Growth rates of permutations with a given descent set

Justin Troyka

California State University, Los Angeles

Permutation Patterns
July 7, 2023

Joint work with Mohamed Omar, in progress

In case you can't focus on my talk, here is a combinatorics problem you can work on instead.

There are n married couples, each comprising one woman and one man. These $2n$ people are dancing in a circle. Very concerned with modesty, they arrange themselves so that no woman is next to a man she is not married to. How many possible arrangements are there?

Cyclic permutations of an arrangement count as different. If you like, start with $n = 3$.

Introduction

If w is a $\{0, 1\}$ word whose length is at least k, then we let $w[k]$ denote the word formed by the first k letters of w.

- Example: $w = 011010011001\ldots$ and $w[5] = 01101$.

The descent word of $\pi \in S_n$ is the word $\mathrm{Des}(\pi) = w_1 \ldots w_{n-1}$ such that $w_i = 1$ if i is a descent and $w_i = 0$ if i is an ascent.

- Example: $\pi = 314952687$ and $\mathrm{Des}(\pi) = 10011001$.

The peak word of $\pi \in S_n$ is the word $\mathrm{Pk}(\pi) = w_1 \ldots w_{n-2}$ such that $w_i = 1$ if $i + 1$ is a peak of π and $w_i = 0$ otherwise.

- Example: $\pi = 314952687$ and $\mathrm{Pk}(\pi) = 0010001$.

Introduction

If w is a $\{0,1\}$ word whose length is at least k, then we let $w[k]$ denote the word formed by the first k letters of w.

- Example: $w = 011010011001\ldots$ and $w[5] = 01101$.

The descent word of $\pi \in S_n$ is the word $\mathrm{Des}(\pi) = w_1 \ldots w_{n-1}$ such that $w_i = 1$ if i is a descent and $w_i = 0$ if i is an ascent.

- Example: $\pi = 314952687$ and $\mathrm{Des}(\pi) = 10011001$.

The peak word of $\pi \in S_n$ is the word $\mathrm{Pk}(\pi) = w_1 \ldots w_{n-2}$ such that $w_i = 1$ if $i+1$ is a peak of π and $w_i = 0$ otherwise.

- Example: $\pi = 314952687$ and $\mathrm{Pk}(\pi) = 0010001$.

Define $d_n(w)$ to be the number of permutations in S_n with descent word $w[n-1]$.

Define $p_n(w)$ to be the number of permutations in S_n with peak word $w[n-2]$.

Introduction

Define $d_n(w)$ to be the number of permutations in S_n with descent word $w[n-1]$.

- Example: $d_6(000100010001\ldots) = ?$
  $= \#\{\pi \in S_6 \colon \ \mathrm{Des}(\pi) = 00010\}$

1235|46	1236|45	1245|36	1346|25	1256|34
1345|26	1346|25	1356|24	1456|23	2345|16
2346|15	2356|14	2456|13	3456|12	

$d_6(000100010001\ldots) = 14.$

Define $p_n(w)$ to be the number of permutations in S_n with peak word $w[n-2]$.

Introduction

Define $d_n(w)$ to be the number of permutations in S_n with descent word $w[n-1]$.

- Example: $d_6(000100010001\ldots) = ?$
  $= \#\{\pi \in S_6 : \ \mathrm{Des}(\pi) = 00010\}$

1235|46	1236|45	1245|36	1346|25	1256|34
1345|26	1346|25	1356|24	1456|23	2345|16
2346|15	2356|14	2456|13	3456|12	

$d_6(000100010001\ldots) = 14.$

Define $p_n(w)$ to be the number of permutations in S_n with peak word $w[n-2]$.

Our goal: understand $d_n(w)$ and $p_n(w)$ for a given infinite word w, especially the asymptotics as $n \to \infty$.

Introduction

Suppose w has finitely many $1'$s. Now $p_n(w)$ is the peak polynomial for w, a polynomial in n, well studied (Billey Burdzy Sagan 2013, Billey Fahrbach Talmage 2016, Diaz-Lopez Harris Insko Omar 2017, etc.). Similarly, $d_n(w)$ is the descent polynomial for w, also a polynomial in n, somewhat less studied (Kantarcı Oğuz 2018, Diaz-Lopez Harris Insko Omar Sagan 2019).

When w is allowed to have infinitely many $0'$s and infinitely many $1'$s, interesting things can happen. The numbers must be greater than polynomial. There is some prior work (see the abstract), including my thesis research (2018).

I. Introduction (done)
II. The growth rate of a descent word or peak word
III. Other results and further questions

I. Introduction (done)
II. **The growth rate of a descent word or peak word**
III. Other results and further questions

II. The growth rate of a descent word or peak word

- $d_n(w)$ is maximized by $w = 010101\ldots$ and $w = 101010\ldots$ (Viennot 1979).
- $d_n(010101\ldots)$ counts the alternating permutations and equals the Euler number, E_n.
- $E_n \sim \frac{4}{\pi}\left(\frac{2}{\pi}\right)^n n!$.
- If w is chosen uniformly at random, then $\mathbb{E}[d_n(w)] = 2\left(\frac{1}{2}\right)^n n!$.

