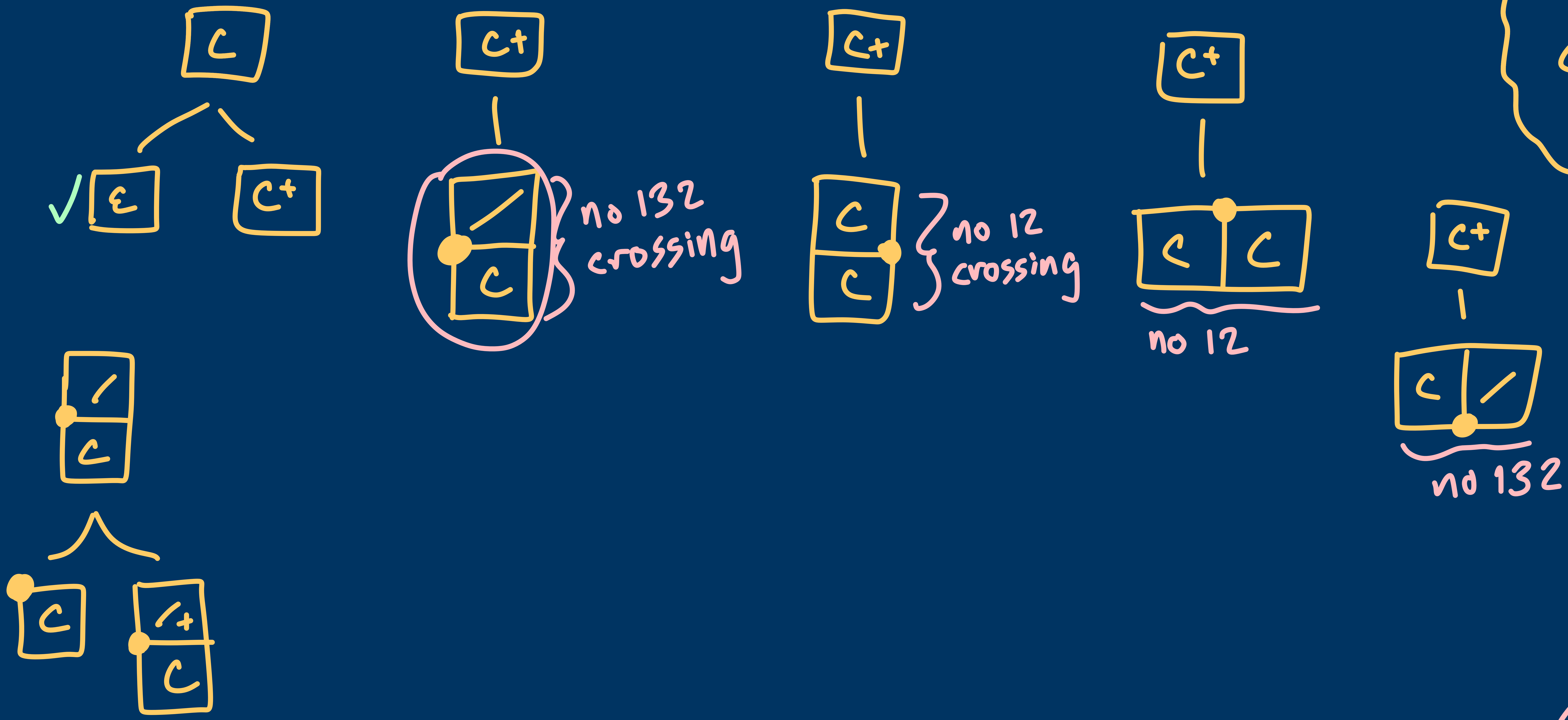

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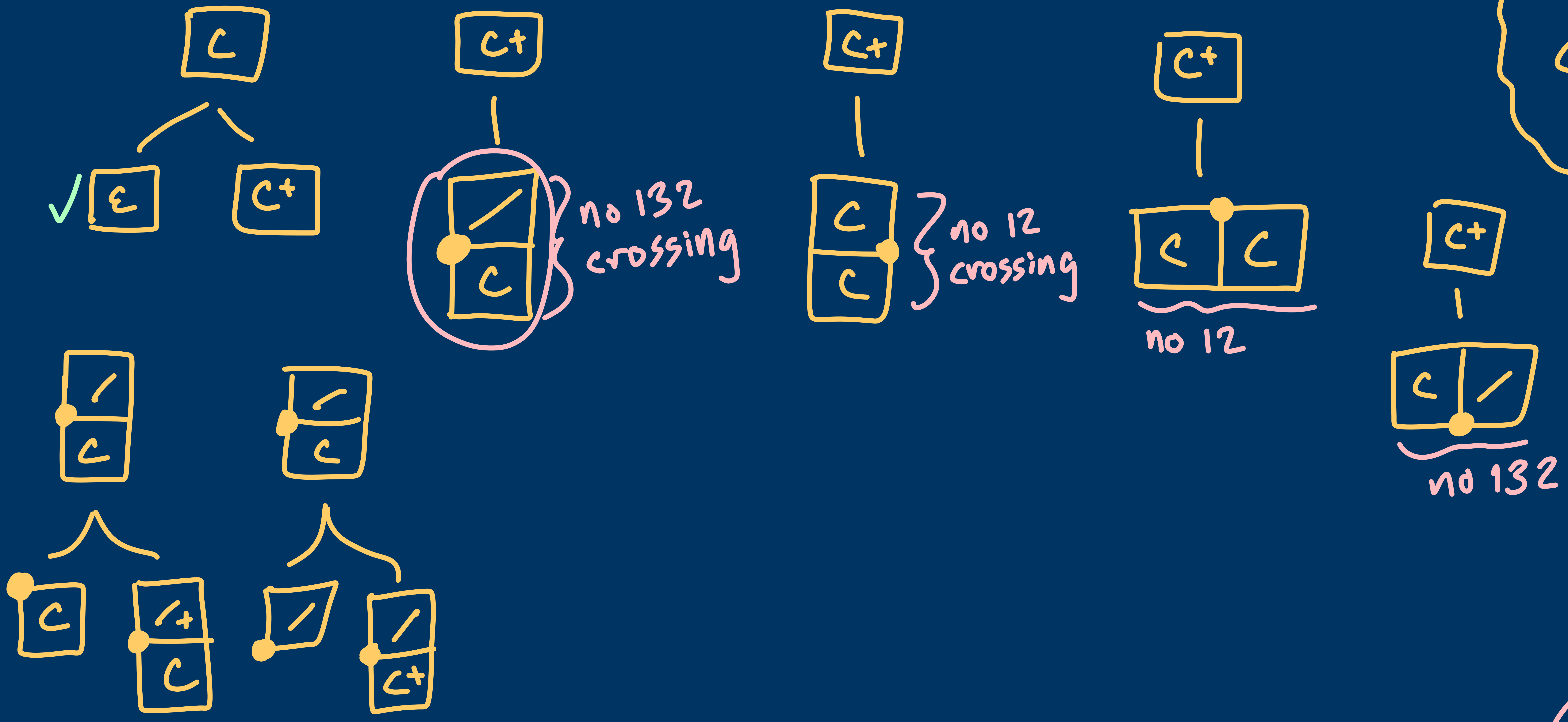

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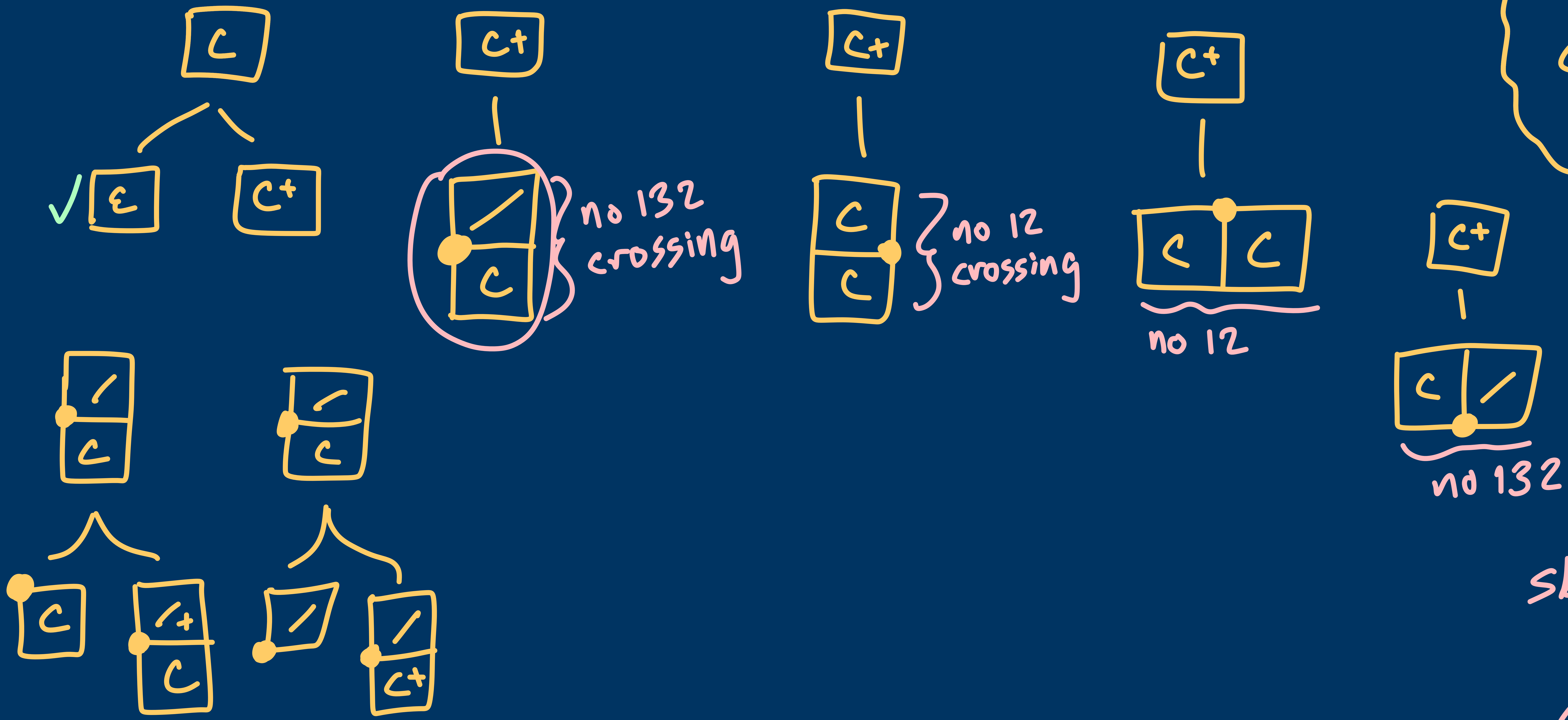

skipping ahead...

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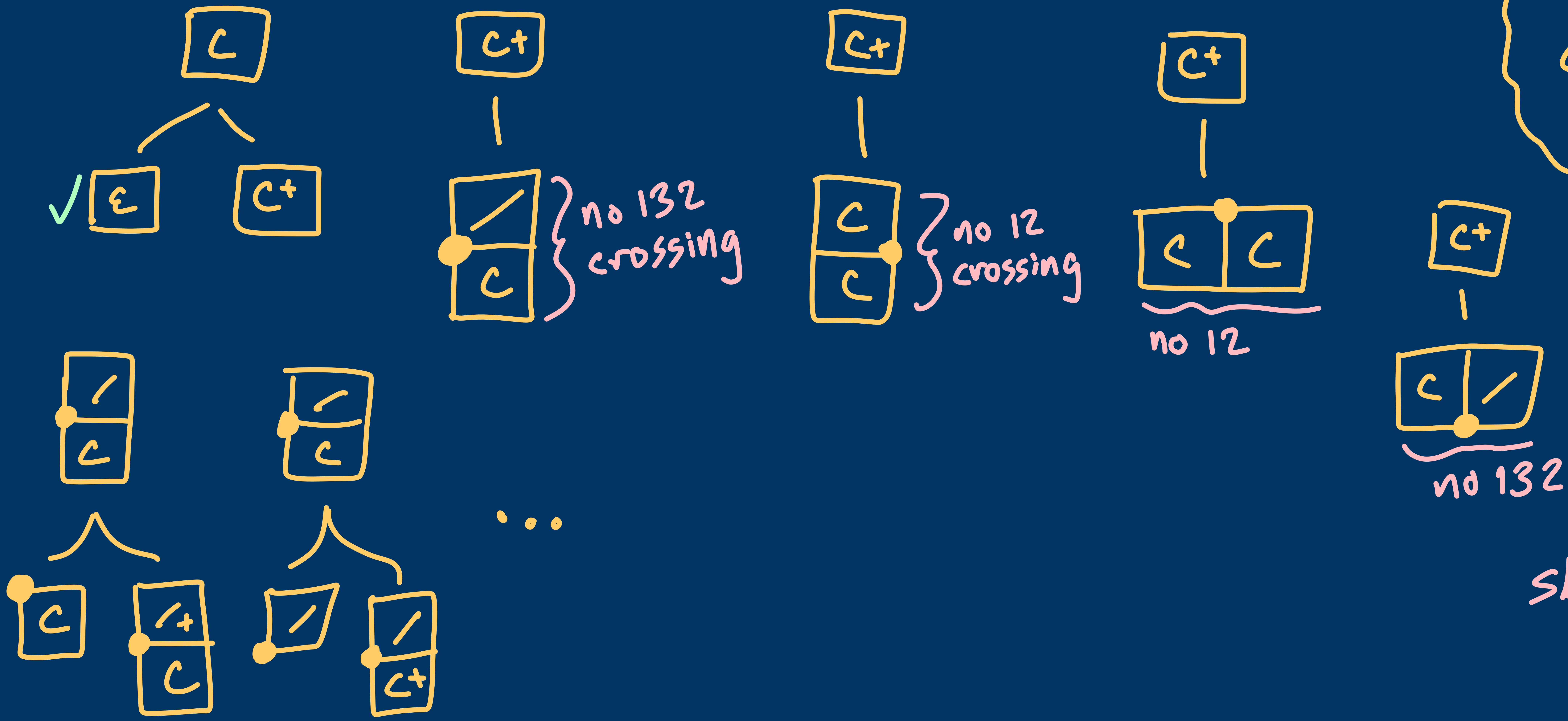

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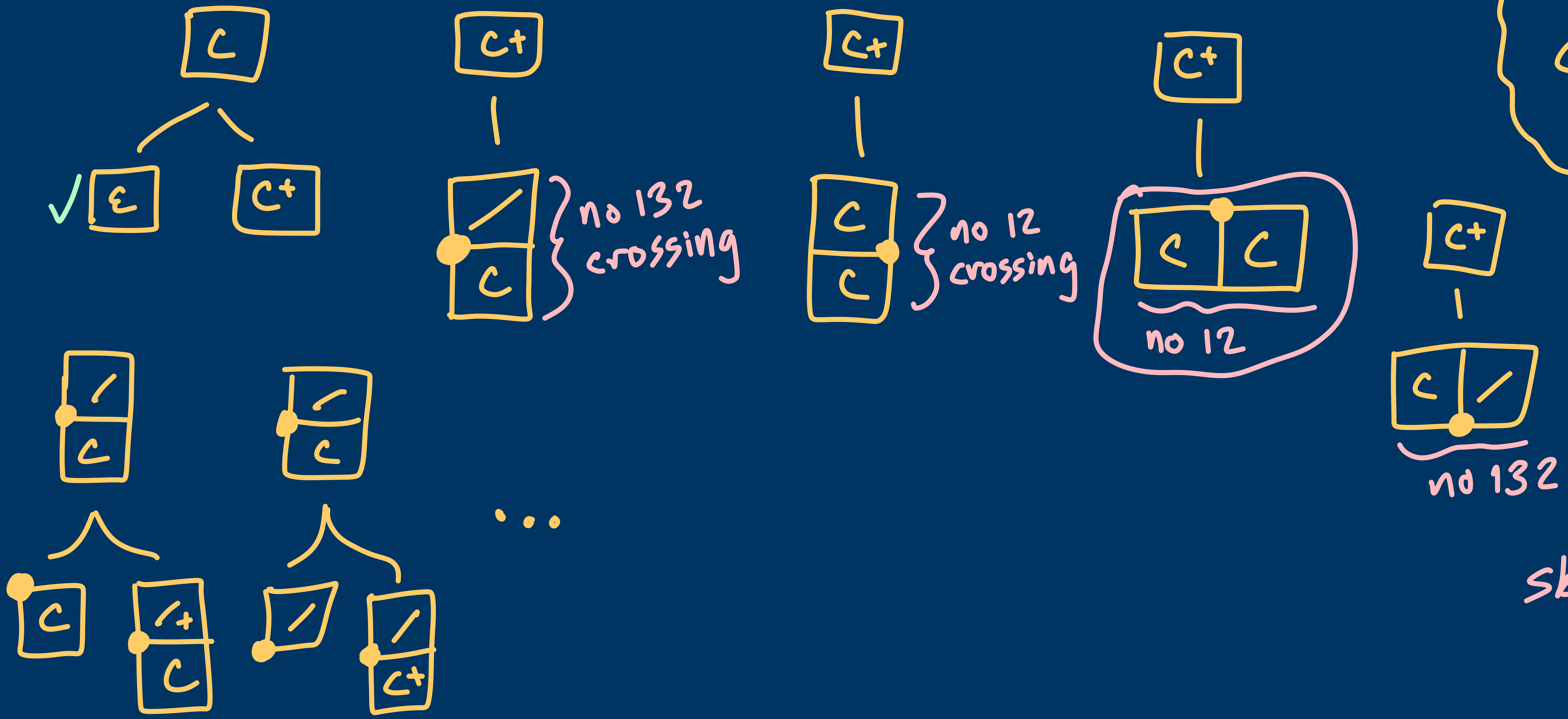

