Descent distribution on Catalan words avoiding ordered pairs of Relations

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Joint work: Jean-Luc Baril

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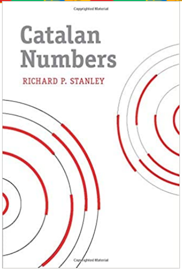
Dijon, France

What is a Catalan word?



Exercise 80:

Definition

A Catalan word $w = w_1 w_2 \cdots w_n$ is one over the set of non-negative integers satisfying $w_1 = 0$ and $0 \leq w_i \leq w_{i-1} + 1$ for $i = 2, \ldots, n$.

0 1 2 1 2 2 0 1 2 3 4

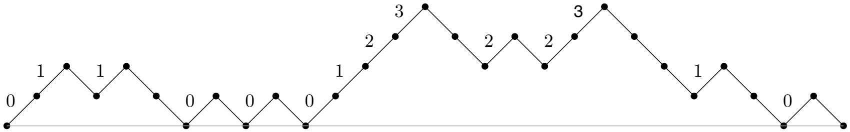
- $C_n :=$ set of **Catalan words** of length n, and $C = \bigcup_{n\geq 0} C_n$.

$$C_4 = \{\texttt{0000},\ \texttt{0001},\ \texttt{0010},\ \texttt{0100},\ \texttt{0011},\ \texttt{0101},\ \texttt{0110},\ \texttt{0111},\ \texttt{0012},\ \texttt{0112},\ \texttt{0120},\ \texttt{0121},\ \texttt{0122},\ \texttt{0123}\}.$$

- The set C_n is enumerated by the Catalan number

$$C_n = \frac{1}{n+1}\binom{2n}{n}.$$

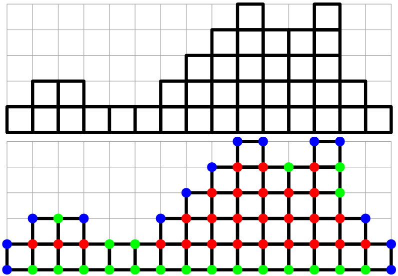
Bijection with Dyck paths:



Catalan word (length 14):

01100012322310

- Toufik Mansour and Vincent Vajnovszki (2013): Catalan words were studied in the context of exhaustive generation of Gray codes for growth-restricted words.
- Jean-Luc Baril, Sergey Kirgizov, and Vincent Vajnovszki (2018) study the distribution of descents on restricted Catalan words avoiding a pattern of length at most three.
- Diana Toquica, Toufik Mansour and JLR (2021) study several combinatorial statistics on the polyominoes associated the Catalan words.



Catalan words avoiding ordered pairs of relations

Motivated by...

- Megan Martinez and Carla Savage (2018) carried out the systematic study of inversion sequences avoiding triples of relations.

> ... for a fixed triple of binary relations (ρ_1, ρ_2, ρ_3), we study the set $I_n(\rho_1, \rho_2, \rho_3)$ consisting of those $e \in I_n$ with no $i < j < k$ such that $e_i \rho_1 e_j, e_j \rho_2 e_k$, and $e_i \rho_3 e_k$...
>
> $$\rho_i \in \{<, >, \leq, \geq, =, \neq, -\}$$

- Juan Auli and Sergi Elizalde (2019). Consecutive patterns in inversion sequences avoiding patterns of relations.
- Arissap Sapounakis, Ioannis Tasoulas, Panagiotis Tsikouras (2007). Counting strings in Dyck paths.
- . . .

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We will say that a Catalan word w contains the pattern $p = (X, Y)$ if there exists $i \geq 1$ such that $w_i \; X \; w_{i+1}$ and $w_{i+1} \; Y \; w_{i+2}$.

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We say that w avoids the consecutive pattern p whenever w does not contain the consecutive pattern p.

