Pattern Avoidance in Alternating Sign Matrices

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The Plan:

Introduction to Alternating Sign Matrices

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Pattern avoidance in ASMs determined by the left key

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Left Key avoidance of 213 and 123

Introduction to Alternating Sign Matrices

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Definition

An *alternating sign matrix* (ASM) is a square matrix where

1. Each entry is one of $0, 1, -1$.
2. Every row and column sum to 1.
3. The non-zero entries of each row and column alternate in sign.

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Johansson and Linusson (2007) explored this version of pattern avoidance, determining the following:

1. The $n \times n$ ASMs avoiding 132 are enumerated by the large Schr oder numbers (offset by one).
2. A complete classification for the enumeration of $n \times n$ ASMs for all pairs of patterns of length three.
3. Numerical evidence for the $n \times n$ ASMs avoiding 123.
4. Numerical evidence for the $n \times n$ ASMs avoiding patterns of length four (seven symmetry classes).

Pattern avoidance in ASMs determined by the left key

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The key is found by systematically removing the -1s of an alternating sign matrix while maintaining an alternating sign matrix at each stage.

The key is the resulting permutation matrix at the final stage.

This version of ASM pattern avoidance is (symmetrically equivalent) to the case of ASMs whose Gog words avoid 312 which was studied by Ayyer, Cori, and Gouyou-Beauchamps (2011).

Reframing in terms of left key avoidance that we study (consistent with Aval [1]), would relate the specific triangles Ayyer, Cori, and Gouyou-Beauchamps [2] study to alternating sign matrices whose key avoids 213.

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1. A -1 entry is removable if there are no other -1 entries that are weakly Northwest of it.
2. For each removable -1,
   2.1 Consider the Ferrer's shape that results using the -1 as the Southeast corner, the next 1 entry to the North of the removable -1 as the Northeast corner, the next 1 to the West of the removable -1 as the Southwest corner and any 1s that appear in this rectangle as corners in the Ferrers diagram.

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   2.2 Replace the North-most 1 from the Ferrer's diagram with a 0. For each subsequent 1, place a new 1 in the row of the previously replaced 1 and column of the current 1, then replace the old 1 with a 0. Finally replace the -1 with a 0.

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The resulting permutation matrix determines the key.

Example

Determining the left key of an alternating sign matrix:

$$\begin{pmatrix} 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & -1 & 1 \\ 0 & 1 & -1 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 \end{pmatrix}$$

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$$\to \begin{pmatrix} 0 & 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 1 & -\mathbf{1} & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 \end{pmatrix}$$

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Known monotone triangles associated with ASMs

Associate monotone triangles with alternating sign matrices as follows:

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- Create a 0-1 matrix by summing rows from the bottom to the current row.

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- Create a 0-1 matrix by summing rows from the bottom to the current row.
- Record the columns where the 1s sit from bottom to top in the matrix to create the triangle from top to bottom.

Example

Obtaining a monotone triangle from an alternating sign matrix:

$$\begin{pmatrix} 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & -1 & 1 \\ 0 & 0 & 0 & 1 & 0 \\ 1 & -1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \end{pmatrix}$$

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Gapless monotone triangles are in bijection with alternating sign matrices avoiding 213.

Left Key avoidance of 213 and 123

Lemma

Removing a removable -1 *maps a decreasing sequence of* 1*s in the ASM to a decreasing sequence of length one less than the original where the largest entry of the original decreasing sequence is removed and all other entries are moved to an earlier position.*

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Let A be an ASM whose key avoids 213*. If A contains* 123 *in the manner studied by Johansson and Linusson, then the key of A also contains* 123*.*

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Theorem

The class of $n \times n$ alternating sign matrices whose key avoids both 123 and 213 is in bijection with the set of gapless monotone triangles of size n with at most two distinct values in any column.

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Since the smallest entry of the column must be k as the columns decrease (gaplessly), there must be an entry of size $k+2$ in the kth column.

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Since the smallest entry of the column must be k as the columns decrease (gaplessly), there must be an entry of size $k+2$ in the kth column.

The row r containing $k+2$ in the kth column is missing two smaller values from $\{1, 2, 3, \ldots, k+1\}$, say a, b. Translating back to A, there must be 1s further up in columns a, b than at least one 1 in column $k+2$.

Case 1: There exist $x < y < r$ such that the A has a 1 in positions $(x, a), (y, b), (r, k+2)$. However, from our second lemma, A cannot contain this 123 pattern.

Case 2: If not, there must exist $x < y < r$ such that the A has a 1 in positions $(x, b), (y, a), (r, k + 2)$.

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This is because any deletions of a smaller entry would indicate we could have chosen a smaller value y (for possibly a different choice of a) and deletions an entry larger than a would require the insertion another larger entry still allowing for a choice of a smaller value of r (for possibly a different choice of k and a).

Inserting a into row y means that entries in the columns contain entries larger than a shift right. Specifically, the entry and b is still missing, consider the column c that contained $b+1$ in row $y+1$.

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The entry $b + 1$ is shifted to column $c + 1$ meaning that column c now contains entry $b - 1$ in row y. However, then the monotone triangle is not gapless in column c contradicting A being a 213-key-avoiding ASM.

There is then a nice bijection between these triangles and a class of inversion sequences known to be counted by the Catalan numbers from work of Martinez and Savage (2018).

Proposition
Gapless monotone triangles with n columns with at most two distinct values in each column are in bijection with inversion sequences avoiding 10.

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Let T_n be the set of gapless monotone triangles with n columns with at most two distinct values in each column. Let M_n be the set of inversion sequences of length n avoiding 10.

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Define $f : T_n \to M_n$ to be such that the ith entry in the resulting inversion sequence is one less than the number of times $n+1-i$ appears in the $(n+1-i)$th column.

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Hence the inversion sequence is weakly increasing, i.e. avoids 10.

Theorem

The number of $n \times n$ *ASMs avoiding both* 213 *and* 123 *is*

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References:

Jean-Christophe Aval.
Keys and alternating sign matrices.
Sém. Lothar. Combin., 59:Art. B59f, 13, 2007/10.

Arvind Ayyer, Robert Cori, and Dominique Gouyou-Beauchamps.
Monotone triangles and 312 pattern avoidance.
Electron. J. Comb., 18(2), 2011.

Robert Johansson and Svante Linusson.
Pattern avoidance in alternating sign matrices.
Ann. Comb., 11(3-4):471–480, 2007.

Alain Lascoux.
Chern and Yang through ice.
preprint.

Megan Martinez and Carla Savage.
Patterns in inversion sequences II: inversion sequences avoiding triples of relations.
J. Integer Seq., 21(2):Art. 18.2.2, 44, 2018.

W. H. Mills, David P. Robbins, and Howard Rumsey, Jr.
Proof of the Macdonald conjecture.
Invent. Math., 66(1):73–87, 1982.

W. H. Mills, David P. Robbins, and Howard Rumsey, Jr.
Alternating sign matrices and descending plane partitions.
J. Combin. Theory Ser. A, 34(3):340–359, 1983.

Thank you!