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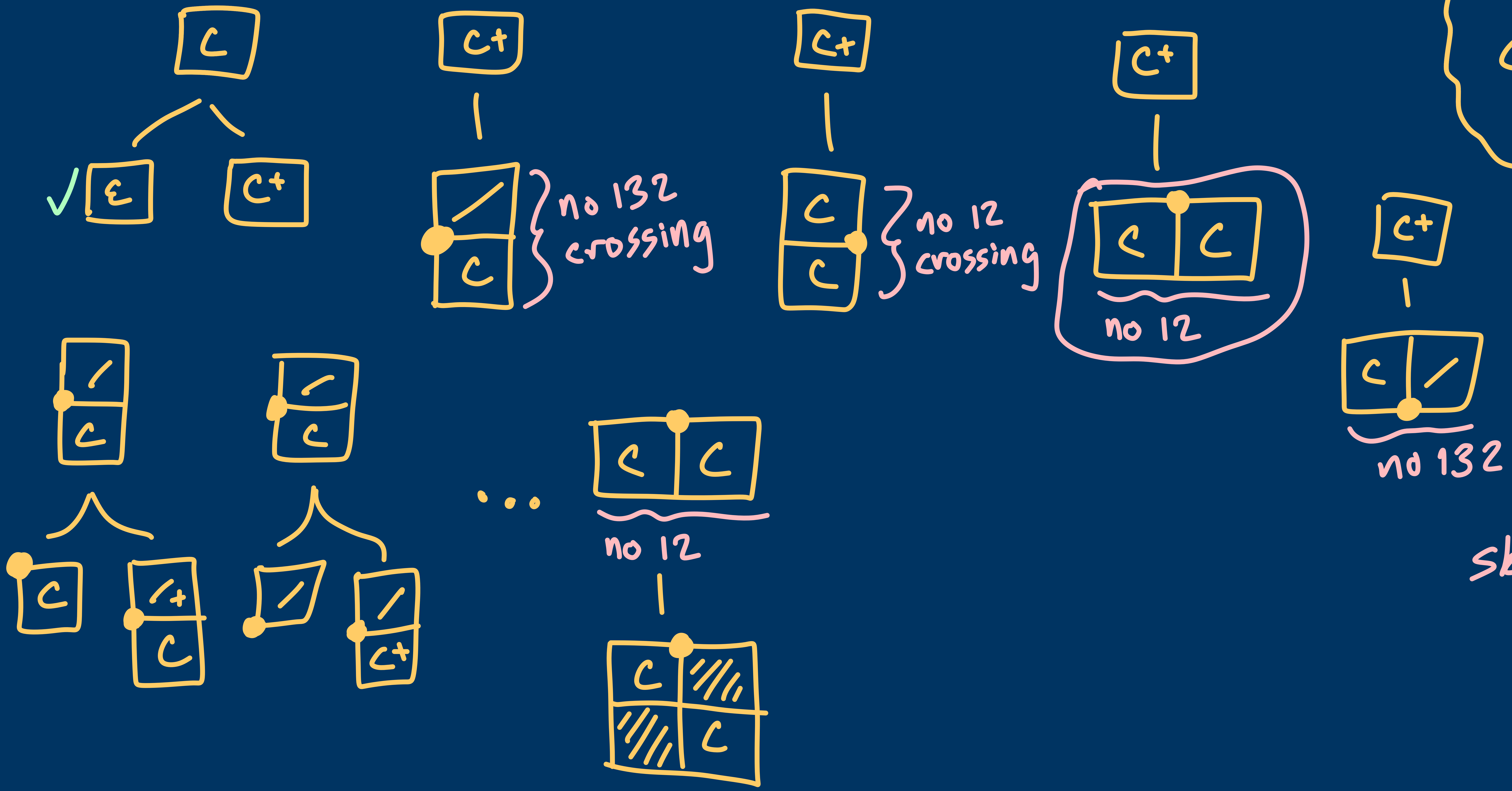

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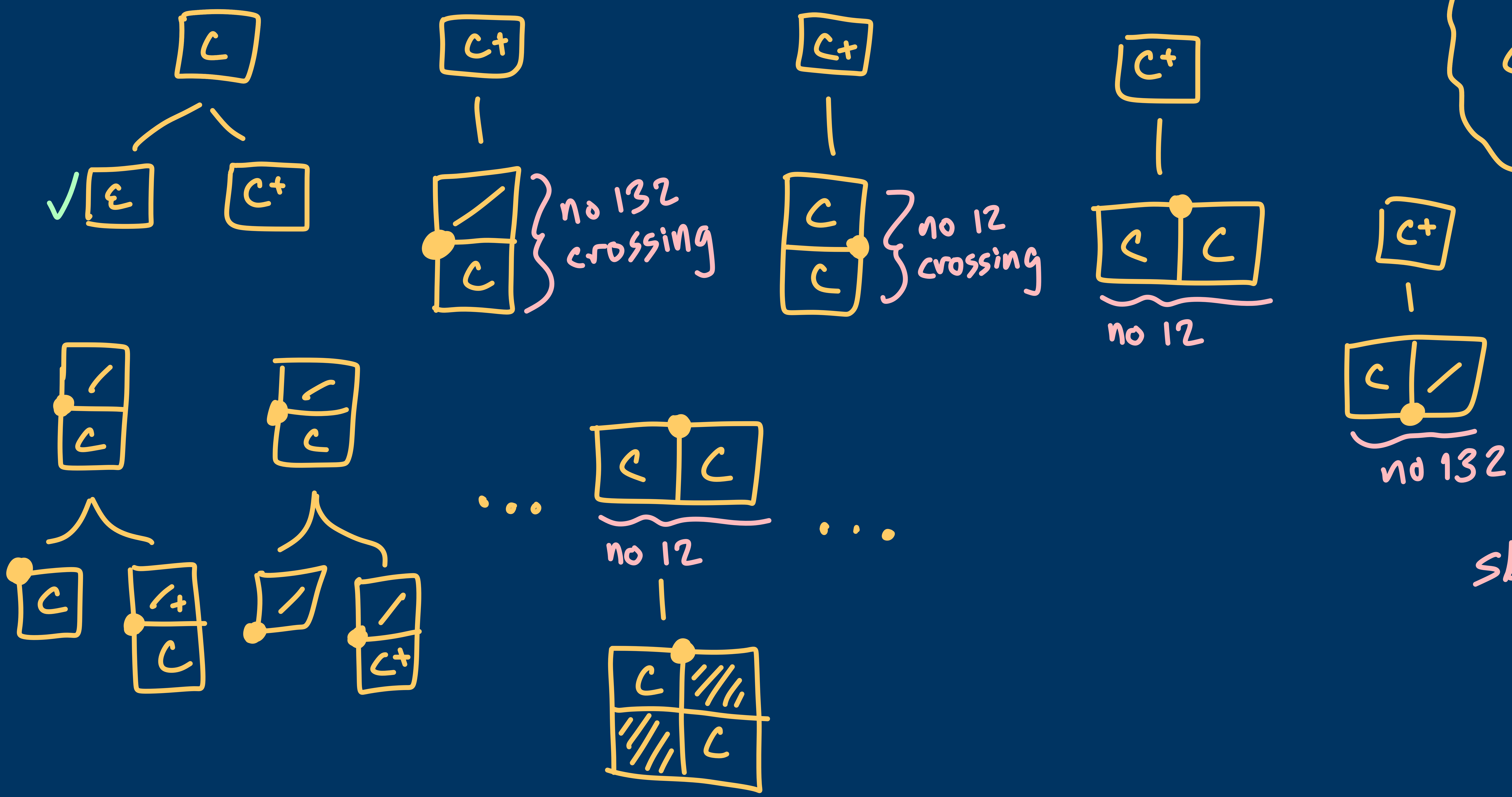

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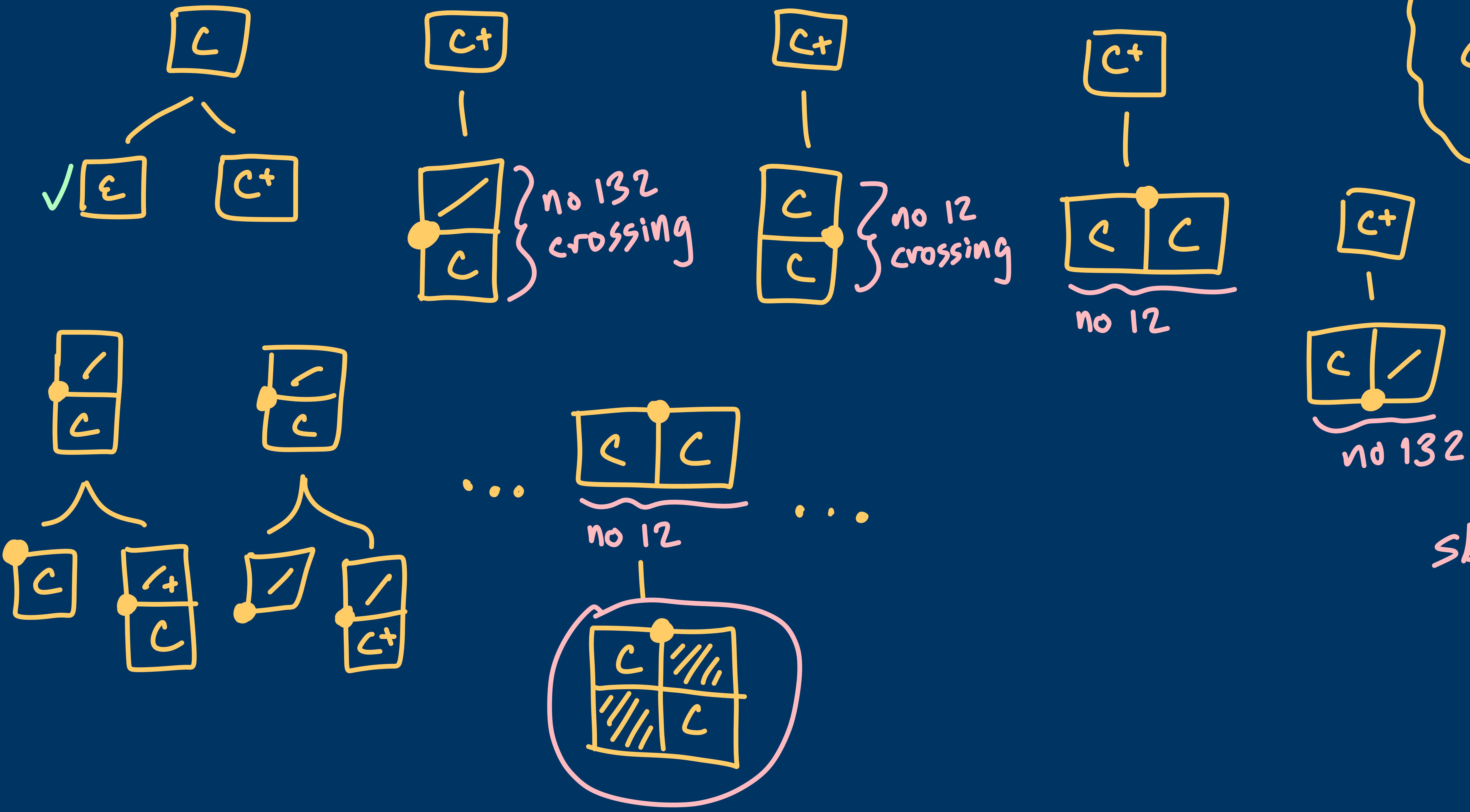

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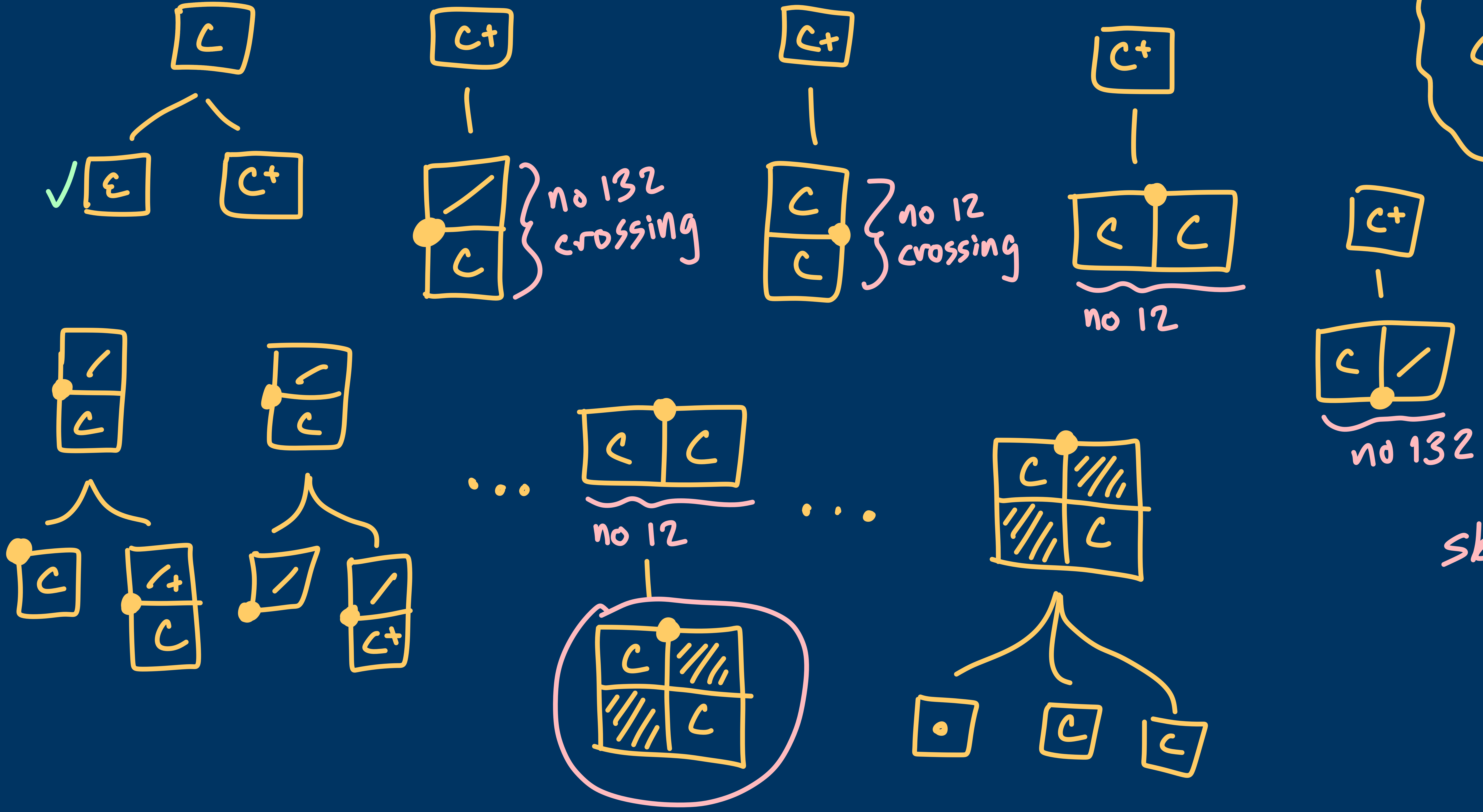

skipping ahead...