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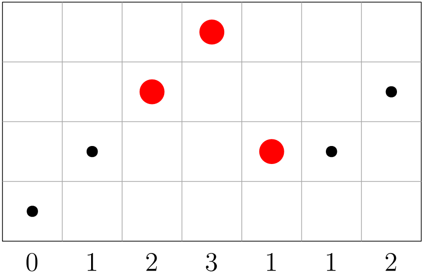
Example

1. The pattern $(\neq, \geq)$ appears twice in the Catalan word **0123112** on the triplets 231 and 311.



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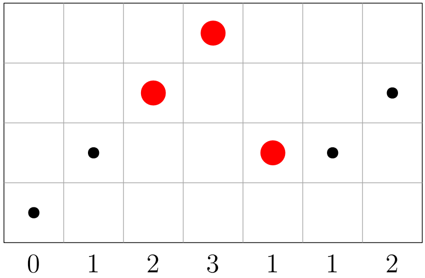

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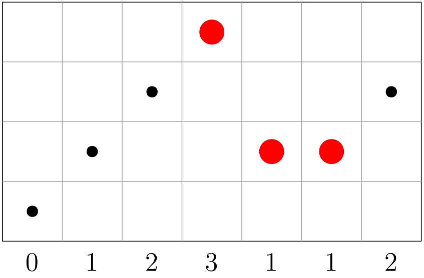

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Example

1. The pattern $(\neq, \geq)$ appears twice in the Catalan word 0123112 on the triplets 231 and 311.
2. The avoidance of $(\neq, \geq)$ on Catalan words is **equivalent** to the avoidance of the four consecutive patterns:

$$\begin{array}{ll} \underline{010} & (0 \neq 1 > 0) \\ \underline{011} & (0 \neq 1 = 1) \\ \underline{100} & (1 > 0 = 0) \\ \underline{210} & (2 \neq 1 > 0) \end{array}$$

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3. The avoidance of $(<, <)$ is equivalent to $\underline{012}$.

We introduce the bivariate generating function

$$\boldsymbol{C}_p(x,y) := \sum_{w\in\mathcal{C}(p)} x^{|w|} y^{\mathsf{des}(w)} = \sum_{n,k\geq 0} \boldsymbol{c}_p(n,k) x^n y^k$$

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Catalan words avoiding the consecutive pattern p

the number of descents in w

- $\boldsymbol{c}_p(n,k)$:= number of Catalan words of length n such that des$(w) = k$.

$$\boldsymbol{C}_p(x) := \sum_{w\in\mathcal{C}(p)} x^{|w|} = \boldsymbol{C}_p(x,1).$$

$$\boldsymbol{D}_p(x) := \left.\frac{\partial \boldsymbol{C}_p(x,y)}{\partial y}\right|_{y=1}.$$

We provide systematically the bivariate generating function for the number of Catalan words avoiding a given pair of relations with respect to the length and the number of descents.

Cases $(=,<)$, $(<,=)$, and $(<,>)$

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There exists a bijection between the Catalan words avoiding $\underline{011}$ and those avoiding $\underline{001}$ preserving the number of descents.

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Bijection (sketch) Replacing, from left to right, each factor $k^j(k+1)$ with the factor $k(k+1)^j$ $(j \geq 2)$.

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$$\boldsymbol{C}_{(=,<)}(x,y) = \boldsymbol{C}_{(<,=)}(x,y)$$

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$$\boxed{\boldsymbol{C}_{(<,>)}(x) = \boldsymbol{C}_{(=,<)}(x) = \boldsymbol{C}_{(<,=)}(x)}$$

Proof.

Let w denote a non-empty Catalan word in $\mathcal{C}(<>)$, and let $w = 0(w'+1)w''$ be the **first return decomposition**, where $w', w'' \in \mathcal{C}(\underline{001})$.

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1. 0α with $\alpha \in \mathcal{C}(<,>)$,
2. $0(\alpha+1)$ with $\alpha \in \mathcal{C}(<,>)$, $\alpha \neq \epsilon$, or
3. $0(\alpha+1)\beta$ where α ends with $a(a+1)$ and $\beta \in \mathcal{C}(<,>)$, $\beta \neq \epsilon$

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Generating Functions:

1. $x\boldsymbol{C}_{(<,>)}(x)$
2. $x(\boldsymbol{C}_{(<,>)}(x) - 1)$
3. $x(\boldsymbol{C}_{(<,>)}(x) - 1)(\boldsymbol{C}_{(<,>)}(x) - 1 - x - x(\boldsymbol{C}_{(<,>)}(x) - 1))$

