Inversion sequences avoiding pairs of patterns

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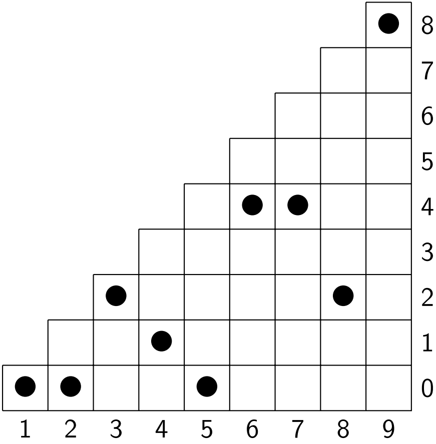
Inversion sequences (1/2)

An inversion sequence is a finite sequence of integers $(\sigma_i)_{i\in\{1,\ldots,n\}}$ such that

$$0 \leqslant \sigma_i < i \quad \forall i \in \{1,\ldots,n\}.$$

Example:

$(0, 0, 2, 1, 0, 4, 4, 2, 8)$:



Inversion sequences (2/2)

$\mathbf{I}_n$ is the set of inversion sequences of length n. $\#\mathbf{I}_n = n!$.

There is a bijection (Lehmer code) which maps a permutation $\pi \in \mathfrak{S}_n$ to the inversion sequence $\sigma \in \mathbf{I}_n$ defined by

$$\sigma_i = \#\{j < i \mid \pi(j) > \pi(i)\} \quad \forall i \in \{1, \ldots, n\}.$$

σ_i counts the inversions of π whose second entry is at position i.

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Example: $(4, 3, 1, 6, 2, 5) \mapsto (0, \ , \ , \ , \ , \ )$

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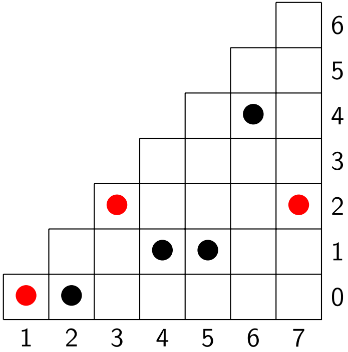
Example: $(4, 3, 1, 6, 2, 5) \mapsto (0, 1, 2, 0, 3, 1)$

- Patterns may have repeated letters.
- We only deal with classical patterns.

Example:

The inversion sequence (0,0,2,1,1,4,2) contains the pattern 011, and avoids 110.



$\mathbf{I}_n(\tau)$ is the set of τ-avoiding inversion sequences of length n.
$\mathbf{I}(\tau) = \bigsqcup_n \mathbf{I}_n(\tau)$.

Examples:

- $\mathbf{I}(01) = \{(0),\ (0,0),\ (0,0,0),\ (0,0,0,0),\ \ldots\}$
- $\mathbf{I}(00) = \{(0),\ (0,1),\ (0,1,2),\ (0,1,2,3),\ \ldots\}$

Context and summary of results

First papers:

- Mansour–Shattuck 2015,
- Corteel–Martinez–Savage–Weselcouch 2016.

Both study inversion sequences avoiding one pattern of length 3.

Subsequent works (by Lin–Yan, Martinez–Savage, Mansour–Yıldırım, among others) left the enumeration open for only one single pattern, and 23 pairs of patterns of length 3.

We solve the enumeration for all 24 cases with polynomial time algorithms, using recurrence formulas derived from four different constructions of inversion sequences.

Inversion sequences avoiding one pattern of length 2 or 3

Pattern θ	$\#\mathbf{I}_n(\theta)$ for $n = 1, \ldots, 7$	Comment	OEIS
00 or 01	1, 1, 1, 1, 1, 1, 1	Constant	A000012
10	1, 2, 5, 14, 42, 132, 429	Catalan numbers	A000108
000	1, 2, 5, 16, 61, 272, 1385	Euler zigzag numbers	A000111
001	1, 2, 4, 8, 16, 32, 64	2^{n-1}	A000079
010	1, 2, 5, 15, 53, 215, 979		A263779
011	1, 2, 5, 15, 52, 203, 877	Bell numbers	A000110
012	1, 2, 5, 13, 34, 89, 233	Fibonacci$(2n-1)$	A001519
021	1, 2, 6, 22, 90, 394, 1806	Large Schröder numbers	A006318
100	1, 2, 6, 23, 106, 565, 3399		A263780
101 or 110	1, 2, 6, 23, 105, 549, 3207	$\#\mathbf{S}_n(1\underline{23}4)$	A113227
102	1, 2, 6, 22, 89, 381, 1694		A200753
120	1, 2, 6, 23, 103, 515, 2803		A263778
201 or 210	1, 2, 6, 24, 118, 674, 4306		A263777