empty or not

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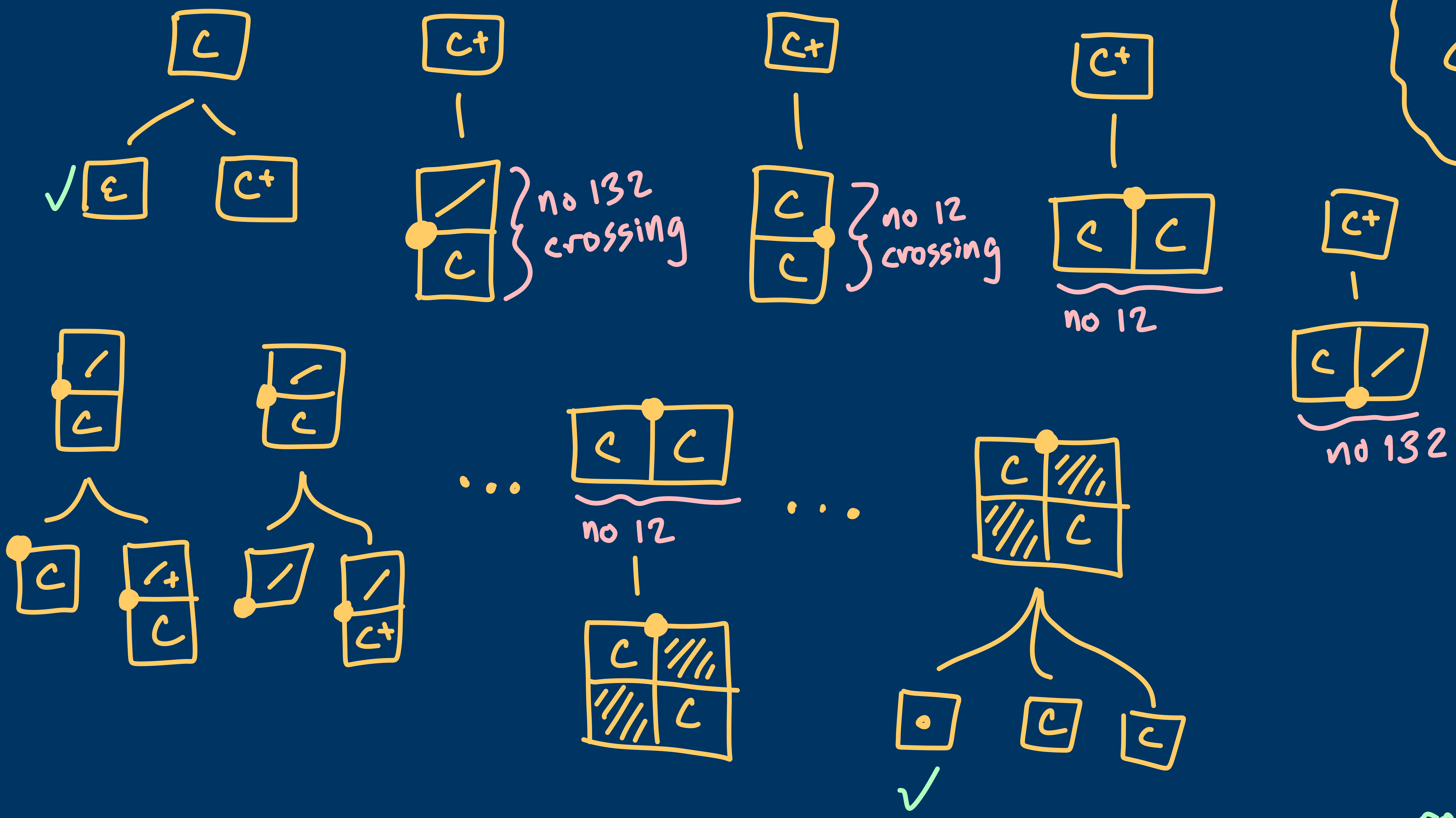

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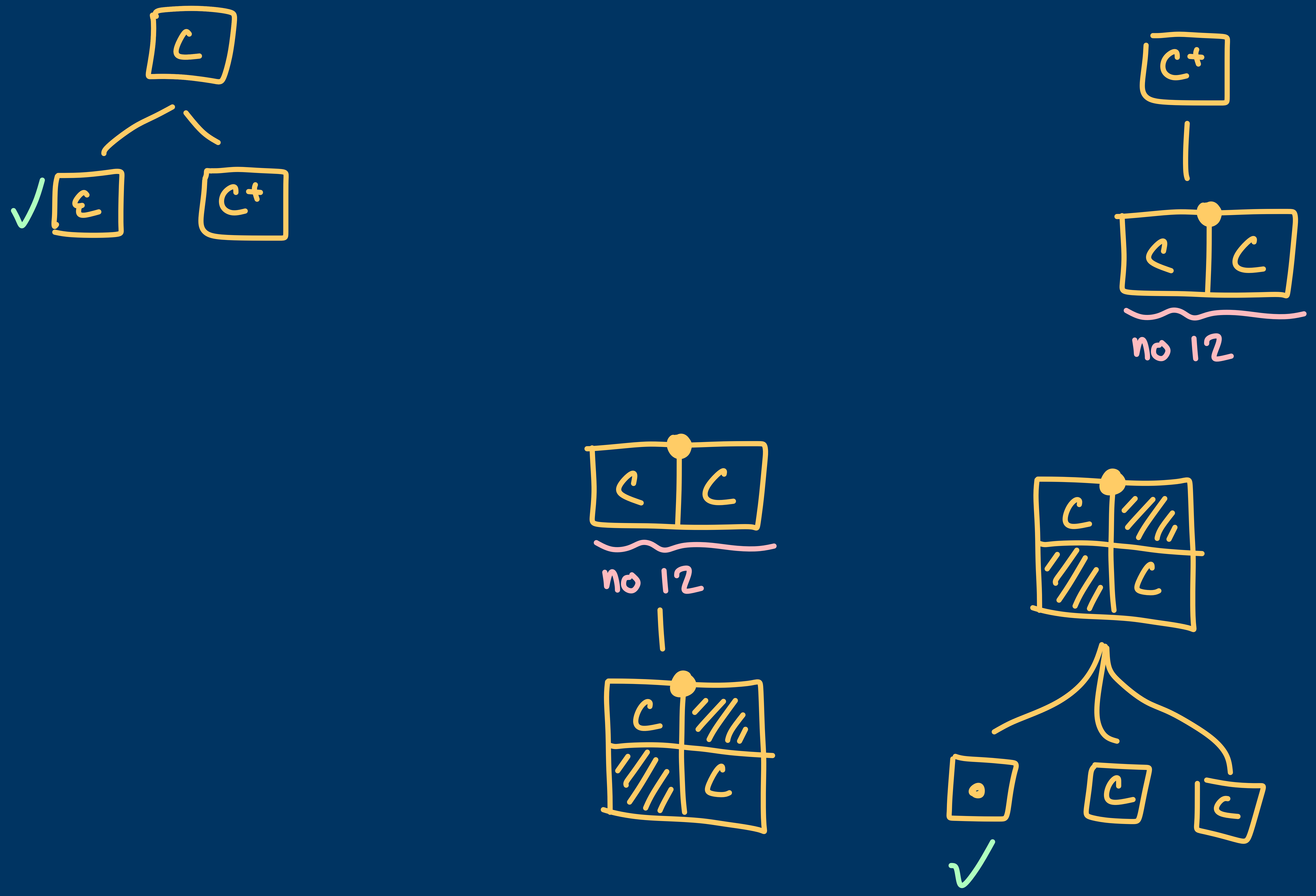

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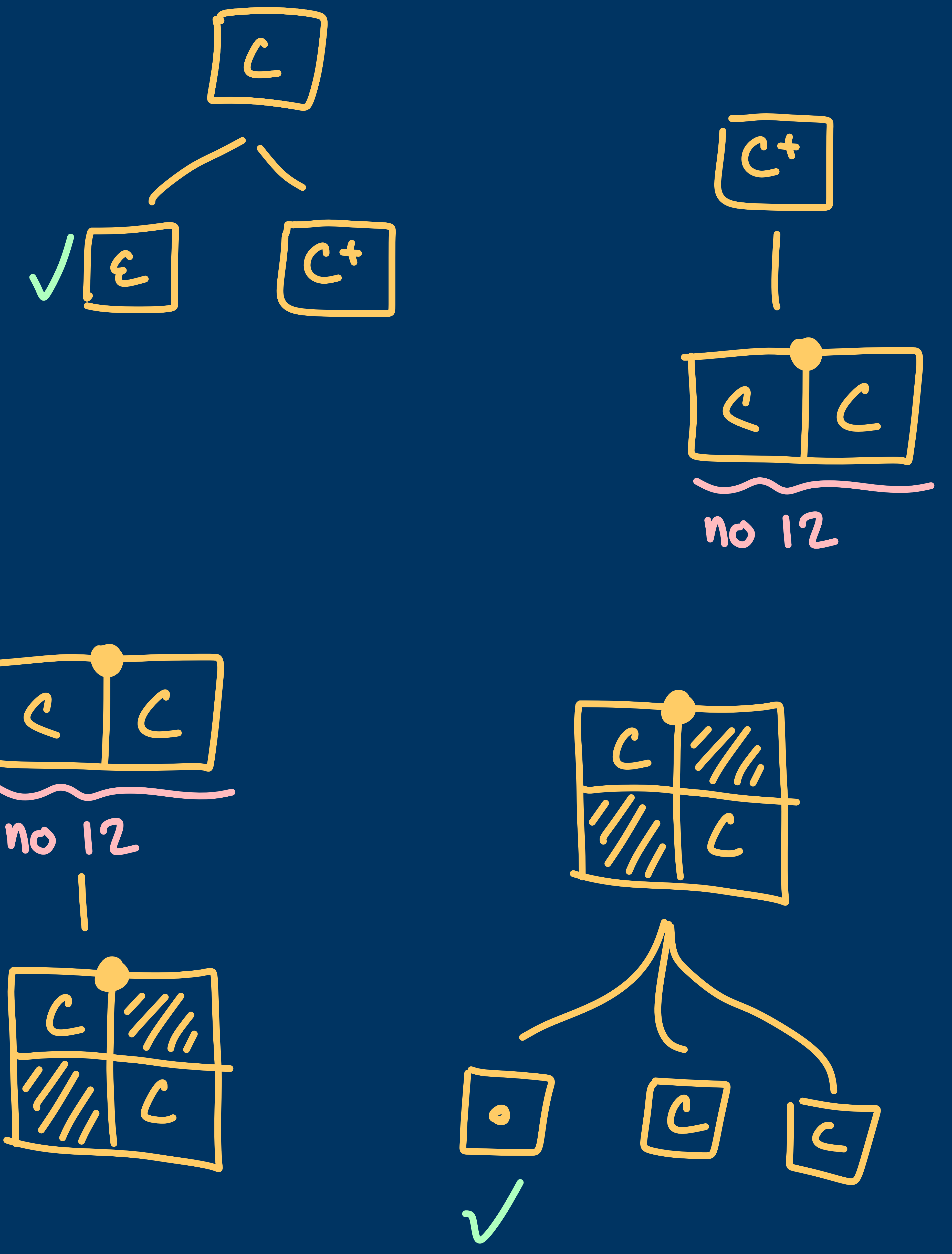

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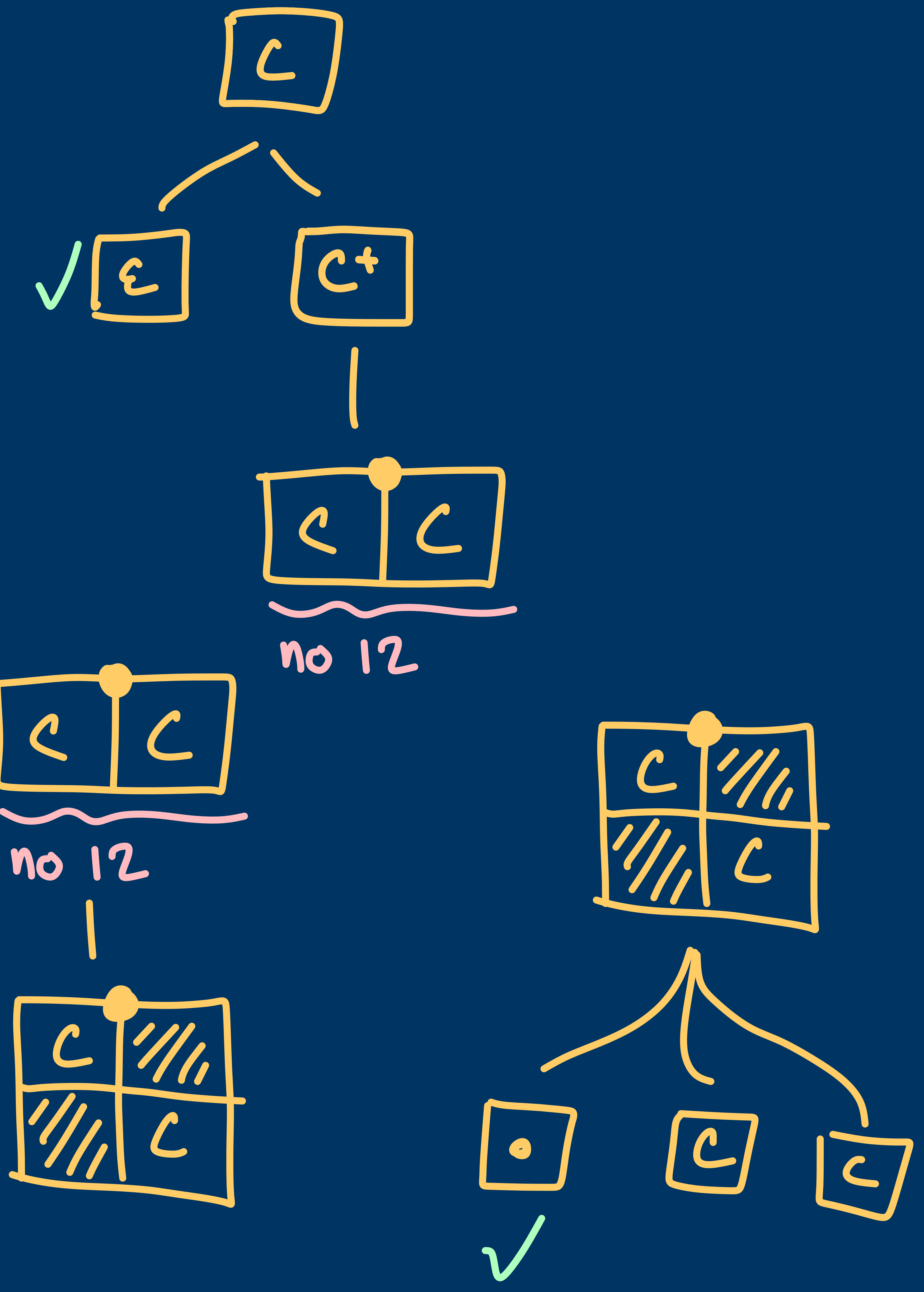

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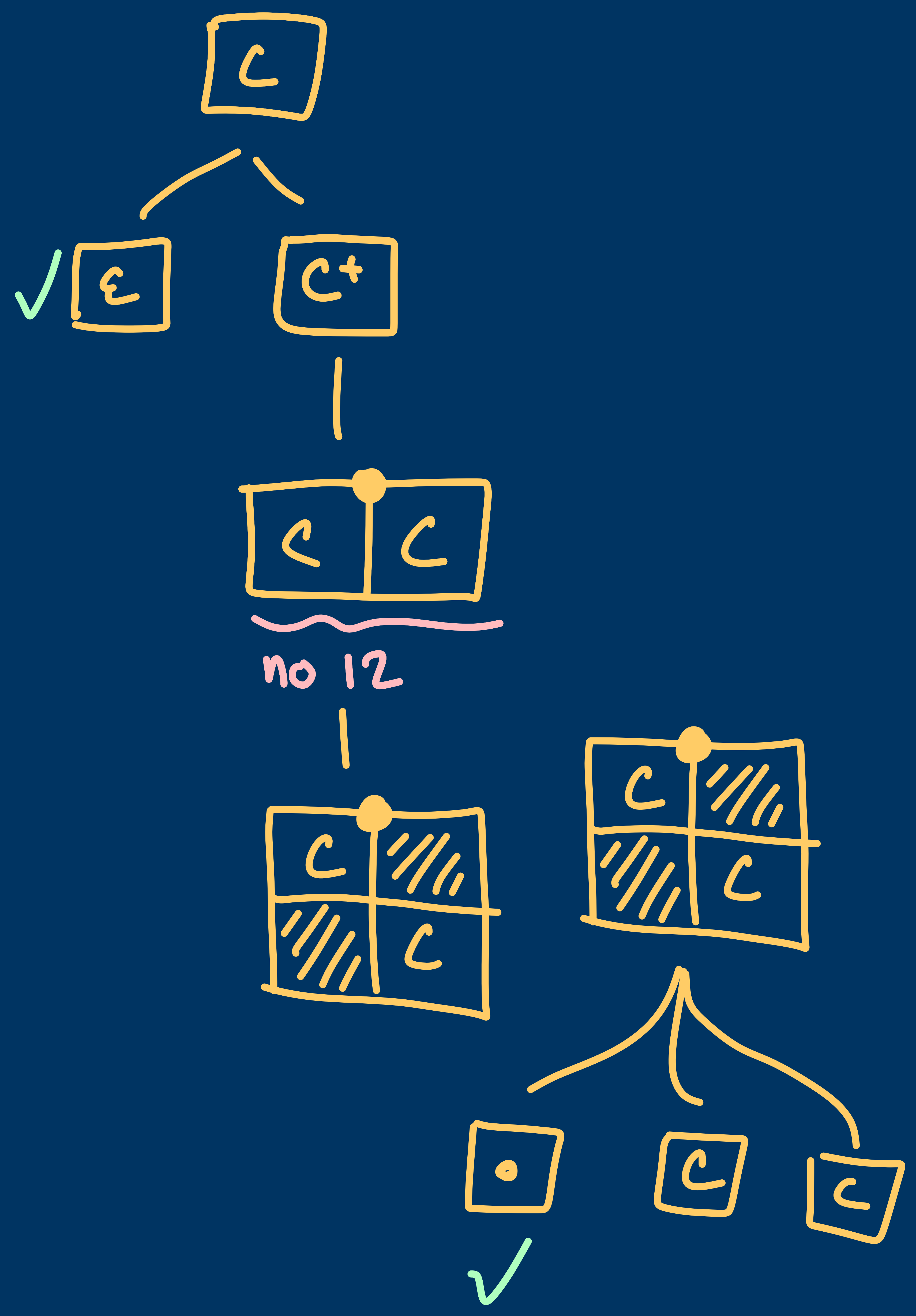