□

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$$\begin{aligned}\boldsymbol{C}_{(<,>)}(x) &= 1 + x\boldsymbol{C}_{(<,>)}(x) + x(\boldsymbol{C}_{(<,>)}(x) - 1)\\ &+ x(\boldsymbol{C}_{(<,>)}(x) - 1)(\boldsymbol{C}_{(<,>)}(x) - 1 - x - x(\boldsymbol{C}_{(<,>)}(x) - 1)),\end{aligned}$$

□

Cases $(=,<)$, $(<,=)$, and $(<,>)$

The sets $\mathcal{C}(=,<)$ and $\boldsymbol{C}(<,>) = \boldsymbol{C}(\underline{010},\underline{120})$ are in one-to-one correspondence, but the number of descents cannot be preserved.

$$\mathcal{C}_4(<,>) = \{0000, 0001, 0011, 0012, 0110, 0111, 0112, 0122, 0123\}.$$

$$\mathcal{C}_4(=,<) = \{0000, 0100, 0101, 0110, 0111, 0120, 0121, 0122, 0123\}.$$

Example: Cases $(=,<)$, $(<,=)$, and $(<,>)$

Theorem

The g.f. $\boldsymbol{C}_{(=,<)}(x,y)$ and $\boldsymbol{C}_{(<,=)}(x,y)$ are given by

$$\frac{(1-x+2xy)(1-x)-\sqrt{(1-x)^4-(1-x)4x^2y}}{2xy(1-x)}.$$

Theorem

We have

$$\boldsymbol{C}_{(<,>)}(x,y)=\frac{1-2x+2xy-x^2y-\sqrt{1-4x+4x^2-2x^2y+x^4y^2}}{2xy(1-x)}.$$

Corollary

$$\boldsymbol{c}_{(<,>)}(n) = \sum_{k=0}^{\lfloor (n-1)/2 \rfloor} \frac{(-1)^k}{n-k} \binom{n-k}{k} \binom{2n-3k}{n-2k-1}, \; n \geq 1.$$

Corollary

$$c_{(<,>)}(n) = \sum_{k=0}^{\lfloor (n-1)/2 \rfloor} \frac{(-1)^k}{n-k} \binom{n-k}{k} \binom{2n-3k}{n-2k-1}, \quad n \geq 1.$$

Catalan words and Dyck path:

- $c_{(<,=)}(n)$:= counts the Dyck paths of semilength n avoiding $UUDU$.

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Catalan words and Dyck path:

- $c_{(<,=)}(n)$:= counts the Dyck paths of semilength n avoiding $UUDU$.
- $c_{(=,<)}(n)$:= counts the Dyck paths of semilength n avoiding $UDUU$.
- But, not all the cases have an *easy* interpretation in terms of Dyck paths.

Corollary

The g.f. for the total number of descents on $\mathcal{C}(<,>)$ *is*

$$\boldsymbol{D}_{(<,>)}(x) = \frac{1-4x+3x^2-(1-2x)\sqrt{1-4x+2x^2+x^4}}{2(1-x)x\sqrt{1-4x+2x^2+x^4}}.$$

The series expansion of $\boldsymbol{D}_{(<,>)}(x)$ is

$$x^4+6x^5+26x^6+100x^7+363x^8+1277x^9+O(x^{10}),$$

where the coefficient sequence does not appear in OEIS.

Summarizing...

Constante cases:

(X,Y)	Cardinality of $\mathcal{C}_n(X,Y)$, $n \geq 1$
$(\leq,\geq),(\leq,\neq)$	1, 2, 2, 2, ...
$(\leq,\leq)$	1, 2, 1, 2, 1, 2, ...
$(\neq,\leq)$	1, 2, 3, 3, 3, ...

