A RAMSEY-LIKE CONJECTURE FOR DIVIDING A CIRCLE (PROBLEM SESSION)

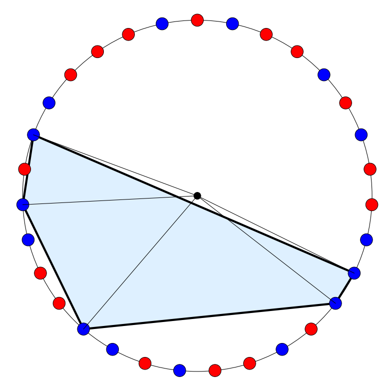
Walter Stromquist *Bryn Mawr College*

Start with 31 points evenly spaced on a circle, each arbitrarily colored blue or red. Is it always possible to find 5 of these points, all of the same color, that divide the circle into arcs proportional to $1,2,4,8,16$ (in some order)?

The answer is yes, as can be verified by checking every possible coloring. The question was inspired by a problem proposed by Robert Tauraso in the American Mathematical Monthly [1]. The 31-point result leads to a nice solution to that problem. But the more interesting question is whether the result generalizes.

> **Conjecture.** Let $k \geq 1$. Given 2^k-1 points, evenly spaced on a circle and each arbitrarily colored blue or red, it is possible to select k of the points, all of the same color, that divide the circle into arcs proportional to $1,2,4,\ldots,2^{k-1}$ (in some order).

The conjecture is trivially true when $k=1$ and $k=2$, and is easy to prove when $k=3$ or $k=4$. Computer proofs exist when $k \leq 7$. Does the conjecture hold in general?



In this example, the selected points are shown as vertices of a pentagon. The arc lengths appear in the order $1,16,2,4,8$.

Some curiosities occur. We cannot require that the arc lengths occur in any particular cyclic order. It is possible that there are "winners" (successful selections of k points) of both colors. It is also possible that the only winners are of the minority color. If $k \geq 3$ then for any coloring, the number of winners is even. The minimum number of winners (over all colorings) is 2 (for 7 points), 4 (for 15 points), and 10 (for 31 points).

János Pach reports [2] that the conjecture would be false if any other finite sequence were substituted for the initial powers of 2. It was his group who used a SAT solver to verify the conjecture when $k=7$.

References

[1] R. Tauraso, *Problem 12251,* Amer. Math. Monthly **128**, 5, p. 467 (May, 2021).

[2] G. Damásdi, N. Frankl, J. Pach, D. Pálvölgyi, *Monochromatic Configurations on a Circle*, in preparation.