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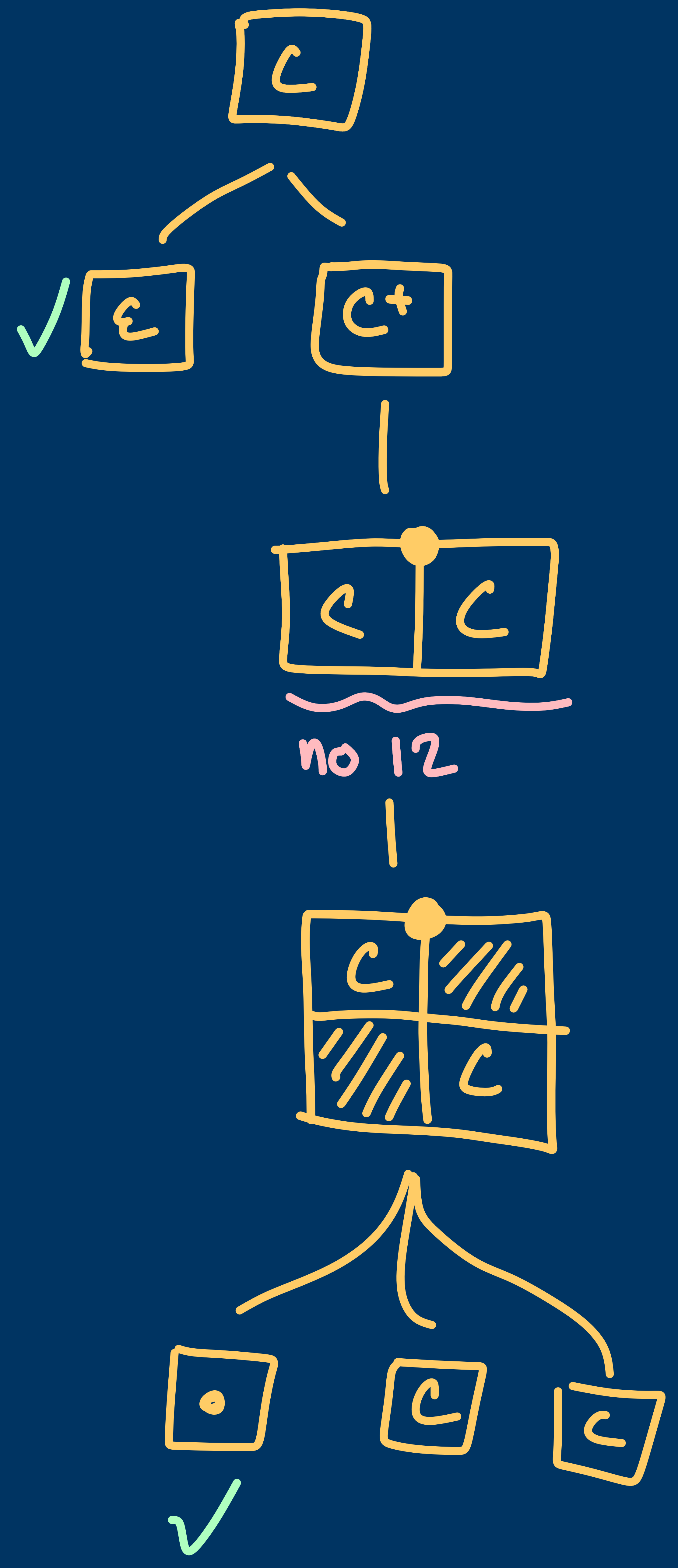

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Combinatorial Exploration

General outline:

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        - ...
- Each time you apply a strategy to a set, you make a puzzle piece. Search the pile of puzzle pieces for a subset that makes a combinatorial specification. If you find one, you win!

  polynomial-time counting algorithm, system of equations for the GF,
  uniform random sampling routine, exhaustive generation (but slow)

Tilings

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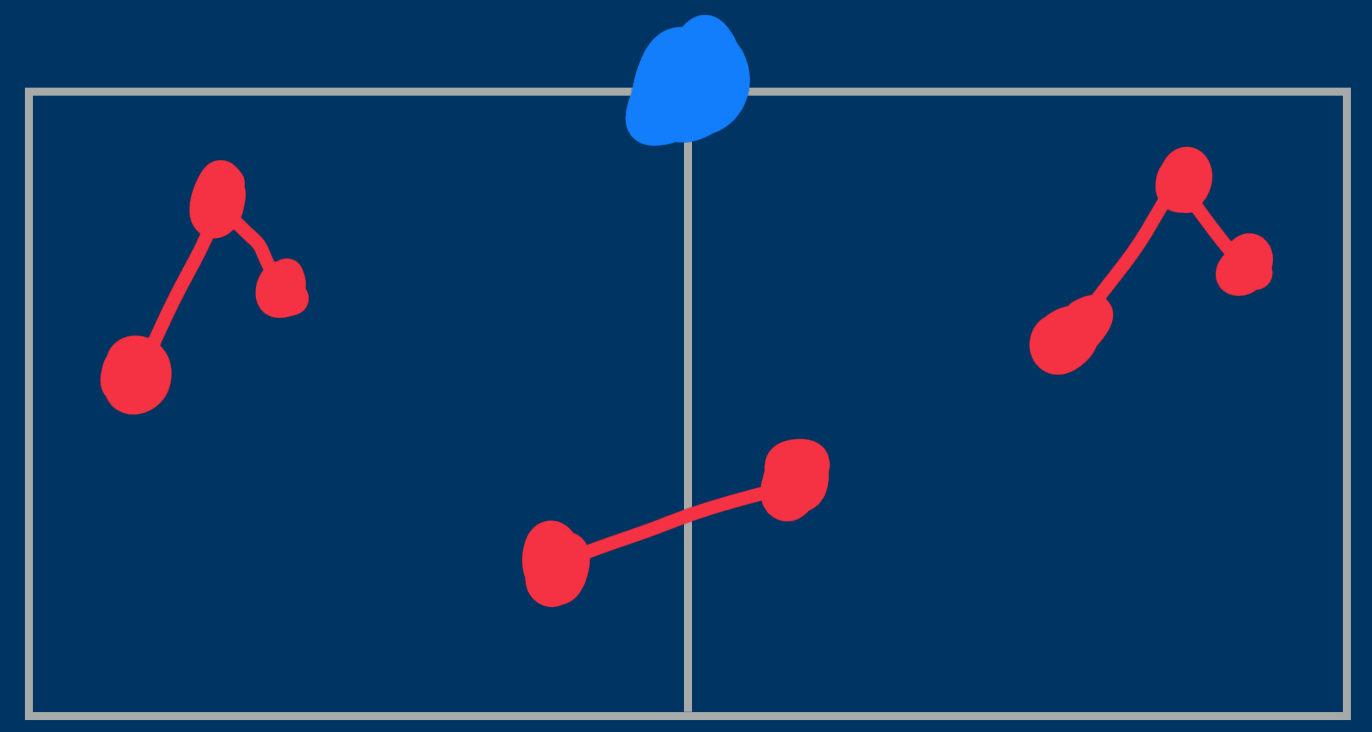
So with any computational method, you have to decide on a finite representation for some sets of permutations.

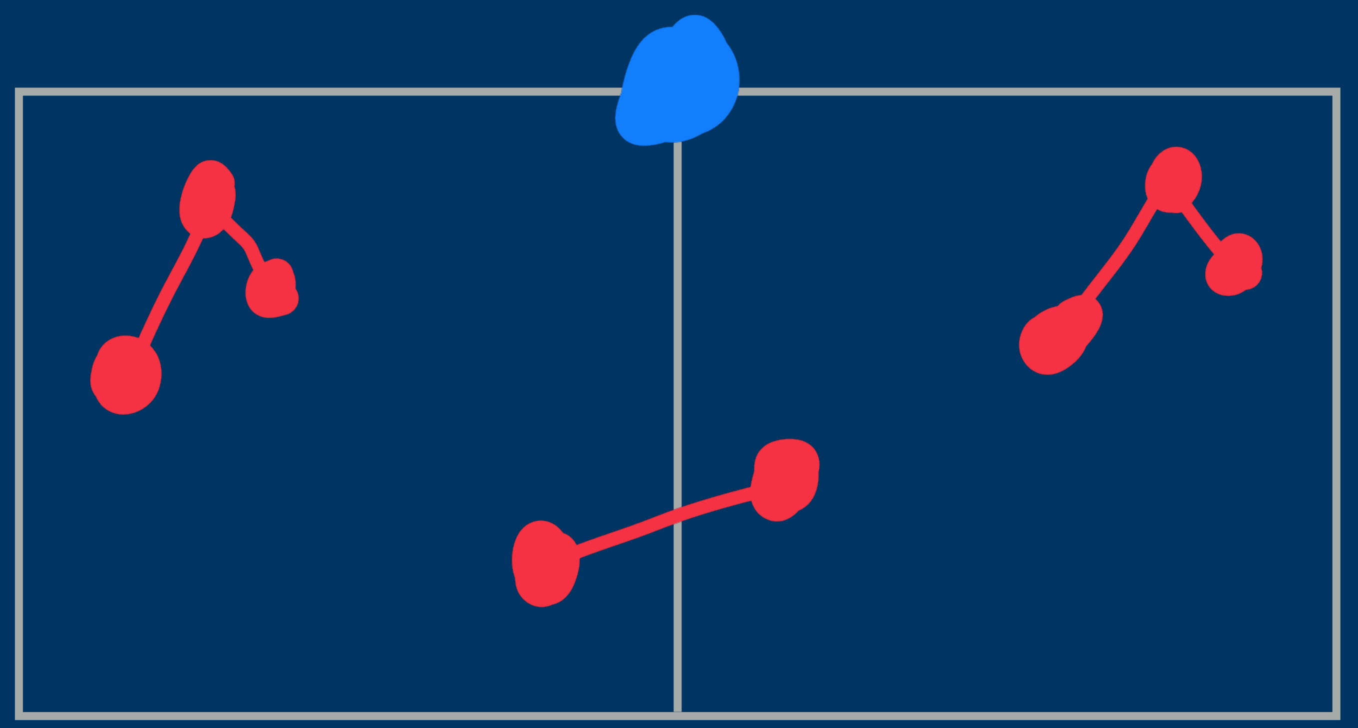
One really good idea we had, after a whole lot of really bad ideas, is a representation called a “Tiling”.

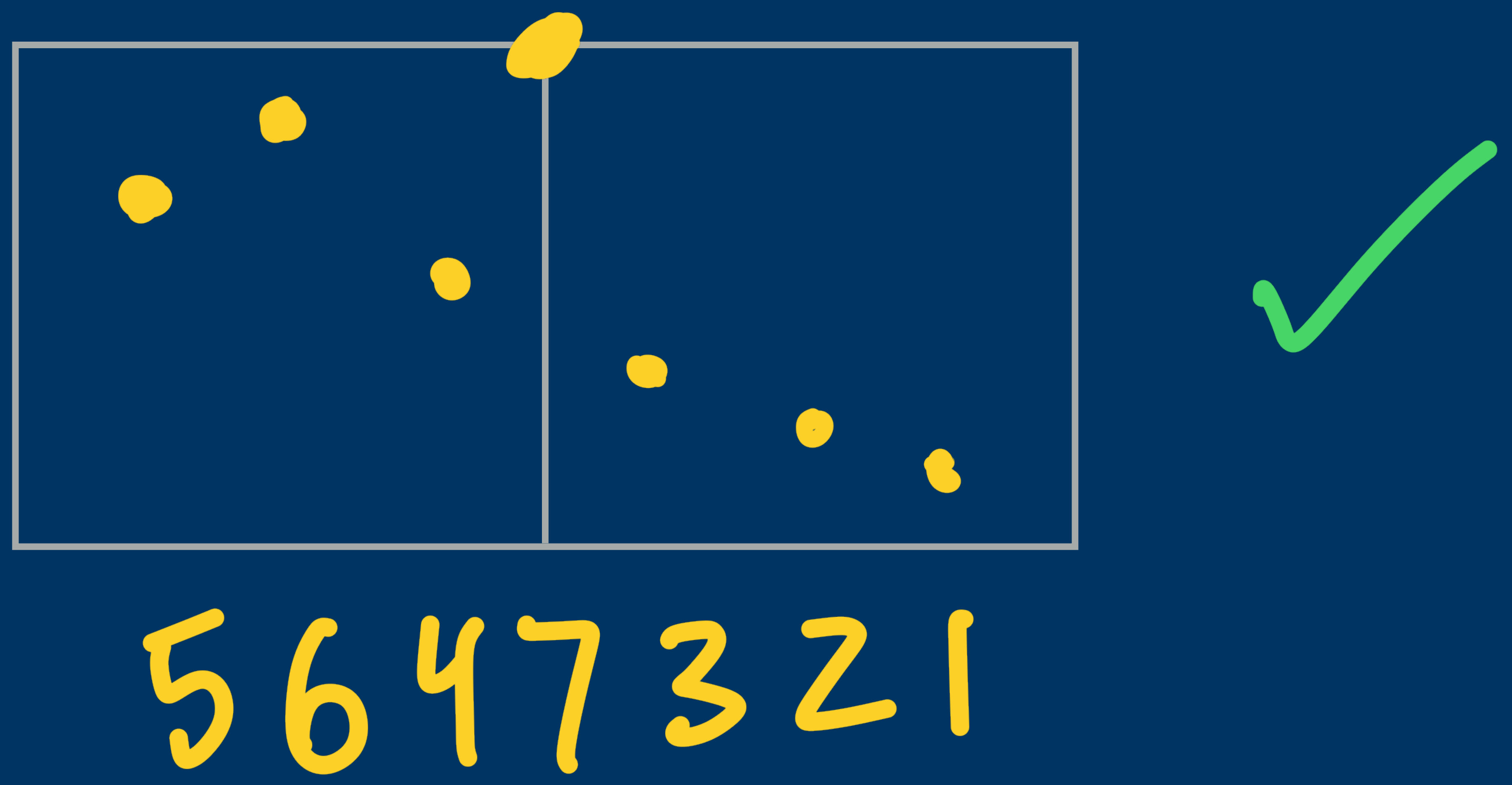
A tiling is a grid of cells that has “obstructions” that tell you patterns that can’t appear, and “requirements” that tell you patterns that must appear.

