Equidistribution of Destop on pattern-avoiding permutation classes

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Descents, descent tops, descent bottoms

Permutation $\pi \in S_n$

$$\pi = \ldots\ a_i\ a_{i+1}\ \ldots\ , \qquad a_i > a_{i+1}$$

Then

- i is a descent (position) of π,
- a_i is descent top of π,
- a_{i+1} is a descent bottom of π.

Define

- $\mathrm{Des}\,\pi$ = set of descents of π,
- $\mathrm{Destop}\,\pi$ = set of descent tops of π,
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Similar notation for ascents ($a_i < a_{i+1}$): Asc, Ascbot, Asctop

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f-Wilf-equivalence

Let f be a permutation statistic. We say that patterns σ and τ are f-Wilf-equivalent if there a bijection Θ between $\mathrm{Av}_n(\sigma)$ and $\mathrm{Av}_n(\tau)$ (avoiders of σ and avoiders of τ) that preserves the f statistic, i.e.

$$f = f \circ \Theta.$$

Conjectures

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The non-singleton Destop-*Wilf-equivalence classes in* S_4 *are:*

- $1243 \sim 3412$,
- $1423 \sim 2413$,
- $2143 \sim 3421$,
- $2314 \sim 3124$,
- $2431 \sim 3142 \sim 3241 \sim 4132$.

Checked: Holds for avoiders of size $\leqslant 10$.

Stronger Conjectures I

Observe: $\{3142, 3241, 4132\}$ is preserved under reversal of complement.

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Observe: 2314 and 3124 end with the largest letter.

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$231 \oplus \sigma \sim 312 \oplus \sigma$ *are* Destop-*shape-Wilf equivalent for any permutation* σ.

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Motivation

- 132 ~ 231 and 132 ~ 312 are Des- and Destop-Wilf-equivalent.
- 231 ~ 312 are Des- and (Destop, Desbot)-Wilf equivalent.
- Stankova, West, '02: 231 ~ 312 are shape-Wilf-equivalent.
- Bloom, '14: 1423 ~ 2413 are Des-Wilf-equivalent.
- Conjectures for Wilf-equivalence of the same patterns on Dumont permutations of the first kind, i.e. permutations with Destop = {all even entries}:
  - B., Jones, '16: 2143 ~ 3421 on $\mathfrak{D}^1$,
  - Archer, Lauderdale, '19: the rest on $\mathfrak{D}^1$.
- (Des, Destop) is not jointly equidistributed for any pair of S_4 (or S_3) patterns. So, we can only preserve Des or Destop, but not both.

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References

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