Generating trees

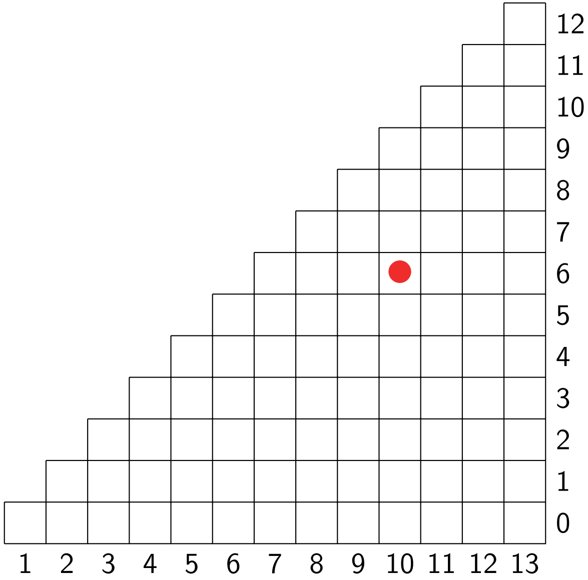
A generating tree construction creates objects of size $n+1$ from objects of size n.

One simple approach: take an inversion sequence $\sigma \in \mathbf{I}_n$, and insert an entry $i \in \{0, \ldots, n\}$ at the end to create a sequence $\sigma \cdot i \in \mathbf{I}_{n+1}$.

When avoiding patterns, some values of i are forbidden. The problem comes down to finding efficient ways to count such forbidden values.

We only use this construction for inversion sequences avoiding $\{000, 100\}$, since all other pairs it could be applied to were already solved (most of them by Mansour and Yıldırım).

Building a sequence around its first maximum



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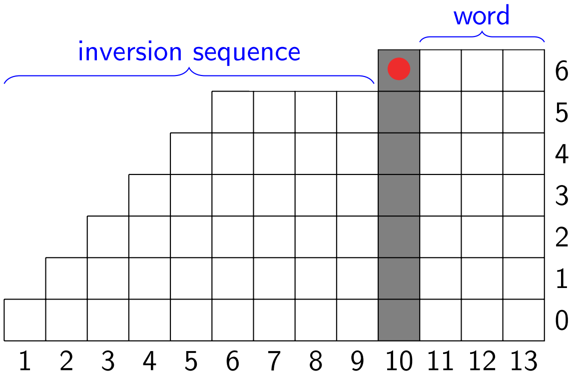

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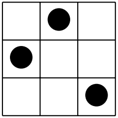
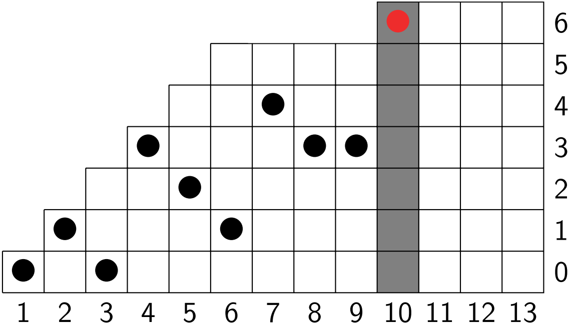
Avoiding the pattern 120





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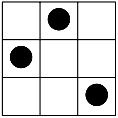

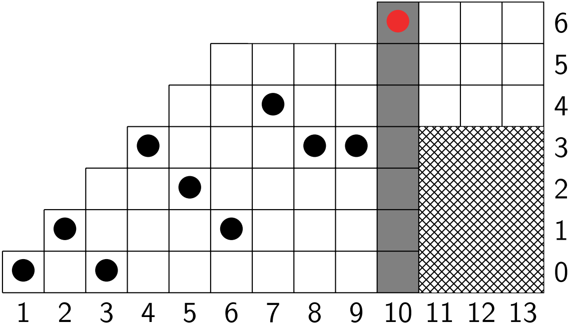

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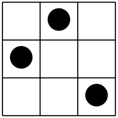