Tilings



Tilings





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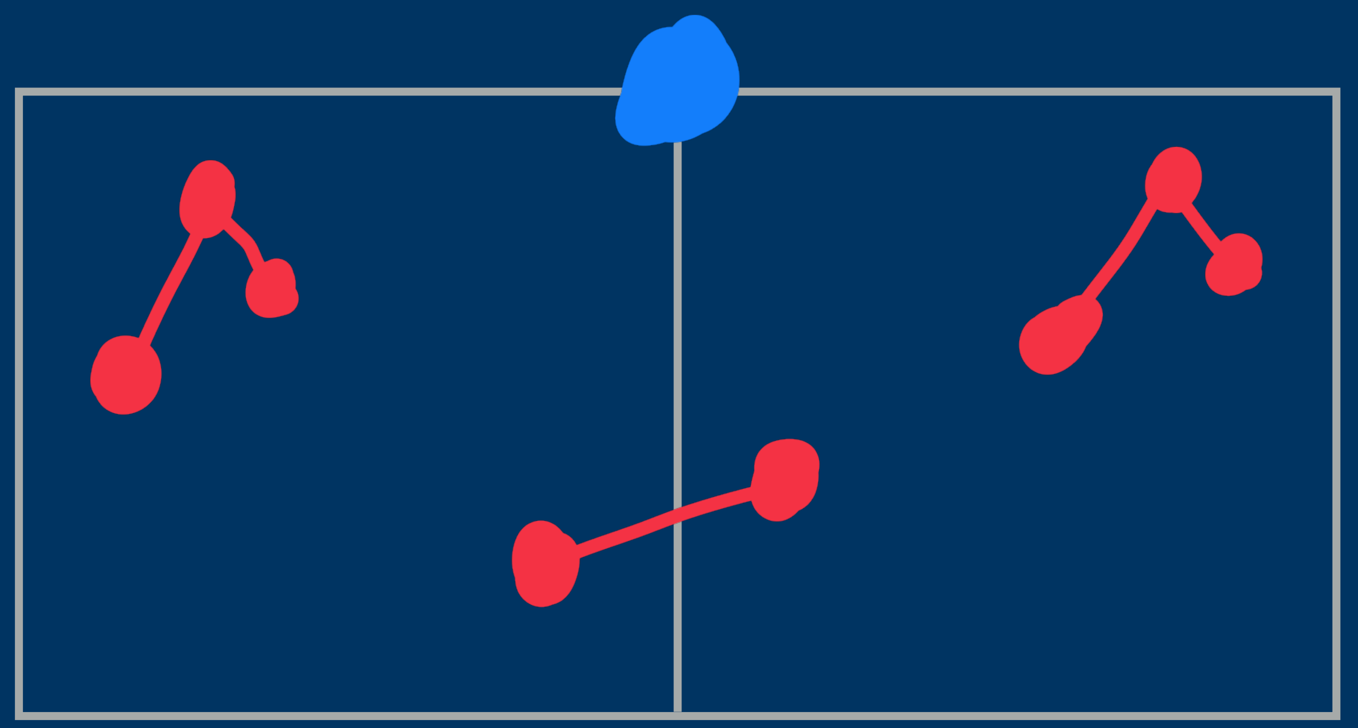
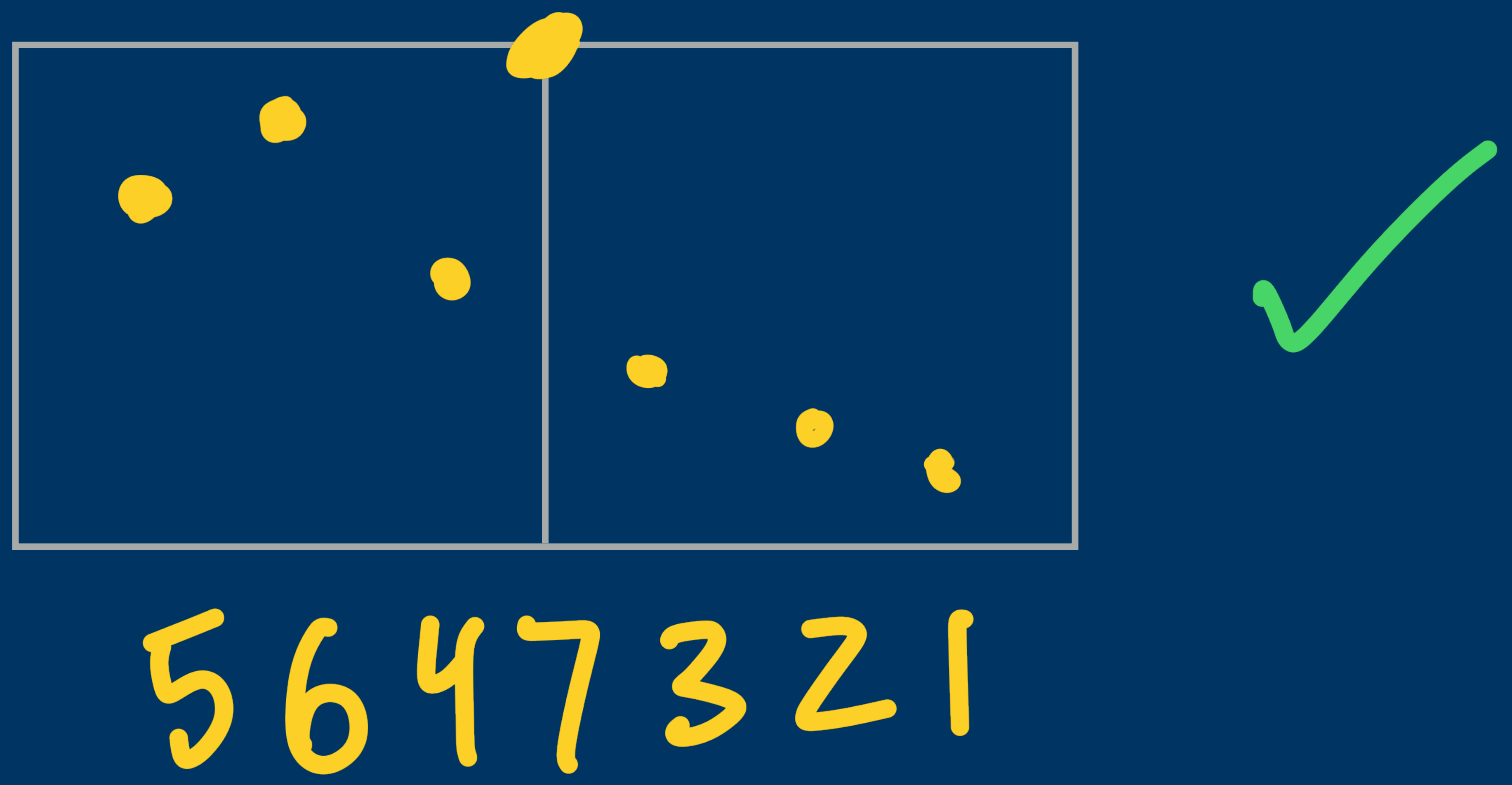
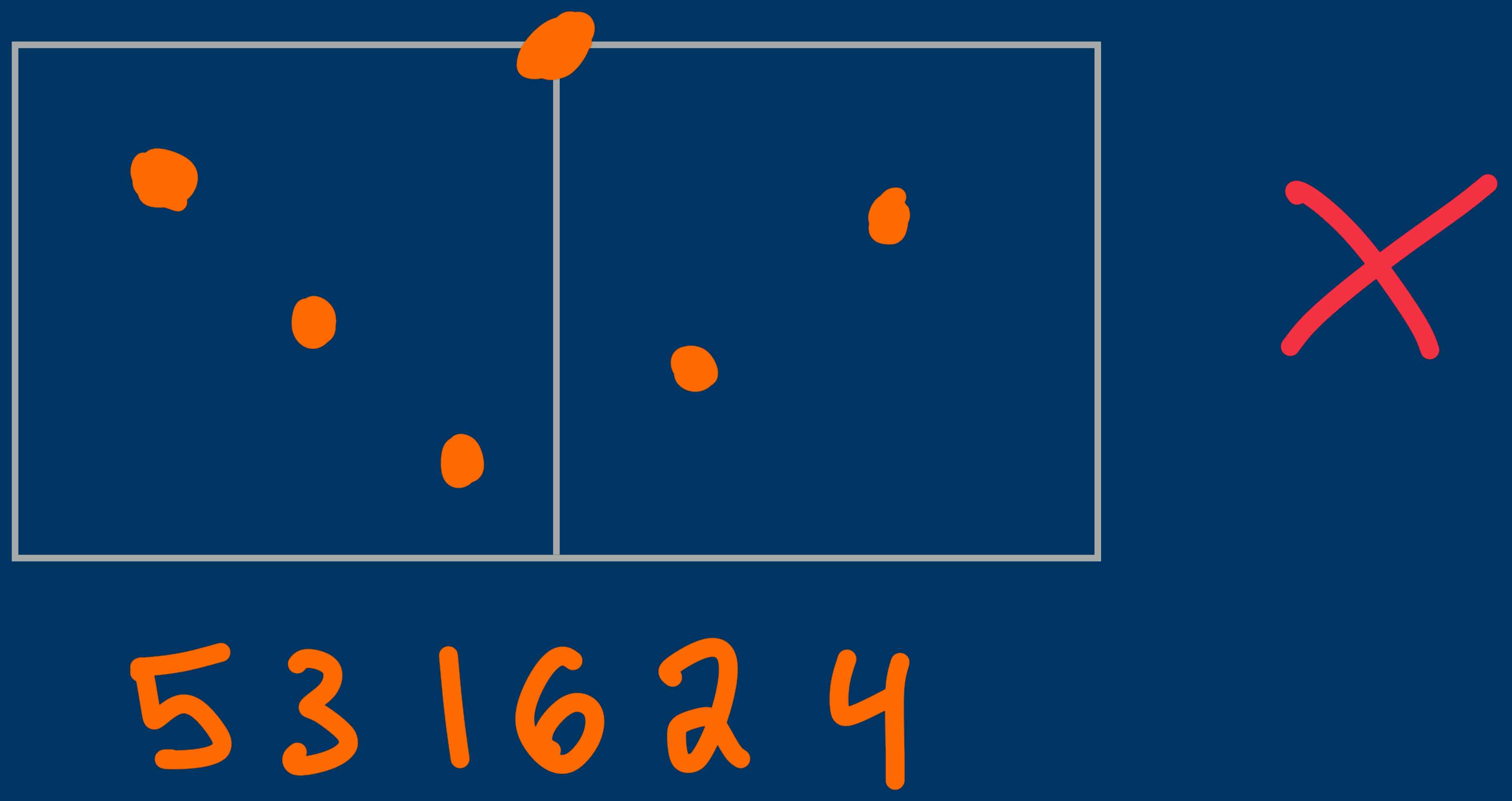




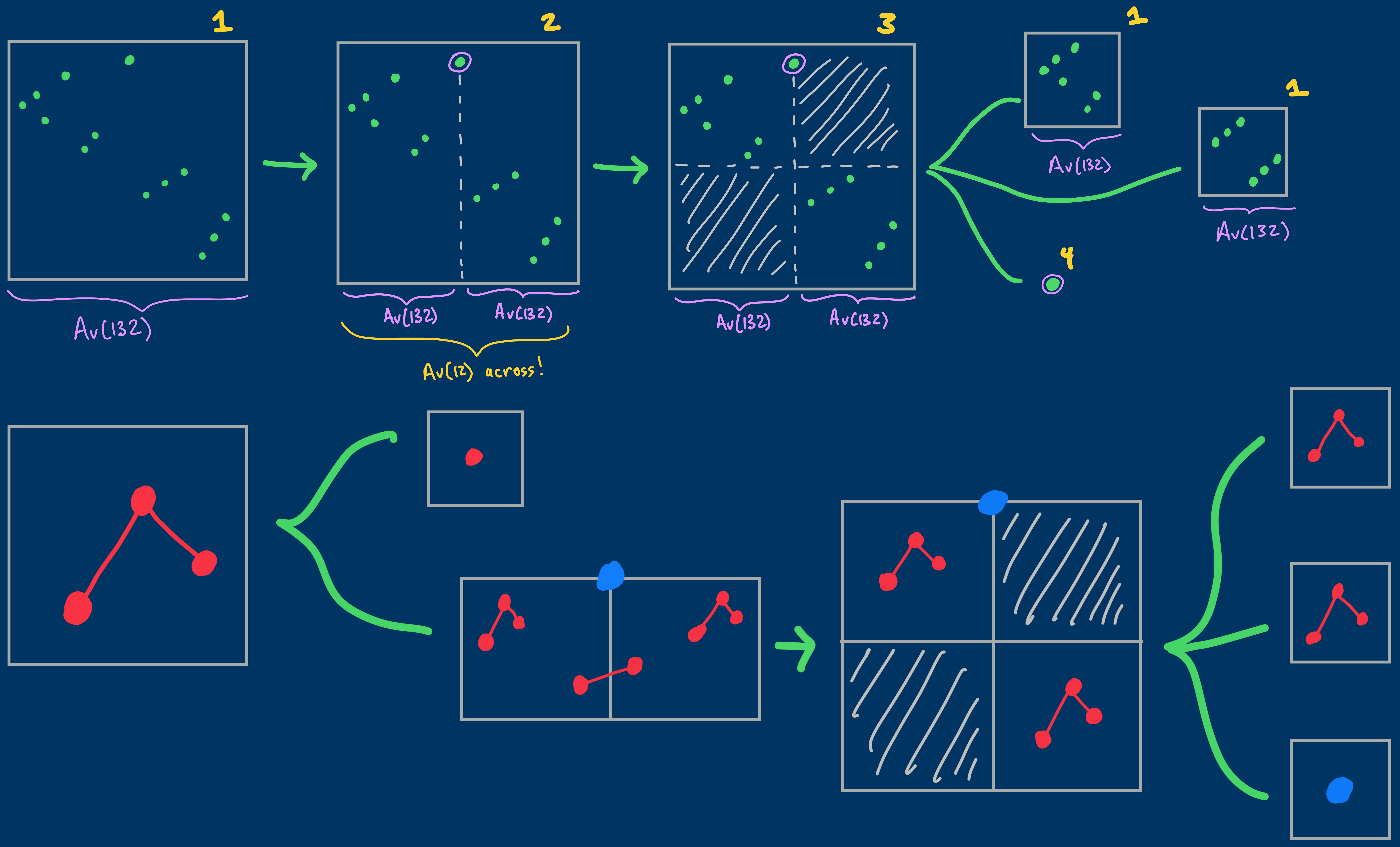


Tilings

The key innovation is that as you perform strategies on tilings, you can keep track of exactly where bad patterns can be formed.

So unlike most other methods, you don’t have to constantly generate permutations at every step to recompute this, which makes applying the strategies very fast.

Combinatorial Exploration



Combinatorial Exploration

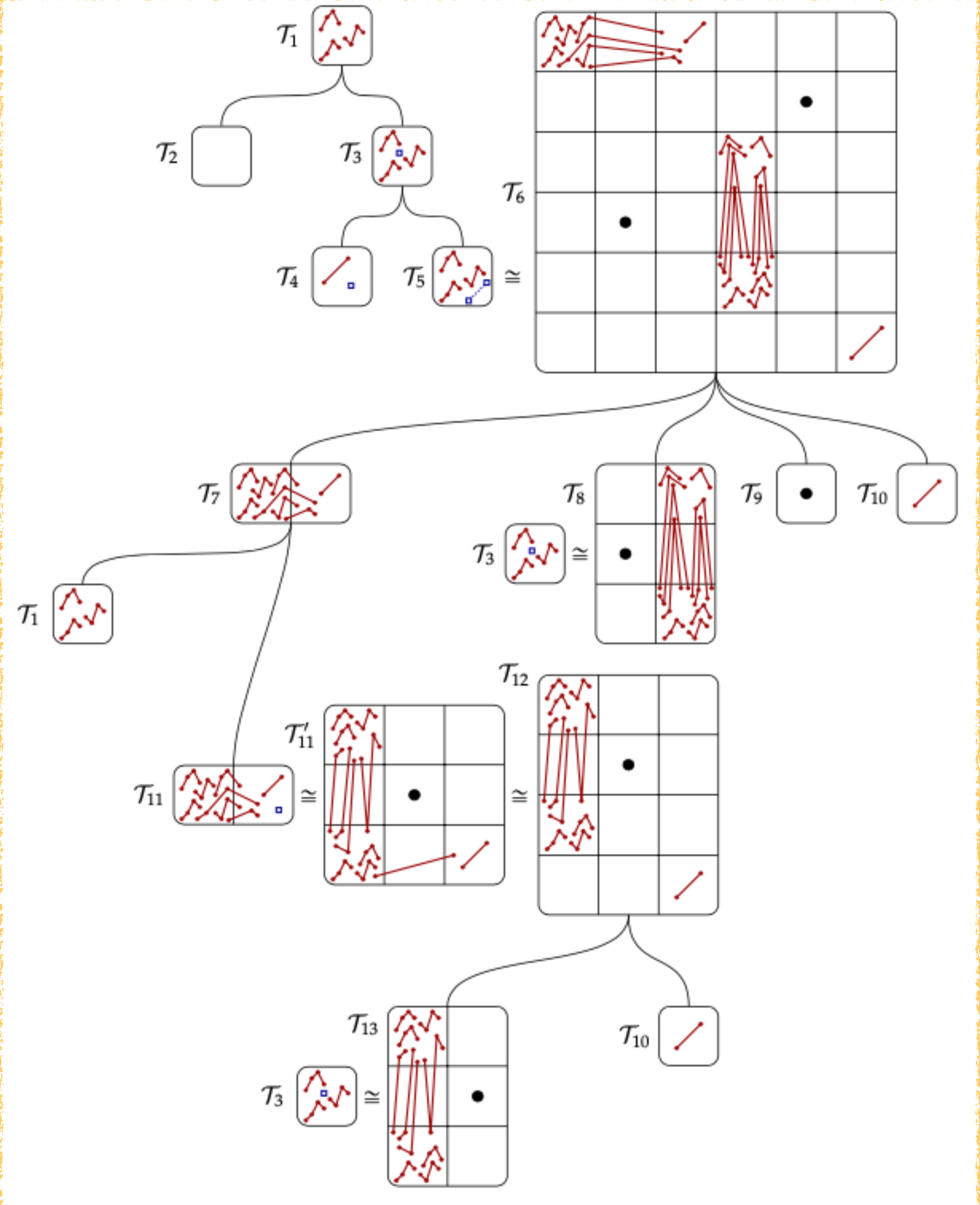


Figure 24: A pictorial representation of the combinatorial specification found by Combinatorial Exploration for Av(1243, 1342, 2143).

Av(1243, 1342, 2143)

The algorithm generates about 5,400 rules before it finds this subset of 10 rules that makes a rigorous specification.

55 seconds

Combinatorial Exploration

We can find combinatorial specifications for:

- 6 out of 7 of the classes avoiding 1 pattern of length 4

  First direct enumerations of $\text{Av}(1342)$ and $\text{Av}(2413)$

- All 56 classes avoiding 2 patterns of length 4

  3 are conjectured to be non-D-finite

  can derive the algebraic GF for the other 53

- All 317 classes avoiding 3 patterns of length 4

- All classes avoiding 4 or more patterns of length 4

Combinatorial Exploration

We can find combinatorial specifications for:

- 1324-avoiding domino permutations
- Preimage of $\mathrm{Av}(321)$ under West-stack-sorting

  $\mathrm{Av}(34251, 35241, 45231)$
- LCI Schubert Varieties

  $\mathrm{Av}(52341, 53241, 52431, 35142, 42513, 351624)$
- "Box classes" like $\mathrm{Av}(1\square 2\square 3)$ and $\mathrm{Av}(1\square\square 32)$
- "POP classes"
- Permutations corresponding to Schubert varieties with a complete parabolic bundle structure

  $\mathrm{Av}(3412, 52341, 635241)$

Combinatorial Exploration

https://permpal.com

PermPAL Home Examples Search Random

The Permutation Pattern Avoidance Library (PermPAL)

PermPAL is a database of algorithmically-derived theorems about permutation classes.

The Combinatorial Exploration framework produces rigorously verified combinatorial specifications for families of combinatorial objects. These specifications then lead to generating functions, counting sequence, polynomial-time counting algorithms, random sampling procedures, and more.

This database contains 23,845 permutation classes for which specifications have been automatically found. This includes many classes that have been previously enumerated by other means and many classes that have not been previously enumerated.

Some Notables Successes:

- 6 out of 7 of the principal classes of length 4
- all 56 symmetry classes avoiding two patterns of length 4
- all 317 symmetry classes avoiding three patterns of length 4
- the "domino set" used by Bevan, Brignall, Elvey Price, and Pantone to investigate Av(1324)
- the class Av(3412, 52341, 635241) of Alland and Richmond corresponding a type of Schubert variety
- the class Av(2341, 3421, 4231, 52143) equal to the (Av(12), Av(21))-staircase (see Albert, Pantone, and Vatter), which appears to be non-D-finite
- all of the permutation classes counted by the Schröder numbers conjectured by Eric Egge
- the class Av(34251, 35241, 45231), equal to the preimage of Av(321) under the West-stack-sorting operation (see Defant)

Section 2.4 of the article Combinatorial Exploration: An Algorithmic Framework for Enumeration gives a more comprehensive list of notable results.