(X,Y)	Cardinality of $\mathcal{C}_n(X,Y)$, $n \geq 1$	OEIS
$(=,=)$	$\boldsymbol{c}_{\underline{000}}(n) = \sum_{k=1}^{n} \binom{k}{n-k} m_{k-1}$	A247333
$(=,\geq)$	1, 2, 4, 10, 26, 72, 206, 606, 1820, 5558, ...	A102407
$(\geq,=)$	1, 2, 4, 10, 26, 72, 206, 606, 1820, 5558, ...	A102407
$(=,>)$	$\boldsymbol{C}_{\underline{110}}(x) = \frac{1-2x^2-\sqrt{1-4x+4x^3}}{2x(1-x)}$	A087626
$(>,=)$	$\boldsymbol{C}_{\underline{100}}(x) = \frac{1-2x^2-\sqrt{1-4x+4x^3}}{2x(1-x)}$	A087626
$(=,\leq)$	1, 2, 3, 7, 17, 43, 114, 310, 861, 2433, ...	A143013
$(\leq,=)$	1, 2, 3, 7, 17, 43, 114, 310, 861, 2433, ...	A143013
$(=,<)$	$\boldsymbol{c}_{\underline{001}}(n) = \sum_{k=0}^{\lfloor (n-1)/2 \rfloor} \frac{(-1)^k}{n-k} \binom{n-k}{k} \binom{2n-3k}{n-2k-1}$	A105633
$(<,=)$	$\boldsymbol{c}_{\underline{011}}(n) = \sum_{k=0}^{\lfloor (n-1)/2 \rfloor} \frac{(-1)^k}{n-k} \binom{n-k}{k} \binom{2n-3k}{n-2k-1}$	A105633
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$(=,\neq)$	1, 2, 4, 8, 17, 38, 89, 216, 539, 1374, ...	A086615
$(\neq,=)$	1, 2, 4, 8, 17, 38, 89, 216, 539, 1374, ...	A086615
$(\geq,\geq)$	m_n (Motzkin numbers)	A001006
$(<,<)$	m_n (Motzkin numbers)	A001006

Dyck paths avoiding $UDUDU$

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$(<,>)$	$\sum_{k=0}^{\lfloor (n-1)/2 \rfloor} \frac{(-1)^k}{n-k} \binom{n-k}{k} \binom{2n-3k}{n-2k-1}$	A105633
$(=,\neq)$	1, 2, 4, 8, 17, 38, 89, 216, 539, 1374, ...	A086615
$(\neq,=)$	1, 2, 4, 8, 17, 38, 89, 216, 539, 1374, ...	A086615
$(\geq,\geq)$	m_n (Motzkin numbers)	A001006
$(<,<)$	m_n (Motzkin numbers)	A001006

It had no combinatorial interpre.

(X,Y)	Cardinality of $\mathcal{C}_n(X,Y)$, $n \geq 1$	OEIS
$(=,=)$	$\boldsymbol{c}_{\underline{000}}(n) = \sum_{k=1}^{n} \binom{k}{n-k} m_{k-1}$	A247333
$(=,\geq)$	1, 2, 4, 10, 26, 72, 206, 606, 1820, 5558, ...	A102407
$(\geq,=)$	1, 2, 4, 10, 26, 72, 206, 606, 1820, 5558, ...	A102407
$(=,>)$	$\boldsymbol{C}_{110}(x) = \frac{1-2x^2-\sqrt{1-4x+4x^3}}{\ldots}$	A087626
$(>,=)$	$\boldsymbol{C}_{\underline{100}}(x) \ldots \frac{\ldots}{2x(1-x)}$	[illegible]
$(=,\leq)$	1, 2, 3, 7, 17, 43, 114, 310, 861, 2433, ...	A143013
$(\leq,=)$	1, 2, 3, 7, 17, 43, 114, 310, 861, 2433, ...	A143013
$(=,<)$	$\boldsymbol{c}_{\underline{001}}(n) = \sum_{k=0}^{\lfloor (n-1)/2 \rfloor} \frac{(-1)^k}{n-k} \binom{n-k}{k} \binom{2n-3k}{n-2k-1}$	A105633
$(<,=)$	$\boldsymbol{c}_{\underline{011}}(n) = \sum_{k=0}^{\lfloor (n-1)/2 \rfloor} \frac{(-1)^k}{n-k} \binom{n-k}{k} \binom{2n-3k}{n-2k-1}$	A105633
$(<,>)$	$\sum_{k=0}^{\lfloor (n-1)/2 \rfloor} \frac{(-1)^k}{n-k} \binom{n-k}{k} \binom{2n-3k}{n-2k-1}$	A105633
$(=,\neq)$	1, 2, 4, 8, 17, 38, 89, 216, 539, 1374, ...	A086615
$(\neq,=)$	1, 2, 4, 8, 17, 38, 89, 216, 539, 1374, ...	A086615
$(\geq,\geq)$	m_n (Motzkin numbers)	A001006
$(<,<)$	m_n (Motzkin numbers)	A001006

Motzkin path with 2 kinds of level steps...