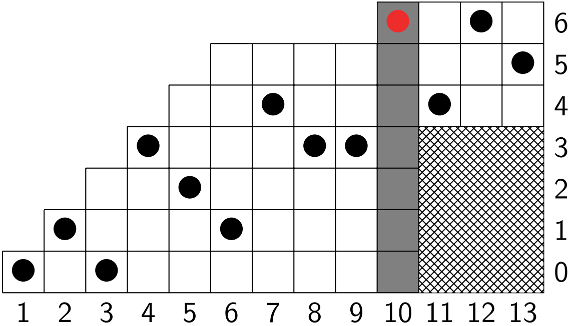

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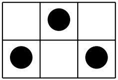
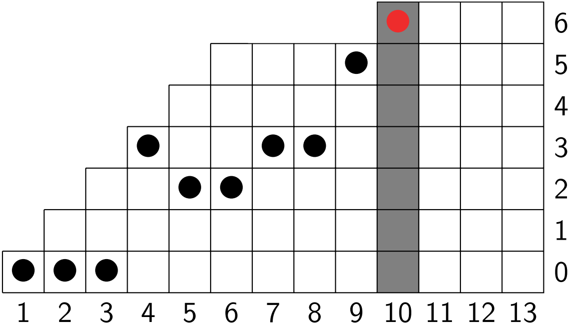
Avoiding the pattern 010 



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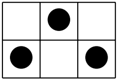

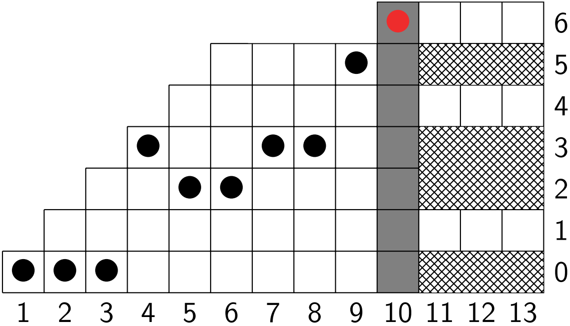

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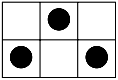
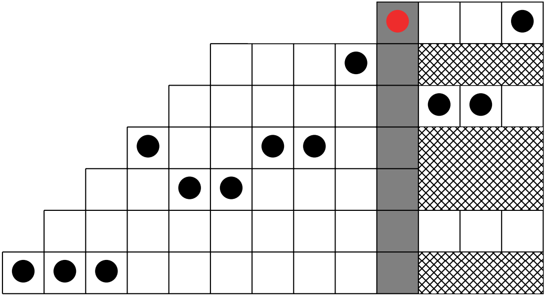
Avoiding the pattern 010 



Avoiding 010

Let $\mathfrak{a}_{n,m,d} = \#\{\sigma \in \mathbf{I}_n(010) \mid \max(\sigma) = m,\ \mathrm{dist}(\sigma) = d\}$.
Then for all $n, m, d > 0$,

$$\mathfrak{a}_{n,m,d} = \sum_{p=m+1}^{n} \sum_{i=0}^{d-1} \sum_{j=0}^{m-1} \mathfrak{a}_{p-1,j,i} \binom{m-i}{d-i-1} \begin{bmatrix} n-p+1 \\ n-p-d+i+2 \end{bmatrix}.$$

- p is the position of the first m,
- i is the number of distinct values to the left of p,
- j is the largest value to the left of p.

There are

- $\mathfrak{a}_{p-1,j,i}$ choices for the inversion sequence on the left,
- $\binom{m-i}{d-i-1}$ choices for the set of letters on the right,
- $\begin{bmatrix} n-p+1 \\ n-p-d+i+2 \end{bmatrix}$ choices for the word over those letters.

Splitting at the first max: concluding remarks

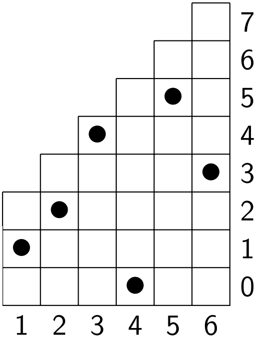
- This method yields good results with many patterns.
- But not all of them (e.g. 102).
- This construction is unpredictable: it can solve the enumeration when avoiding 000 or 210, but not the pair {000, 210}.