The comb_spec_searcher github repository contains the open-source python framework for Combinatorial Exploration, and the tilings github repository contains the code needed to apply it to the field of permutation patterns.

Av(2143, 3412)

View Raw Data

Generating Function

$$\frac{3x-1}{\sqrt{-4x+1}\,(2x-1)}$$

Copy to clipboard: latex Maple sympy Search on PermPAL

Counting Sequence

1, 1, 2, 6, 22, 86, 340, 1340, 5254, 20518, 79932, 311028, 1209916, 4707964, 18330728, ...

Copy 101 terms to clipboard Search on OEIS Search on PermPAL

Recurrence

$a(0) = 1$
$a(1) = 1$
$a(2) = 2$
$$a(n+3) = \frac{12\,(1+2n)a(n)}{n+3} - \frac{2\,(16+13n)a(n+1)}{n+3} + \frac{(19+9n)a(n+2)}{n+3},$$

Copy to clipboard: latex Maple

Implicit Equation for the Generating Function

$$(4x-1)(2x-1)^2F(x)^2 + (3x-1)^2 = 0$$

Copy to clipboard: latex Maple Search on PermPAL

Heatmap

To create this heatmap, we sampled 1,000,000 permutations of length 300 uniformly at random. The color of the point (i, j) represents how many permutations have value j at index i (darker = more).

PermPAL Home Examples Search Random

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Specification 1 | Specification 2 | Specification 3 | Specification 4 | Specification 5

This specification was found using the strategy pack "Row And Col Placements Tracked Fusion Isolated" and has 29 rules.

Found on April 21, 2021.

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Finding the specification took 653 seconds.

Proof Tree

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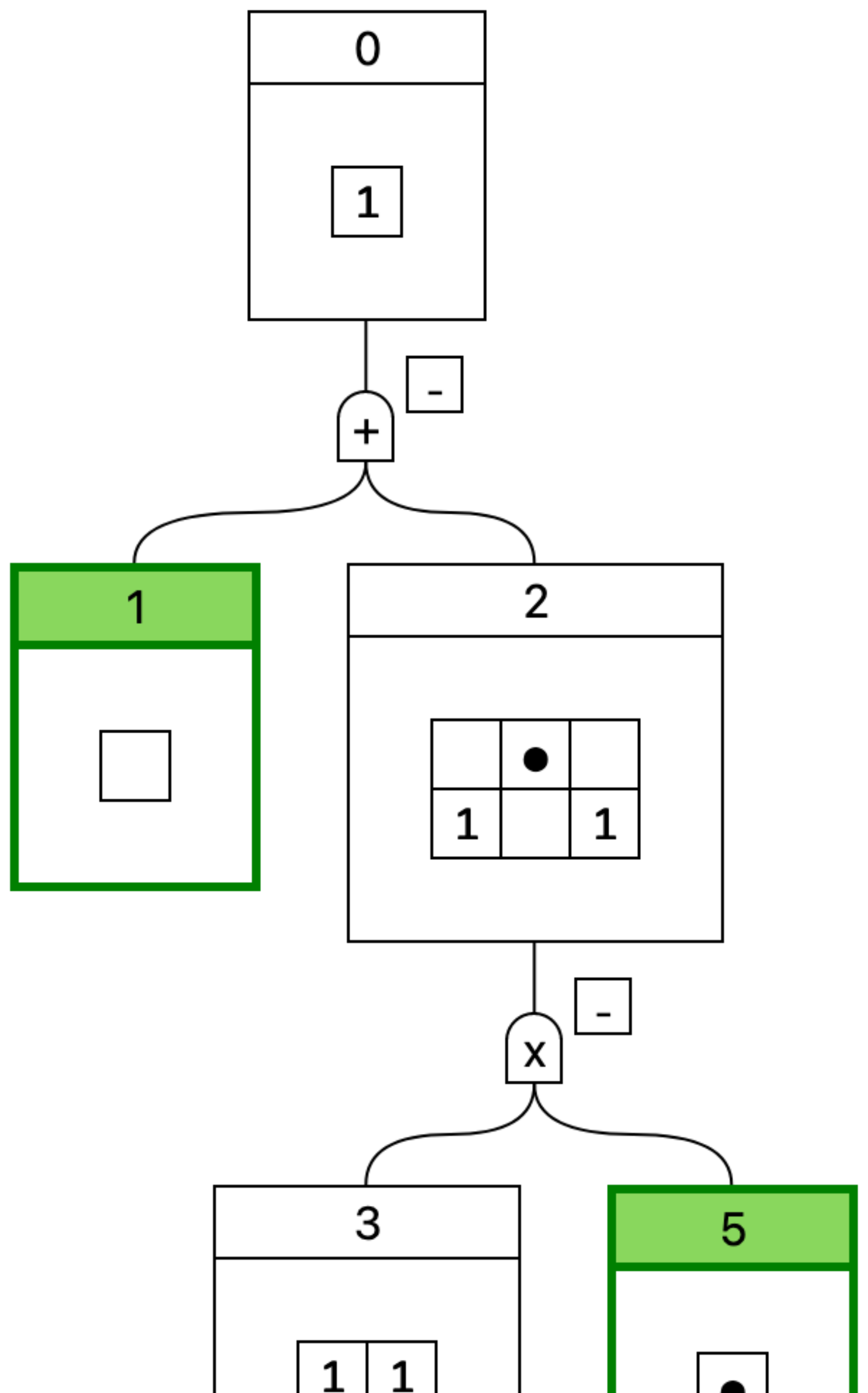
View tree on standalone page.



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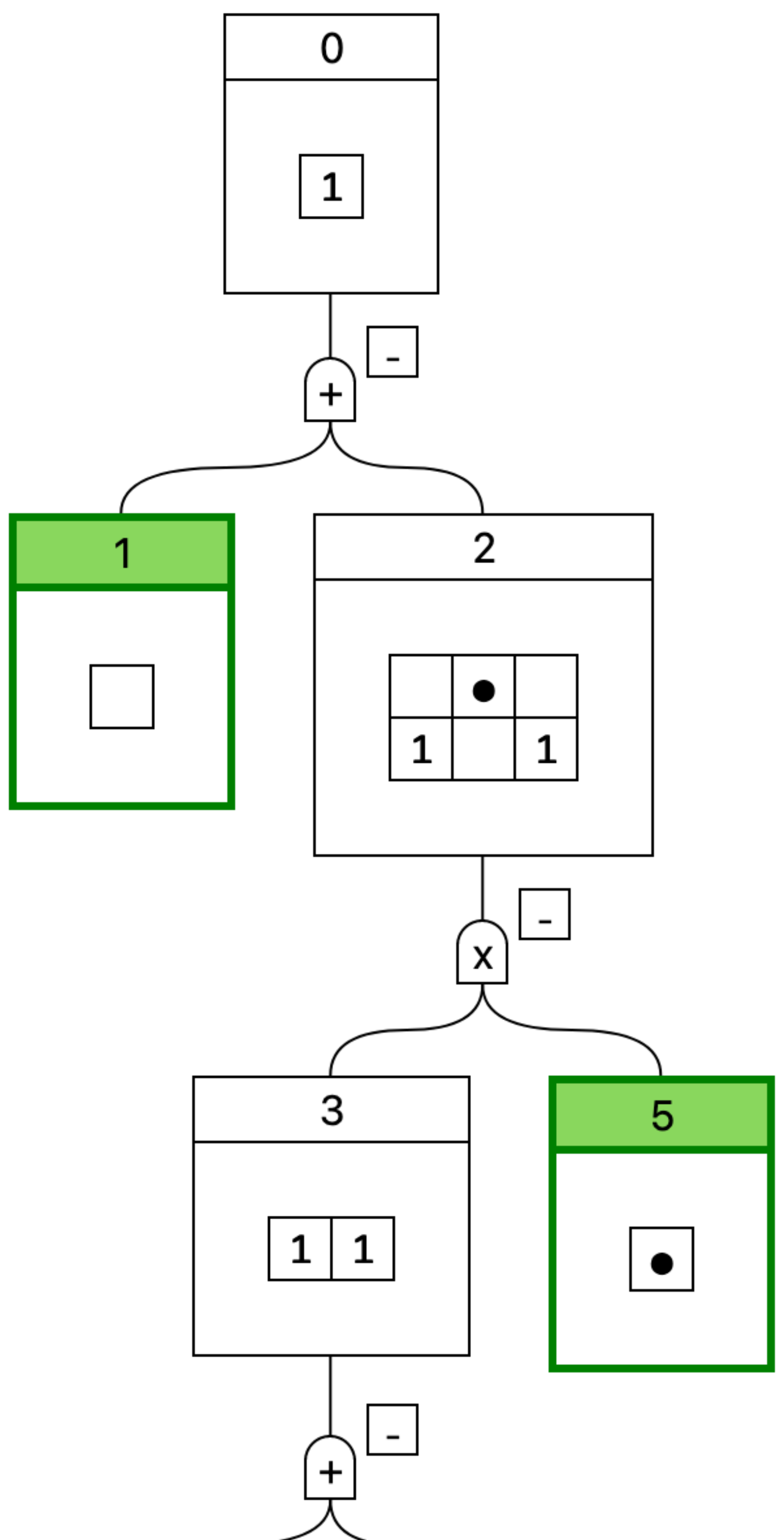
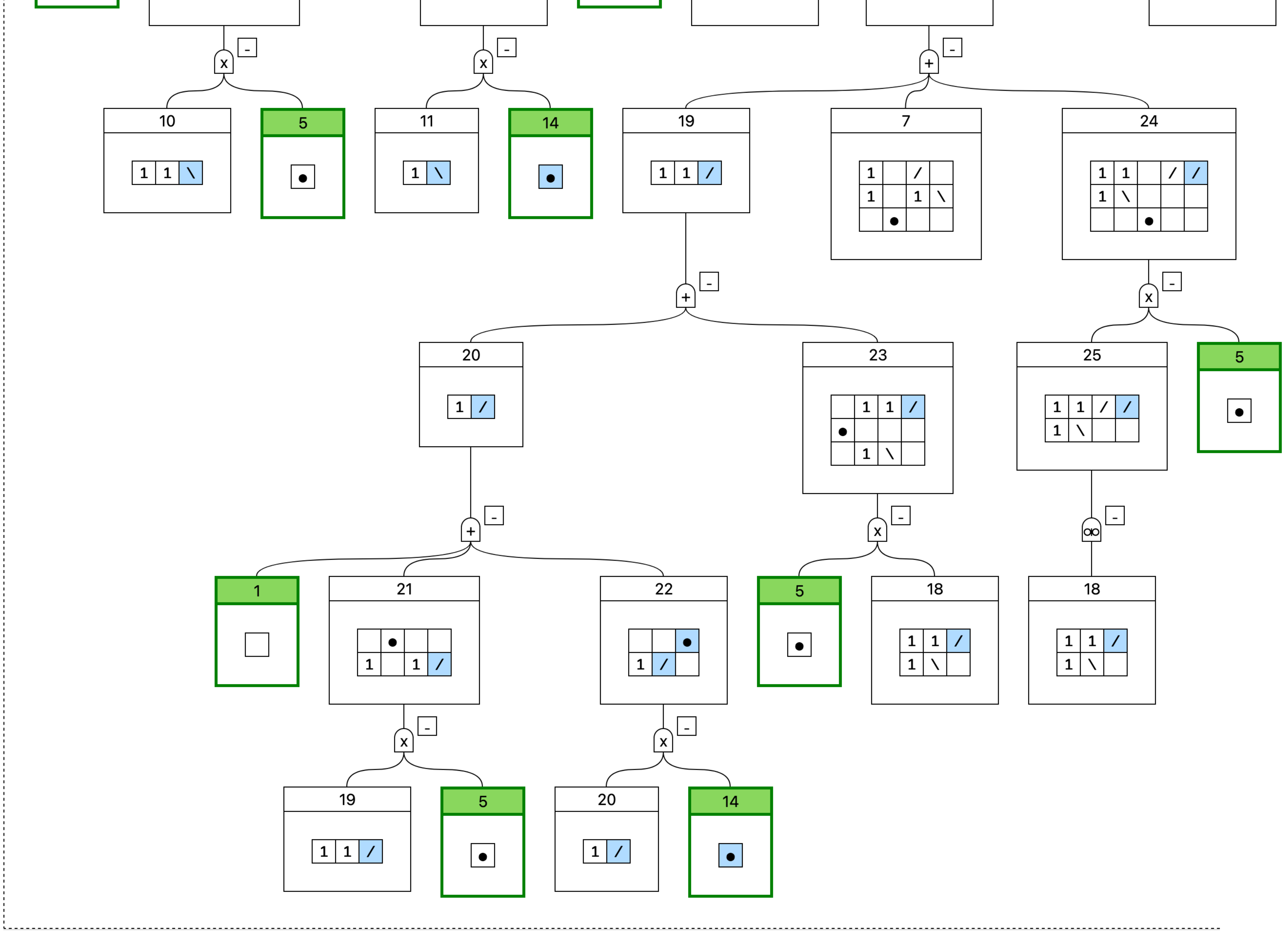
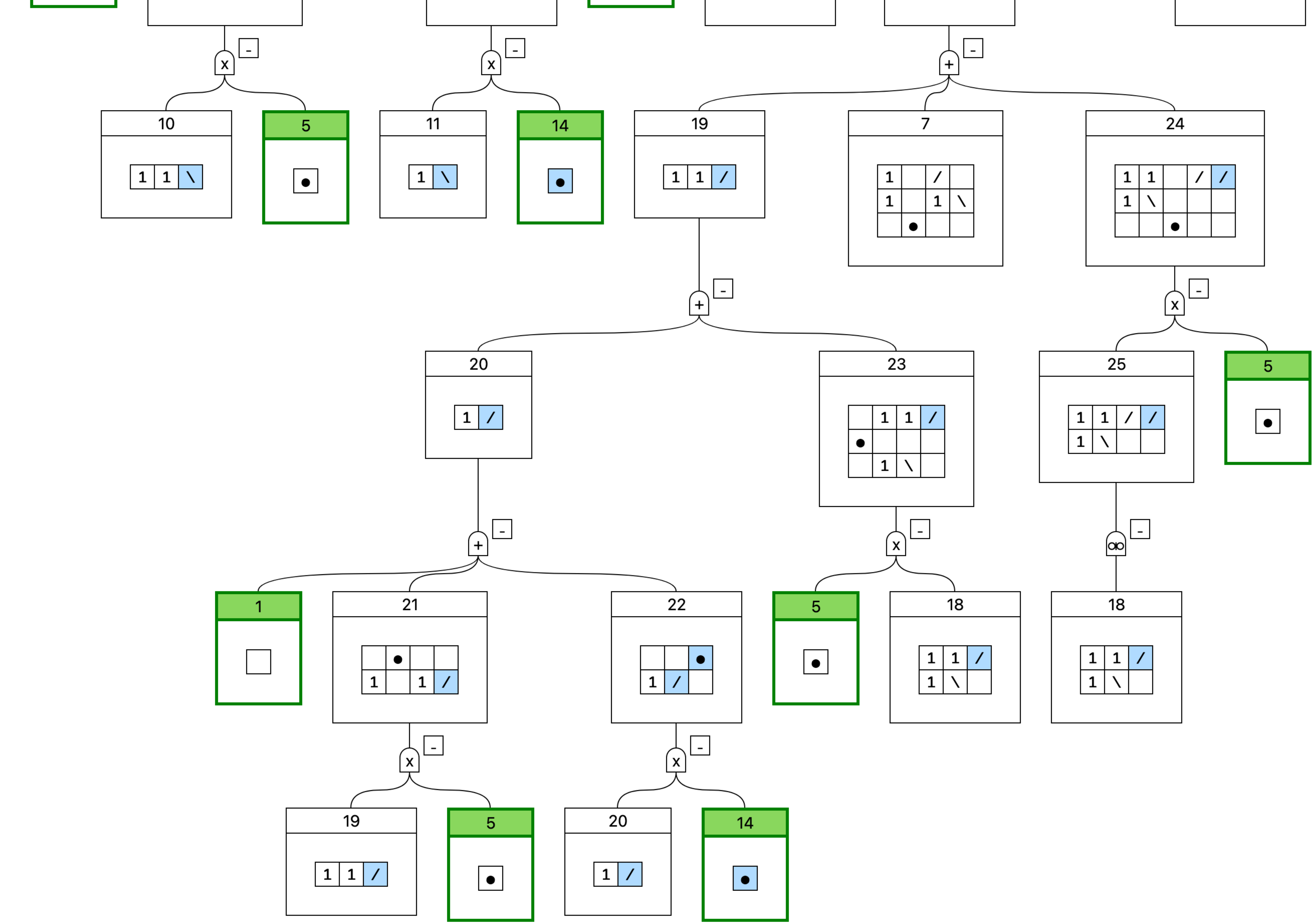