(X,Y)	Cardinality of $\mathcal{C}_n(X,Y)$, $n \geq 1$	OEIS
$(\geq, >)$	1, 2, 5, 13, 35, 97, 275, 794, 2327, 6905, ...	A082582
$(>, \geq)$	1, 2, 5, 13, 35, 97, 275, 794, 23[illegible]	[illegible]
$(>, <)$	1, 2, 5, 13, 35, 97, 275, 794, 2327, 6905, ...	A082582
$(\geq, \leq)$	F_{n+1} (Fibonacci number)	A000045
$(\leq, <)$	F_{n+1} (Fibonacci number)	A000045
$(<, \leq)$	F_{n+1} (Fibonacci number)	A000045
$(\geq, <)$	2^{n-1}	A011782
$(\leq, >)$	2^{n-1}	A011782
$(\geq, \neq)$	$\binom{n}{2}+1$	A000124
$(>, >)$	$\boldsymbol{c}_{\underline{210}}(n) = \sum_{k=0}^{\lfloor n/2 \rfloor} \frac{1}{n-k}\binom{n-k}{k}\binom{n-k}{k+1}2^{}$[illegible]	A159771
$(>, \leq)$	P_{n+1}	A000129

Fibonacci numbers

Pell numbers

$(>, \neq)$	1, 2, 5, 13, 34, 90, 242, 660, 1821, 5073, ...	New
$(<, \geq)$	n	A000027
$(\neq, \geq)$	n	A000027
$(<, \neq)$	1, 2, 3, 6, 12, 25, 54, 119, 267, 608, ...	
$(\neq, >)$	1, 2, 4, 9, 22, 56, 146, 388, 1048, 2869, ...	A152225
$(\neq, <)$	1, 2, 4, 8, 17, 37, 82, 185, 423, 978, ...	A292460
$(\neq, \neq)$	1, 2, 3, 6, 11, 22, 43, 87, 176, 362, ...	A026418

New

Further Driections

- Let $\boldsymbol{c}_p(n,k)$ denote the number of Catalan words (avoiding the consecutive pattern p) of length n, whose last symbol is equal to k.
- Let $\mathcal{T}_p$ be the infinite matrix $\mathcal{T}_p := (\boldsymbol{c}_p(n,k))_{n\geq 1, k\geq 0}$.

Example

The first few rows of the matrix $\mathcal{T}_{\underline{010}}$ are

$$\mathcal{T}_{\underline{010}} = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 2 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ \mathbf{3} & \mathbf{3} & \mathbf{3} & \mathbf{1} & 0 & 0 & 0 & 0 & 0 \\ 9 & 8 & 6 & 4 & 1 & 0 & 0 & 0 & 0 \\ 25 & 25 & 16 & 10 & 5 & 1 & 0 & 0 & 0 \\ 73 & 74 & 51 & 28 & 15 & 6 & 1 & 0 & 0 \\ 223 & 223 & 159 & 91 & 45 & 21 & 7 & 1 & 0 \\ 697 & 696 & 496 & 296 & 150 & 68 & 28 & 8 & 1 \end{pmatrix}$$

Riordan Arrays

Definition

A Riordan array is an infinite lower triangular matrix whose k-th column has generating function $g(x)f(x)^k$ for all $k \geq 0$, for some formal power series $g(x)$ and $f(x)$ with $g(0) \neq 0$, $f(0) = 0$, and $f'(0) \neq 0$. Such a Riordan array is denoted by $(g(x), f(x))$.

$$(g(x), f(x)) =: \begin{pmatrix} l_{00} & & & \\ l_{10} & l_{11} & & \\ l_{20} & l_{21} & l_{22} & \\ l_{30} & l_{31} & l_{32} & l_{33} \\ \vdots & \vdots & \vdots & \vdots \\ g(x) & g(x)f(x) & g(x)f^2(x) & g(x)f^3(x) \end{pmatrix}$$

The product of two Riordan arrays $(g(x), f(x))$ and $(h(x), l(x))$ is defined by

$$(g(x), f(x)) * (h(x), l(x)) = (g(x)h(f(x)), l(f(x))) \,. \tag{1}$$

Under this operation, the set of all Riordan arrays is a group.