II. The growth rate of a descent word or peak word

- $d_n(w)$ is maximized by $w = 010101\ldots$ and $w = 101010\ldots$ (Viennot 1979).
- $d_n(010101\ldots)$ counts the alternating permutations and equals the Euler number, E_n.
- $E_n \sim \frac{4}{\pi}\left(\frac{2}{\pi}\right)^n n!$.
- If w is chosen uniformly at random, then $\mathbb{E}[d_n(w)] = 2\left(\frac{1}{2}\right)^n n!$.

These suggest that, in many cases, $d_n(w)$ is asymptotically of the form $cn^p L^n n!$. We focus on the number L. Define

$$\operatorname{gr} d_n(w) = \lim_{n\to\infty}\left(\frac{d_n(w)}{n!}\right)^{1/n}.$$

II. The growth rate of a descent word or peak word

- $d_n(w)$ is maximized by $w = 010101\ldots$ and $w = 101010\ldots$ (Viennot 1979).
- $d_n(010101\ldots)$ counts the alternating permutations and equals the Euler number, E_n.
- $E_n \sim \frac{4}{\pi}\left(\frac{2}{\pi}\right)^n n!$.
- If w is chosen uniformly at random, then $\mathbb{E}[d_n(w)] = 2\left(\frac{1}{2}\right)^n n!$.

These suggest that, in many cases, $d_n(w)$ is asymptotically of the form $cn^p L^n n!$. We focus on the number L. Define

$$\operatorname{gr} d_n(w) = \lim_{n\to\infty}\left(\frac{d_n(w)}{n!}\right)^{1/n}.$$

- $0 \leqslant \operatorname{gr} d_n(w) \leqslant 2/\pi \approx 0.637$.

II. The growth rate of a descent word or peak word

- $p_n(w)$ is maximized by $w = 001001001\ldots$, with slight adjustments when $n \equiv 1 \pmod 3$. If w maximizes $p_n(w)$, then $p_n(w) = c\, 3^{-n/3} n!$, where c is determined by $n \bmod 3$. (Kasraoui 2012, Billey Fahrbach Talmage 2016).
- Just as before, we can write

$$\operatorname{gr} p_n(w) = \lim_{n\to\infty} \left(\frac{p_n(w)}{n!} \right)^{1/n}.$$

- $0 \leqslant \operatorname{gr} p_n(w) \leqslant 1/\sqrt[3]{3} \approx 0.693.$

II. The growth rate of a descent word or peak word

- $p_n(w)$ is maximized by $w = 001001001\ldots$, with slight adjustments when $n \equiv 1 \pmod 3$. If w maximizes $p_n(w)$, then $p_n(w) = c\,3^{-n/3} n!$, where c is determined by $n \bmod 3$. (Kasraoui 2012, Billey Fahrbach Talmage 2016).
- Just as before, we can write

$$\operatorname{gr} p_n(w) = \lim_{n\to\infty} \left(\frac{p_n(w)}{n!}\right)^{1/n}.$$

- $0 \leqslant \operatorname{gr} p_n(w) \leqslant 1/\sqrt[3]{3} \approx 0.693.$

Summary: *The number of permutations with descent word w or peak word w is at most an exponentially decaying fraction of the total number of permutations, and the base of the exponential decay is* $\leqslant 2/\pi$ *for the descent word and* $\leqslant 1/\sqrt[3]{3}$ *for the peak word.*

II. The growth rate of a descent word or peak word

We say two infinite words $u = u_1 u_2 \ldots$ and $v = v_1 v_2 \ldots$ are equicaudal if $u_a u_{a+1} \ldots = v_b v_{b+1} \ldots$ for some a and b — that is, u and v have the same tail, or the same coda.

- Example: $0101001100110011\ldots$
  and $00010000 1100110011\ldots$ are equicaudal.

Theorem (Omar & T. 2023+): If u and v are equicaudal, then $\operatorname{gr} d_n(u) = \operatorname{gr} d_n(v)$.

II. The growth rate of a descent word or peak word

We say two infinite words $u = u_1 u_2 \ldots$ and $v = v_1 v_2 \ldots$ are equicaudal if $u_a u_{a+1} \ldots = v_b v_{b+1} \ldots$ for some a and b — that is, u and v have the same tail, or the same coda.

- Example: $010100110011001 1\ldots$
  and $000100001100110011\ldots$ are equicaudal.

Theorem (Omar & T. 2023+): If u and v are equicaudal, then $\operatorname{gr} d_n(u) = \operatorname{gr} d_n(v)$.