- Collapse



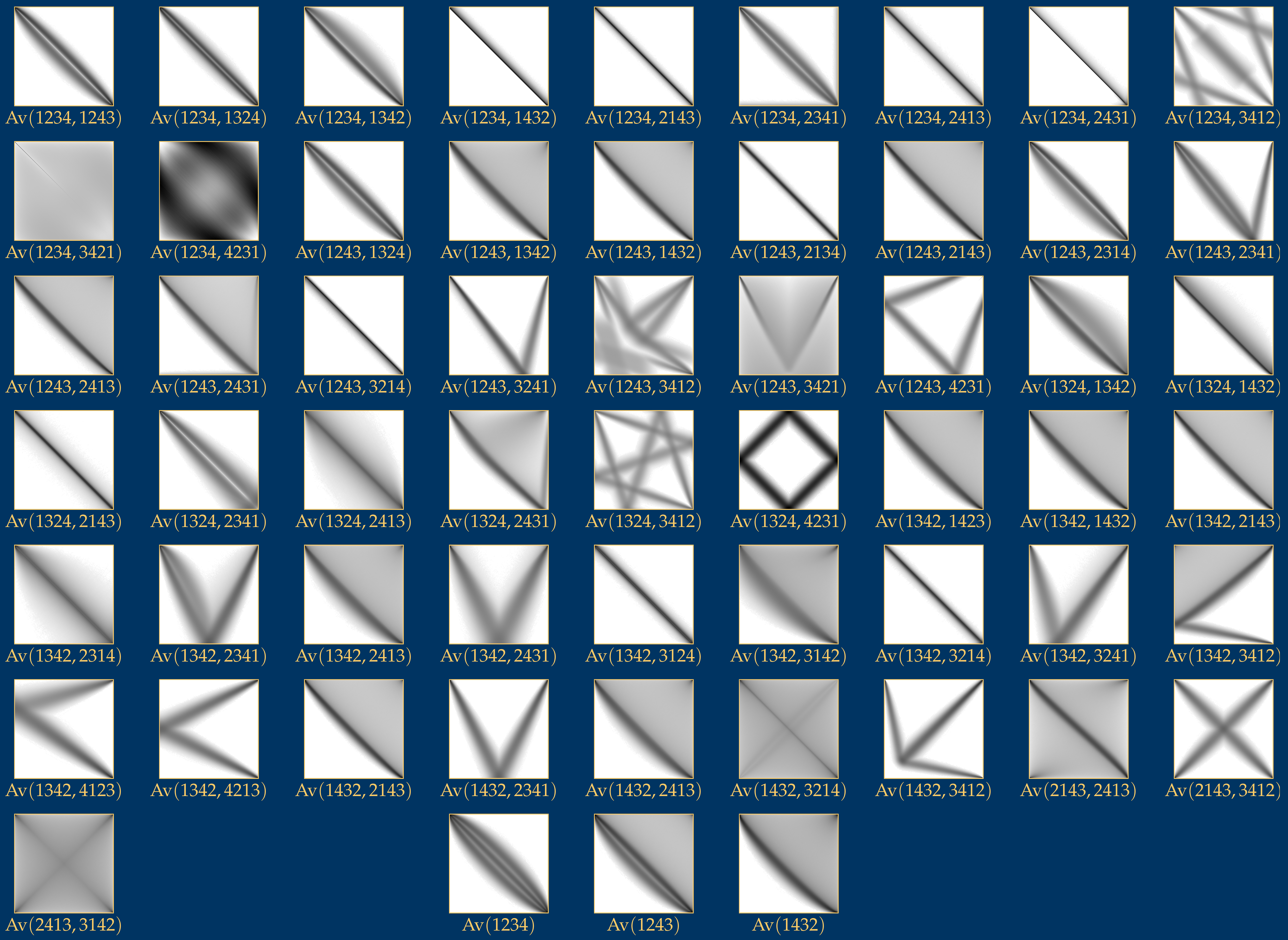
- Collapse

System of Equations

Copy 29 equations to clipboard: latex Maple sympy

$$F_0(x) = F_1(x) + F_2(x)$$
$$F_1(x) = 1$$
$$F_2(x) = F_3(x)F_5(x)$$
$$F_3(x) = F_0(x) + F_4(x)$$
$$F_4(x) = F_5(x)F_6(x)$$
$$F_5(x) = x$$
$$F_6(x) = F_{28}(x) + F_3(x) + F_7(x)$$
$$F_7(x) = F_5(x)F_8(x)$$
$$F_8(x) = F_9(x, 1)$$
$$F_9(x, y) = F_{10}(x, y) + F_{16}(x) + F_{26}(x, y)$$
$$F_{10}(x, y) = F_{11}(x, y) + F_{15}(x, y)$$
$$F_{11}(x, y) = F_1(x) + F_{12}(x, y) + F_{13}(x, y)$$
$$F_{12}(x, y) = F_{10}(x, y)F_5(x)$$
$$F_{13}(x, y) = F_{11}(x, y)F_{14}(x, y)$$
$$F_{14}(x, y) = yx$$
$$F_{15}(x, y) = F_5(x)F_9(x, y)$$
$$F_{16}(x) = F_{17}(x)F_5(x)$$
$$F_{17}(x) = F_{18}(x, 1)$$
$$F_{18}(x, y) = F_{19}(x, y) + F_{24}(x, y) + F_7(x)$$
$$F_{19}(x, y) = F_{20}(x, y) + F_{23}(x, y)$$
$$F_{20}(x, y) = F_1(x) + F_{21}(x, y) + F_{22}(x, y)$$
$$F_{21}(x, y) = F_{19}(x, y)F_5(x)$$
$$F_{22}(x, y) = F_{14}(x, y)F_{20}(x, y)$$
$$F_{23}(x, y) = F_{18}(x, y)F_5(x)$$
$$F_{24}(x, y) = F_{25}(x, y)F_5(x)$$
$$F_{25}(x, y) = -\frac{-yF_{18}(x, y) + F_{18}(x, 1)}{-1 + y}$$
$$F_{26}(x, y) = F_{27}(x, y)F_5(x)$$
$$F_{27}(x, y) = -\frac{-yF_9(x, y) + F_9(x, 1)}{-1 + y}$$
$$F_{28}(x) = F_{17}(x)F_5(x)$$

Combinatorial Exploration



Combinatorial Exploration

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- rigorous definition of "combinatorial strategy" as a component of combinatorial specifications
- algorithm to search giant set of rules for a subset that is a combinatorial specification
- working with gridded versions of objects internally, represented by obstructions and requirements

Combinatorial Exploration is domain-agnostic and can be used in other fields

Combinatorial Exploration

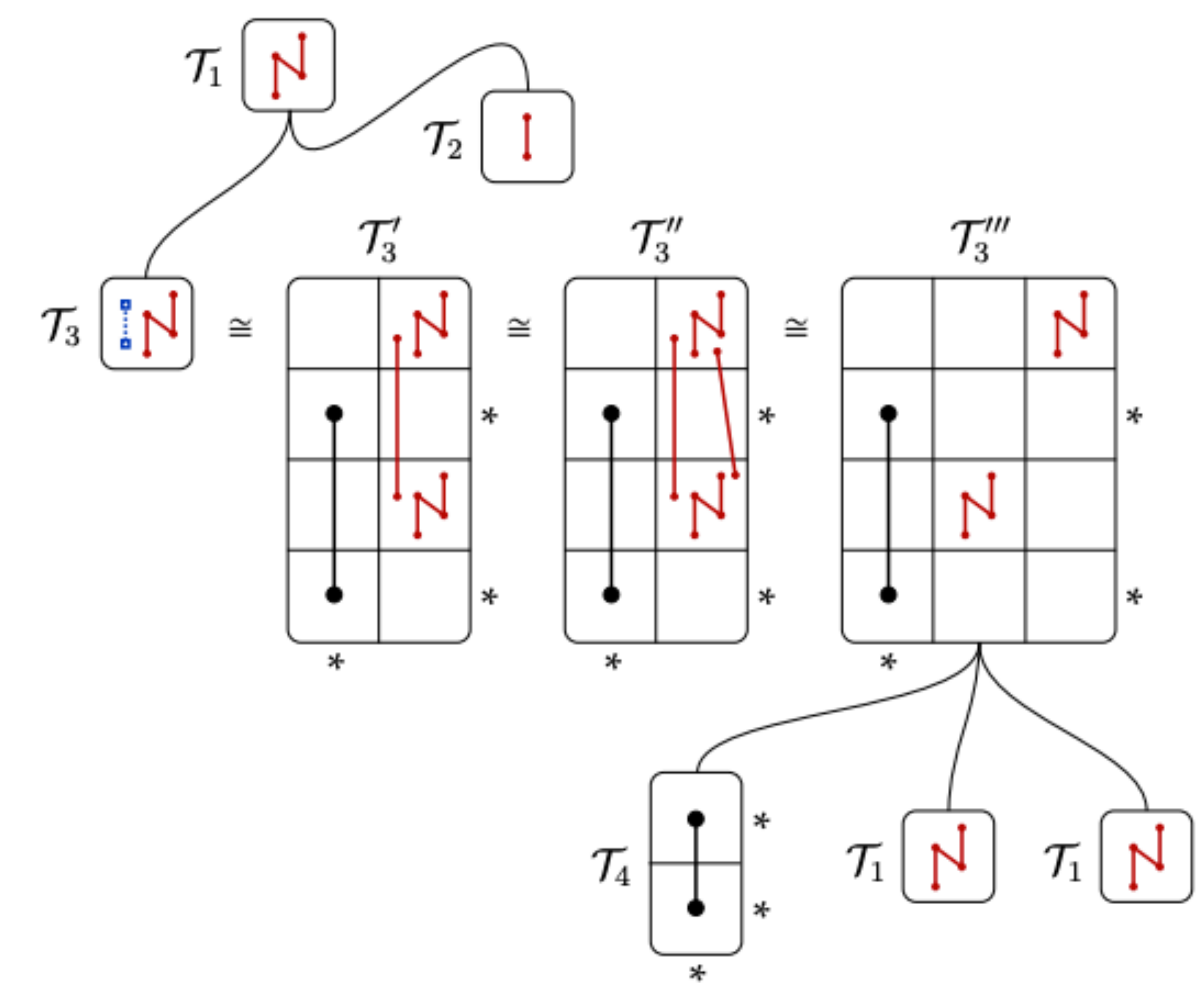


Figure 2.4: A visual representation of the proof tree for noncrossing diagrams $\mathcal{D}$( ).

Enumerative perspectives on chord diagrams

by

Lukas Nabergall

A thesis
presented to the University of Waterloo
in fulfillment of the
thesis requirement for the degree of
Doctor of Philosophy
in
Combinatorics and Optimization

Combinatorial Exploration

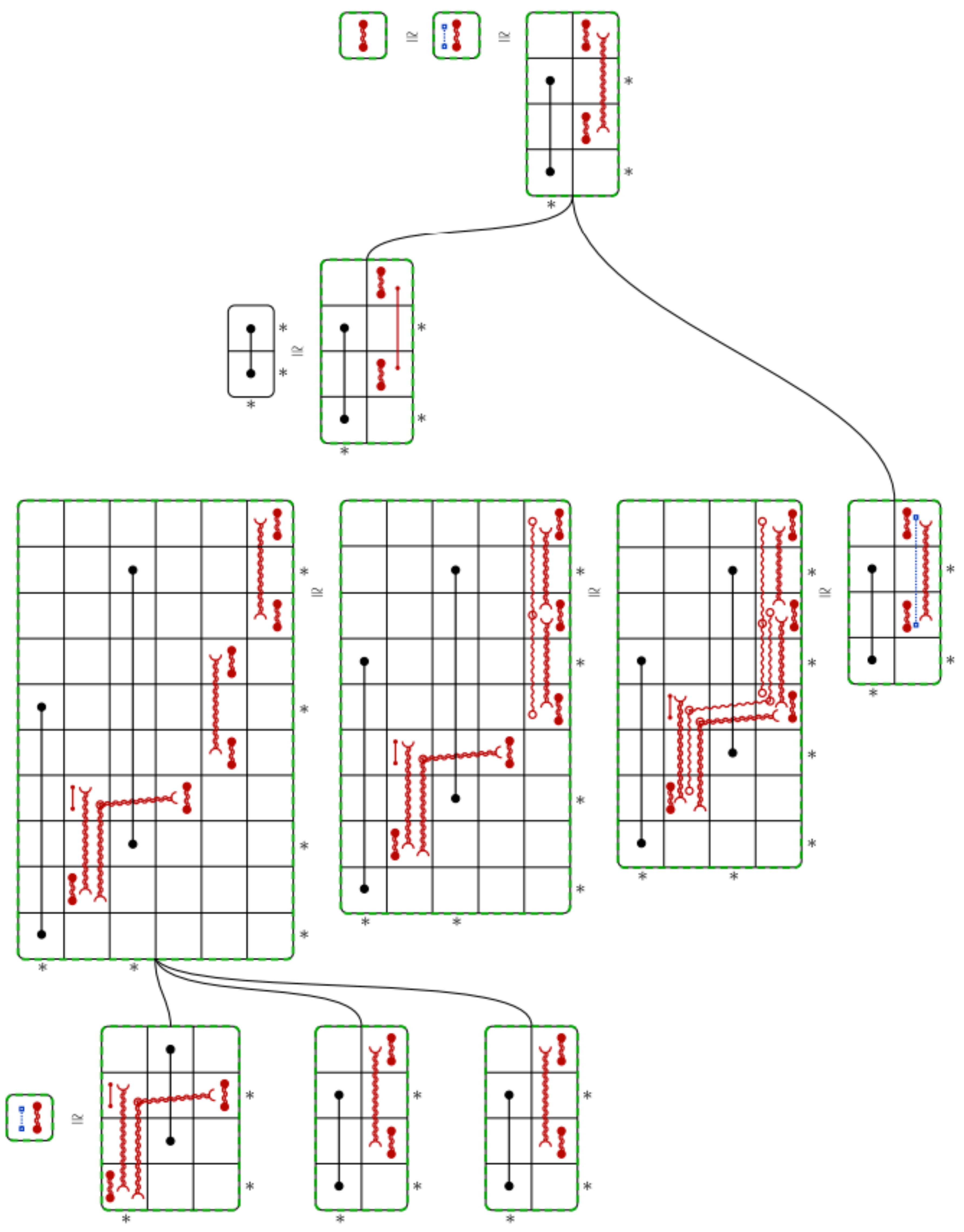
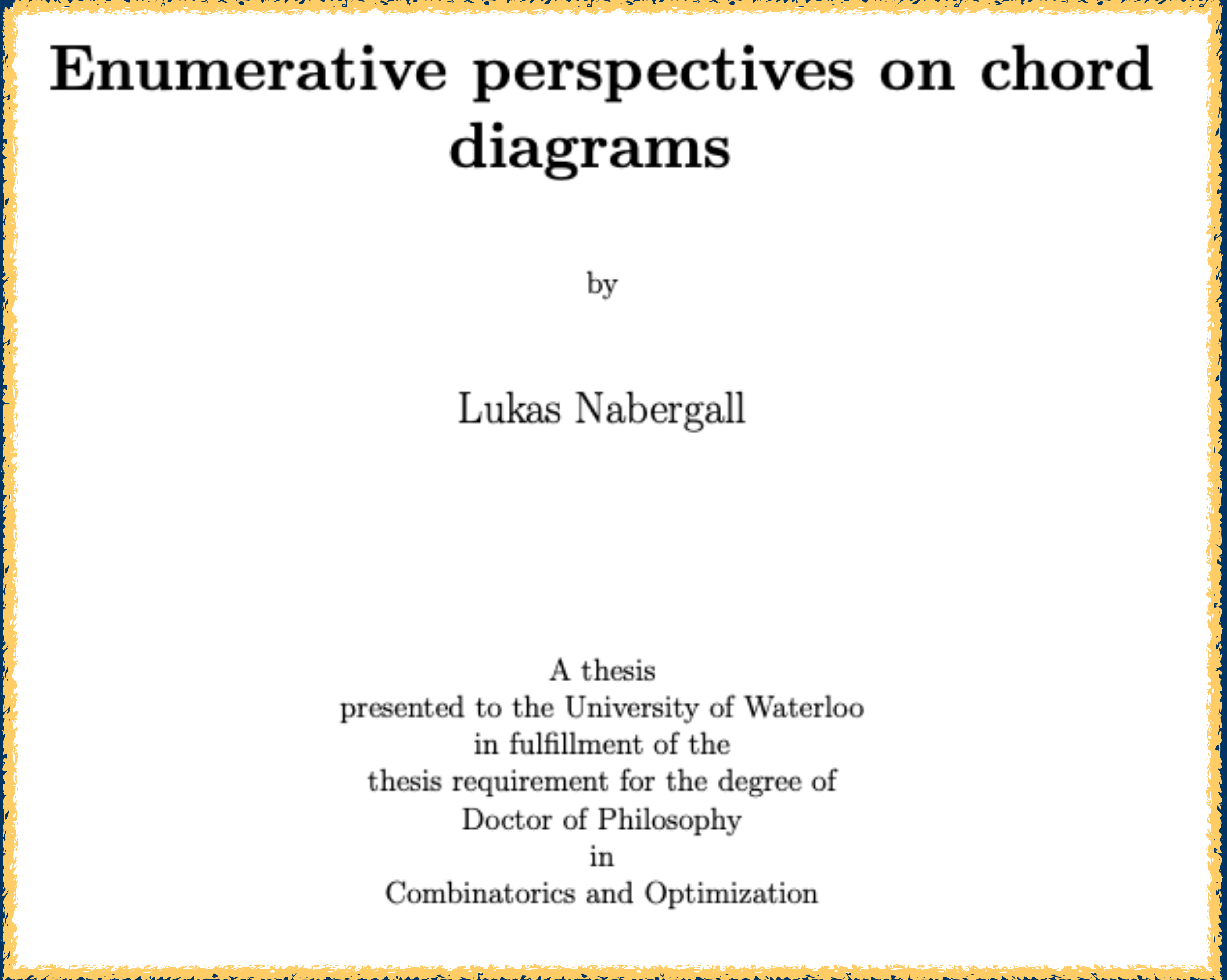