The product of two Riordan arrays $(g(x), f(x))$ and $(h(x), l(x))$ is defined by

$$(g(x), f(x)) * (h(x), l(x)) = (g(x)h(f(x)), l(f(x))) . \qquad (1)$$

Under this operation, the set of all Riordan arrays is a group.

Theorem

The matrix $\mathcal{T}_{\underline{010}}$ is a Riordan array given by

$$\left(1, \frac{1+x^2-\sqrt{1-4x+2x^2-4x^3+x^4}}{2(1+x^2)}\right)$$

$$= \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 2 & 1 & 0 & 0 & 0 & 0 \\ 0 & 3 & 3 & 3 & 1 & 0 & 0 & 0 \\ 0 & 9 & 8 & 6 & 4 & 1 & 0 & 0 \\ 0 & 25 & 25 & 16 & 10 & 5 & 1 & 0 \\ 0 & 73 & 74 & 51 & 28 & 15 & 6 & 1 \end{pmatrix}.$$

Theorem

If C_n denotes the n-th Catalan number, then for $n \geq 2$ and $k \geq 0$,

$$\boldsymbol{c}_{\underline{010}}(n,k) = \sum_{\ell=0}^{n-1} \boldsymbol{c}_{\underline{010}}(n-1, k-1-\ell)\, a_\ell,$$

where

$$a_n := 1 + \sum_{i=0}^{\lfloor \frac{n-1}{2} \rfloor} (-1)^{i+1} \binom{n-i-1}{i} \overline{C}_{n-i-1} \quad \textit{and}$$

$$\overline{C}_n := \begin{cases} C_{\frac{n-1}{2}}, & \textit{if } n \textit{ is odd;} \\ 0, & \textit{otherwise.} \end{cases}$$

Theorem

For $p \in \{\underline{010}, \underline{000}, \underline{210}, \underline{120}, \underline{100}, \underline{110}, \underline{001}, \underline{101}, \underline{120}, \underline{100}, \underline{110}\}$*, the matrices* $\mathcal{T}_p$ *are Riordan arrays.*

Theorem

For $p \in \{\underline{010}, \underline{000}, \underline{210}, \underline{120}, \underline{100}, \underline{110}, \underline{001}, \underline{101}, \underline{120}, \underline{100}, \underline{110}\}$*, the matrices* $\mathcal{T}_p$ *are Riordan arrays.*

The matrix related to the pattern $\underline{012}$ can not be a Riordan array.

Theorem

For $p \in \{\underline{010}, \underline{000}, \underline{210}, \underline{120}, \underline{100}, \underline{110}, \underline{001}, \underline{101}, \underline{120}, \underline{100}, \underline{110}\}$, *the matrices* $\mathcal{T}_p$ *are Riordan arrays.*

The matrix related to the pattern $\underline{012}$ can not be a Riordan array. **Problem:** What is the combinatorial interpretation of the inverse matrix? For example, the absolute value of the second and third columns of the inverse matrix $(\mathcal{M}_{\underline{010}}^{f(x)=1})^{-1}$ are the sequences A104545 (number of Motzkin paths of length n having no consecutive $(1,0)$ steps) and A256169, respectively.

$$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & -2 & 1 & 0 & 0 & 0 & 0 \\ 0 & -3 & 3 & -3 & 1 & 0 & 0 & 0 \\ 0 & 5 & -8 & 6 & -4 & 1 & 0 & 0 \\ 0 & -11 & 17 & -16 & 10 & -5 & 1 & 0 \\ 0 & 25 & -38 & 39 & -28 & 15 & -6 & 1 \end{pmatrix}.$$

Thank you!