We solve the enumeration of inversion sequences avoiding the patterns 010, {000, 120}, {010, 000}, {010, 110}, {010, 120}, {010, 201}, {010, 210}, {011, 120}, {100, 120}, {101, 120}, {110, 120}, {102, 201}, and {120, 201} by splitting at the first maximum.

Shifted inversion sequences

σ is a *h*-shifted inversion sequence of length n if $0 \leqslant \sigma_i < i + h$ for all $i \in \{1, \ldots, n\}$. We denote by $\mathbf{I}_n^h$ the set of such sequences σ.

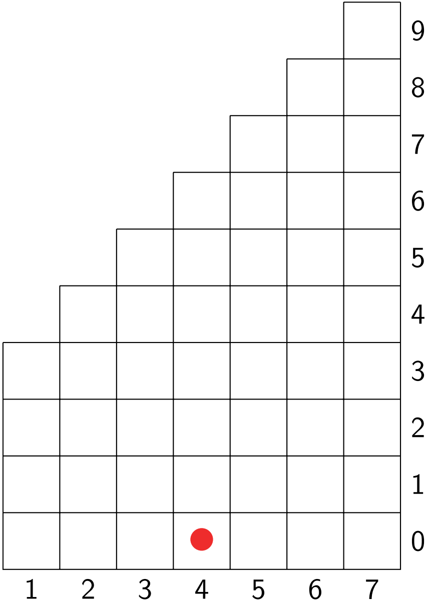
Example:
$(1, 2, 4, 0, 5, 3) \in \mathbf{I}_6^h \quad \forall h \geqslant 2.$



Elements of $\mathbf{I}_n^h$ can be seen as (classical) inversion sequences of length $n + h$ whose first h entries were deleted.

In general, splitting an inversion sequence in two results in an inversion sequence on the left side, and a shifted inversion sequence on the right.

Splitting at the first zero



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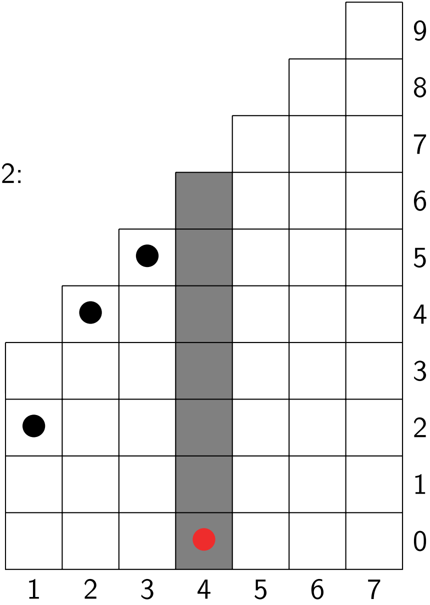
Avoiding 010 and 102:



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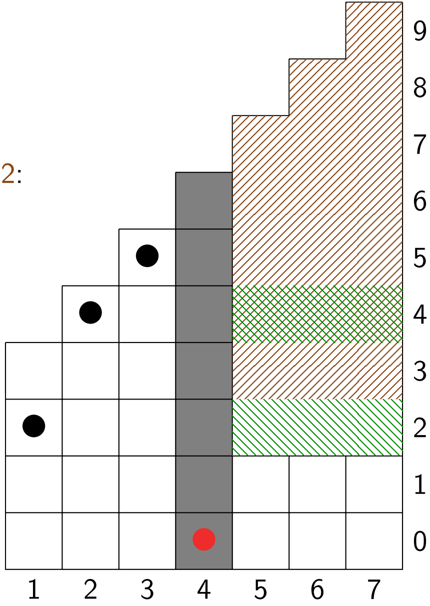

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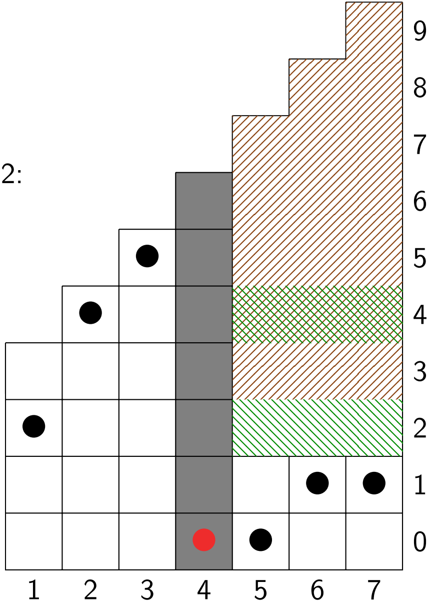