Figure 3.2: A proof tree for the class $\mathcal{C}(T_{\geqslant 3}, B_{\geqslant 3})$ of tree diagrams.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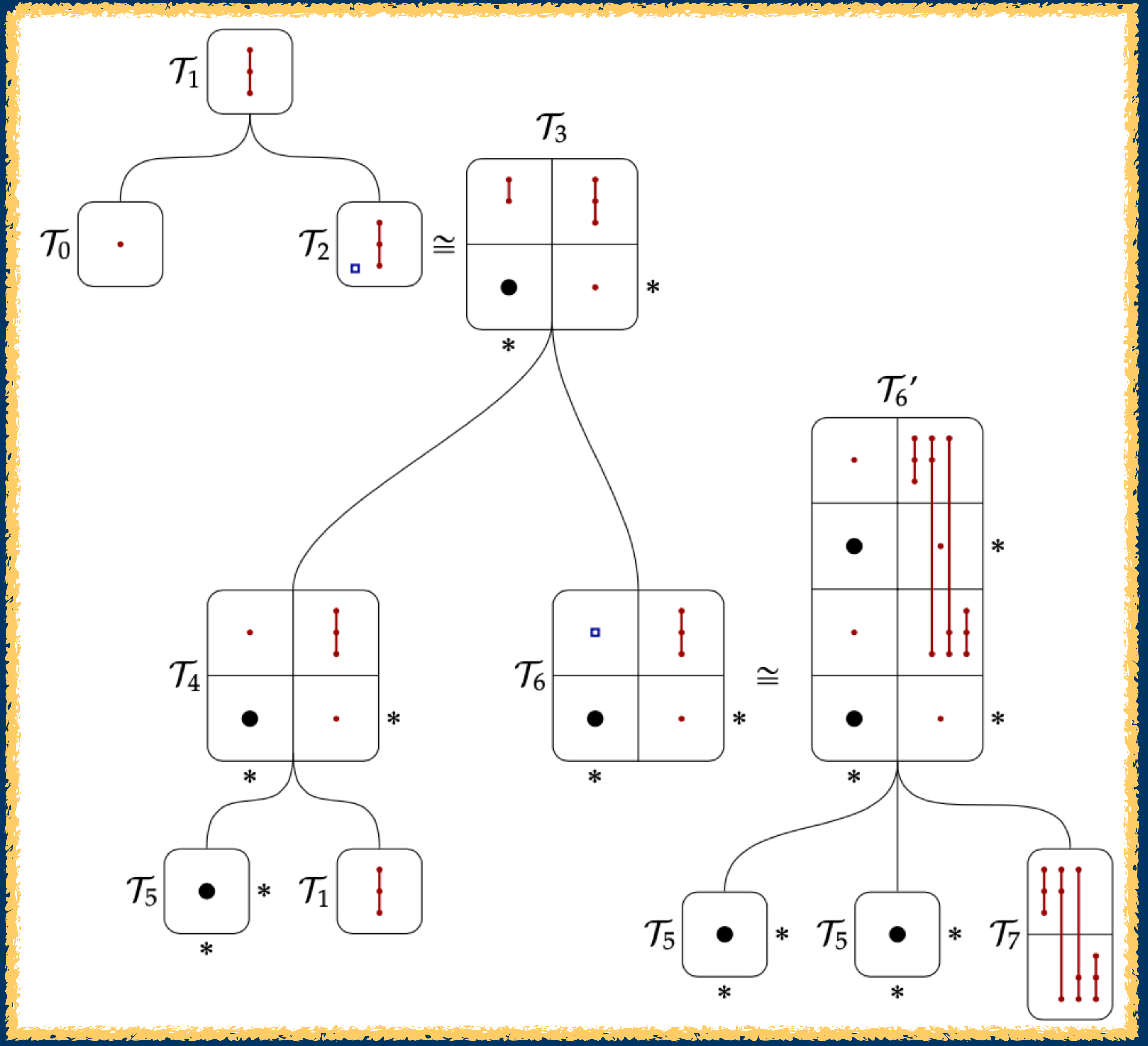
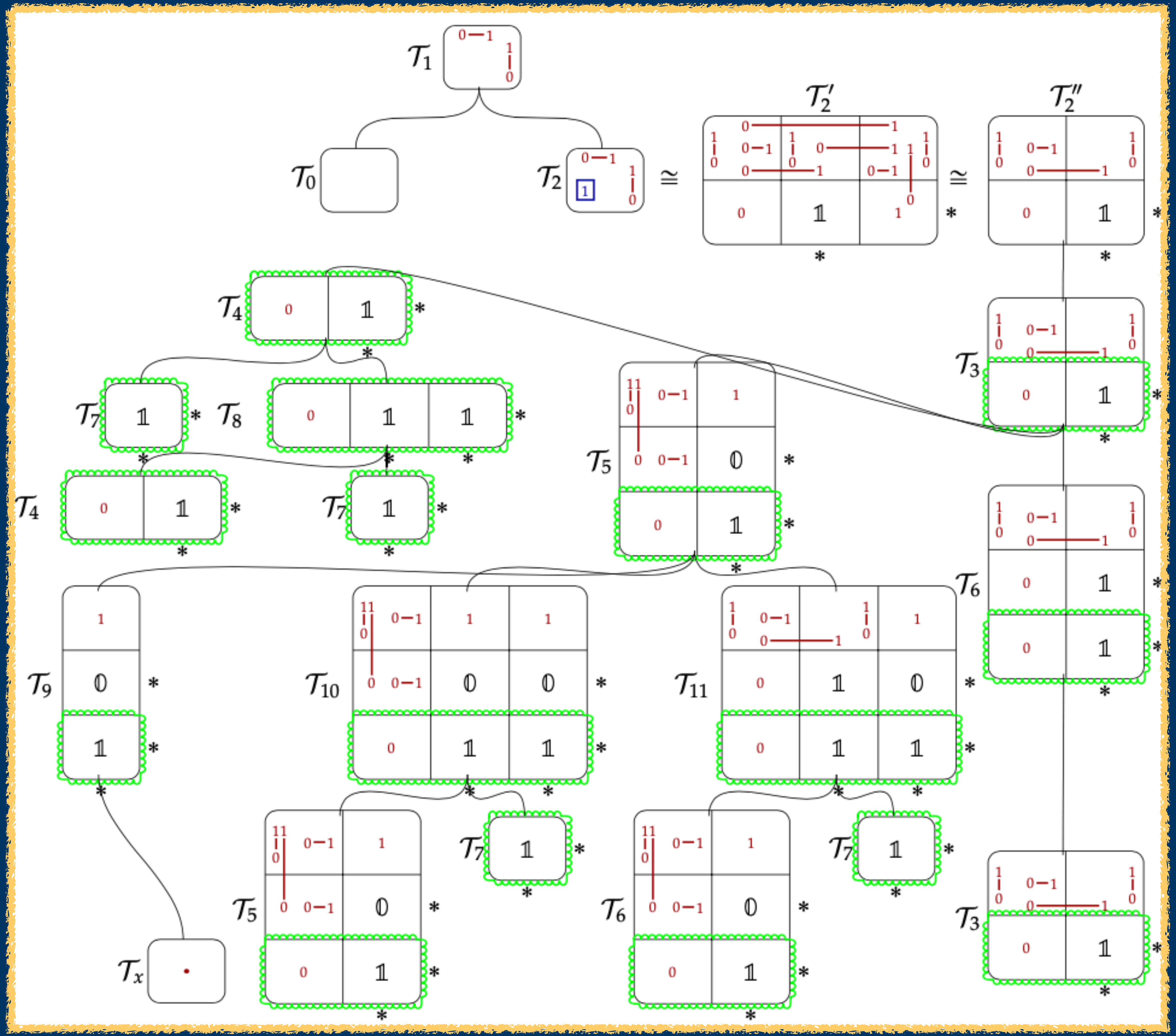
Set Partitions



$$T_1(x) = 1 + (x + x^2)T_1(x) + x^3 \frac{d}{dx} T_1(x)$$

Polyominoes



$$T_1(x) = \prod_{i=1}^{\infty} \frac{1}{1 - x^i}$$

	rigorous	non-rigorous
experimental	- enumeration schemes WILF, WILFPLUS, Flexible Schemes (E) - Combinatorial Exploration (E)	- Struct - BiSC
non-experimental	- generating trees (E) - FINLABEL - ECO Method - Combinatorial Generation - Regular Insertion Enc. - Finite Simples - Poly Classes	- HERB

	rigorous	non-rigorous
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PermLab

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πDD Kuszmaul

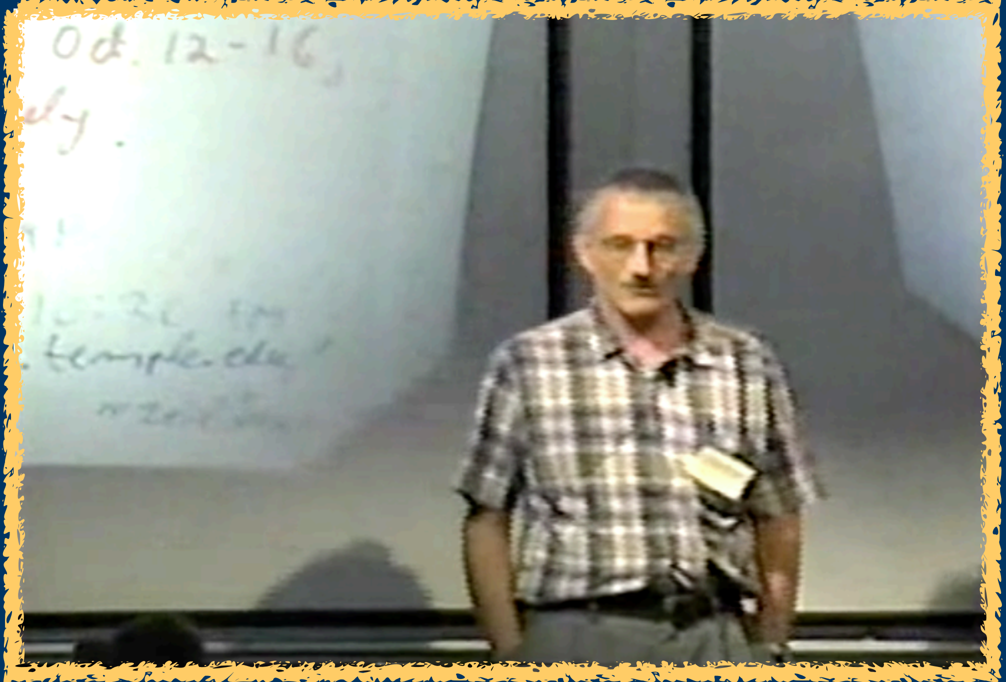
Permutation Patterns: Easy or Hard?

Enumeration Schemes and, More Importantly, Their Automatic Generation

Doron Zeilberger*

Department of Mathematics, Temple University, Philadelphia, PA 19122, USA
zeilberg@math.temple.edu, `http://www.math.temple.edu/~zeilberg`

Received May 27, 1998



Apology. The success rate of the present method, in its present state, is somewhat disappointing. Ekhad was able to reproduce the classical cases and a few new ones, but for most patterns and sets of patterns, it failed to find a scheme (defined below) of reasonable depth. But the present framework for setting up a scheme could be modified and extended in various ways. We do believe that an appropriate enhancement of the present method would yield, if not a 100% success rate, at least close to it.

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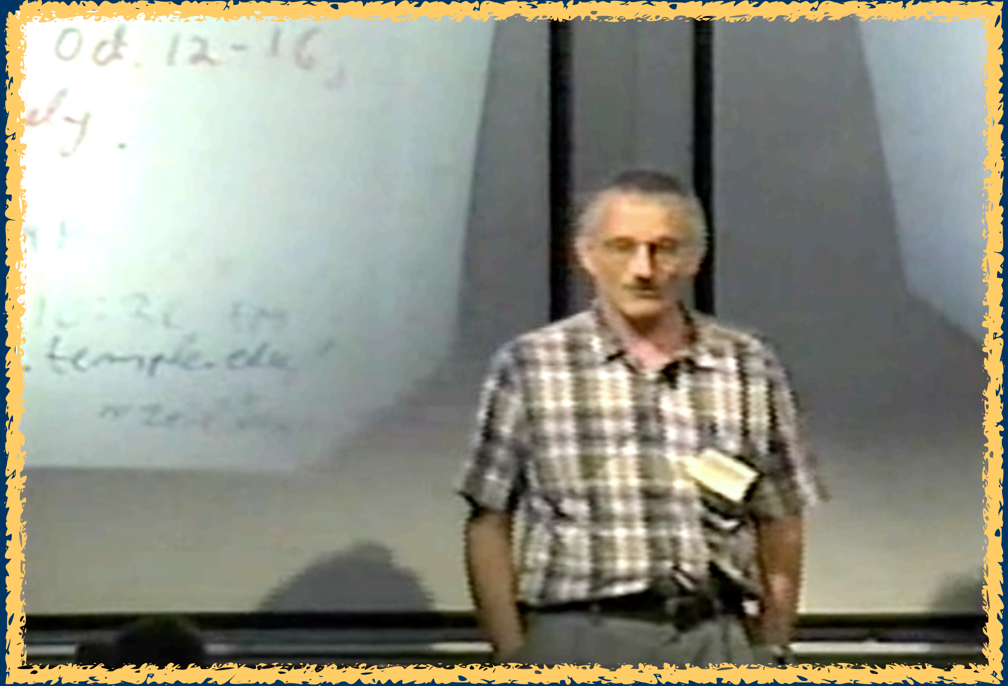

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A bit optimistic!

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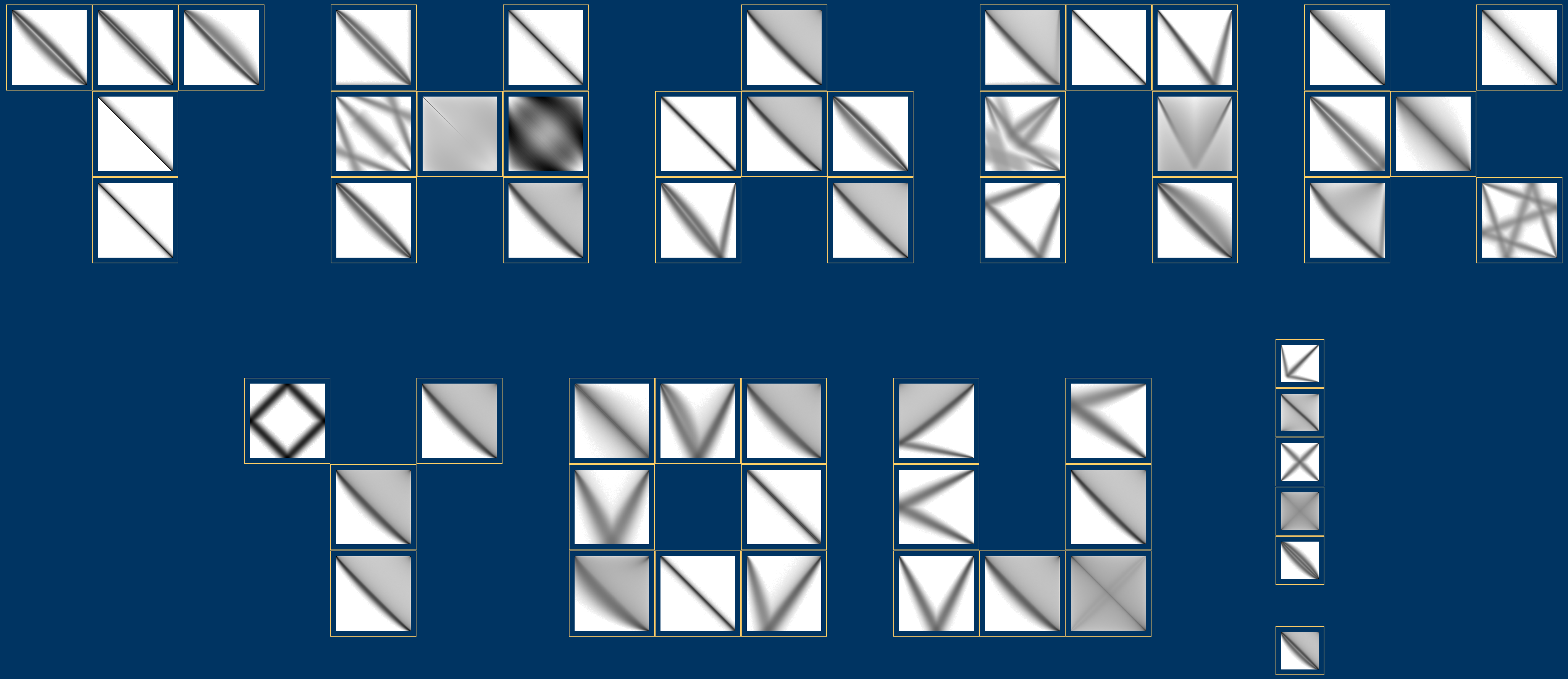
What about the classes that avoid a single pattern of length 5, like $\mathrm{Av}(24135)$?

25 years from now, how will the role of computers in our research be different?


THANK YOU!

10 years and 2 days ago...