Inversion sequences avoiding {010,102}

Let $\mathfrak{a}_{n,h} = \#\mathbf{I}_n^h(010, 102)$,
$\mathfrak{b}_{n,k} = \#\{\omega \in \{1, \ldots, k\}^n(010, 102) \mid \max(\omega) = k\}$. Then

$$\mathfrak{a}_{n,h} = \Big(\sum_{z=1}^{n} \mathfrak{a}_{n-z,h+z-1}\Big) + \mathfrak{a}_{n,h-1} + \Big(\sum_{\ell=1}^{n-1} \mathfrak{a}_{\ell,h-1}\Big)$$
$$+ \sum_{r=1}^{n-2} \sum_{m=1}^{h} \Big(\mathfrak{b}_{r,m} \cdot \sum_{\ell'=0}^{n-r-1} (n - r - \ell' - \delta_{\ell',0}) \cdot \mathfrak{a}_{\ell',h-m-1}\Big)$$

- z is the number of leading zeros,
- ℓ is the left size,
- r is the right size,
- m is the right maximum,
- ℓ' is the left size after removing m.

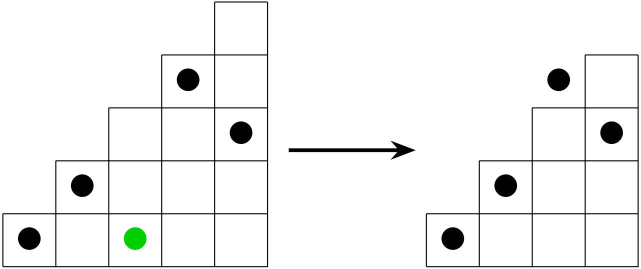
Shifted sequences: concluding remarks

- Similar to the previous method.
- But more complex, since the shift requires an additional parameter.

We solved the enumeration for the pairs of patterns $\{010, 102\}$, $\{100, 102\}$, and $\{102, 210\}$ by splitting shifted inversion sequences at the first zero.

Deleting a term

In general, deleting a term from an inversion sequence does not yield an inversion sequence:



Deleting a maximum always results in an inversion sequence, since all values to its right are lesser or equal.

Avoiding the pair {100, 110} (1/2)

- In a 100-avoiding sequence, all values appearing after the first maximum must be distinct, unless they are repetitions of the maximum.
- In a 110-avoiding sequence, any repetitions of the maximum must appear in a single factor at the end.

Example: $(0, 0, 1, 0, 4, 2, 6, 5, 1, 3, 4, 6, 6) \in \mathbf{I}_{13}(100, 110)$.

Remark: since any repetitions of the maximum must appear at the end, it is sufficient to enumerate sequences which contain a single maximum.

Avoiding the pair {100, 110} (2/2)

Let $\mathfrak{a}_{n,m,p} = \#\{\sigma \in \mathbf{I}_n(100, 110) \mid \sigma_p = m \quad \text{and} \quad \forall i \neq p,\ \sigma_i < m\}$.

If $p \in \{n, n-1\}$, then

$$\mathfrak{a}_{n,m,p} = \sum_{s=0}^{m-1} \sum_{q=s+1}^{n-1} \sum_{r=1}^{n-q} \mathfrak{a}_{n-r,s,q}.$$

Otherwise,

$$\mathfrak{a}_{n,m,p} = \sum_{s=1}^{m-1} \left((n-p) \cdot \mathfrak{a}_{n-1,s,p} + \sum_{q=s+1}^{p-1} (\mathfrak{a}_{n-1,s,q} + \mathfrak{a}_{n-2,s,q})\right).$$

- s is the second maximum of σ,
- q is the position of the first s,
- r is the number of occurrences of s.

Deleting the maxima: concluding remarks

- Can be expressed through generating trees, but requires more parameters.

We solved the enumeration for the pairs {000, 102}, {000, 201}, {000, 210}, {100, 101}, {100, 110}, {101, 210}, and {110, 201} by deleting the maxima.

Perspectives

We have completed the enumeration of inversion sequences avoiding one or two patterns of length 3.
This leads to some new questions:

- What is the asymptotic growth of the associated enumeration sequences?
- What is the nature of their generating functions?

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