Proof idea: Shifting u by one position multiplies $d_n(u)$ by a factor between $1/n$ and n. By induction, $d_n(v)/d_n(u)$ is bounded above and below by rational functions of n. ∎

II. The growth rate of a descent word or peak word

We say two infinite words $u = u_1 u_2 \dots$ and $v = v_1 v_2 \dots$ are equicaudal if $u_a u_{a+1} \dots = v_b v_{b+1} \dots$ for some a and b — that is, u and v have the same tail, or the same coda.

- Example: $0101001100110011\dots$
  and $00010000 1100110011\dots$ are equicaudal.

Theorem (Omar & T. 2023+): If u and v are equicaudal, then $\operatorname{gr} d_n(u) = \operatorname{gr} d_n(v)$.

Proof idea: Shifting u by one position multiplies $d_n(u)$ by a factor between $1/n$ and n. By induction, $d_n(v)/d_n(u)$ is bounded above and below by rational functions of n. ∎

- This extends the fact that $d_n(w)$ is a polynomial of n when w has finitely many 1's, since in this case w and $000\dots$ are equicaudal.

II. The growth rate of a descent word or peak word

Theorem (Omar & T. 2023+): For every $L \in [0, 2/\pi]$, there exists an infinite word w such that $\operatorname{gr} d_n(w) = L$.

Proof idea: We construct w letter by letter, by looking at the value of $g_n = \frac{d_n(w)}{n!} L^{-n}$. Append 0's until g_n is a bit lower than 1; next, append 10's until g_n is a bit higher than 1; repeat.

$$0000010101000000001010101010000000000000\ldots$$

We know that each phase of adding 0's will eventually end, because otherwise w would be equicaudal to $000\ldots$ and thus g_n would go to 0. Similarly, we know that each phase of adding 10's will eventually end, because otherwise w would be equicaudal to $010101\ldots$ and thus g_n would go to infinity. ■

II. The growth rate of a descent word or peak word

Theorem (Omar & T. 2023+): For every $L \in \left[0, 1/\sqrt[3]{3}\right]$, there exists an infinite word w such that $\operatorname{gr} p_n(w) = L$.

We have not yet tried to prove this using a similar technique as for $d_n(w)$, but it will probably work. Our proof goes a different route.

I. Introduction (done)
II. The growth rate of a descent word or peak word
III. **Other results and further questions**

III. Other results and further questions

- Recall that every $L \in [0, 2/\pi]$ is a growth rate of a descent word. This implies that the set

$$\left\{ \left(\frac{d_n(w)}{n!} \right)^{1/n} : w \text{ a binary word}, n \geqslant 0 \right\}$$

is dense in $[0, 2/\pi]$.

- But there is a more direct way to see this:

Theorem (Omar & T. 2023+): For every $L \in [0, 2/\pi]$ and every $\epsilon > 0$, there exist $a, b \geqslant 0$ such that

$$\left| \left(\frac{d_n(0^a(10)^b)}{n!} \right)^{1/n} - L \right| < \epsilon.$$

III. Other results and further questions

Theorem (Omar & T. 2023+): For every $L \in [0, 2/\pi]$ and every $\epsilon > 0$, there exist $a, b \geqslant 0$ such that

$$\left| \left(\frac{d_n(0^a(10)^b)}{n!} \right)^{1/n} - L \right| < \epsilon.$$

III. Other results and further questions

Theorem (Omar & T. 2023+): For every $L \in [0, 2/\pi]$ and every $\epsilon > 0$, there exist $a, b \geqslant 0$ such that

$$\left| \left(\frac{d_n(0^a(10)^b)}{n!} \right)^{1/n} - L \right| < \epsilon.$$

That is a corollary of the following:

Theorem (Omar & T. 2023+): For every $L \in [0, 2/\pi]$, there exists $\gamma > 0$ such that

$$\operatorname{gr} d_n(0^a(10)^b) = L,$$

where $a = \left\lfloor \frac{\gamma n}{\log(n)} \right\rfloor [+1]$ and $b = \frac{n-1-a}{2}$.

- The proof is a fun application of Stirling's approximation.

III. Other results and further questions

Theorem (Omar & T. 2023+): For every $L \in [0, 2/\pi]$ and every $\epsilon > 0$, there exist $a, b \geqslant 0$ such that

$$\left| \left(\frac{d_n(0^a(10)^b)}{n!} \right)^{1/n} - L \right| < \epsilon.$$

That is a corollary of the following:

Theorem (Omar & T. 2023+): For every $L \in [0, 2/\pi]$, there exists $\gamma > 0$ such that

$$\operatorname{gr} d_n(0^a(10)^b) = L,$$

where $a = \left\lfloor \frac{\gamma n}{\log(n)} \right\rfloor [+1]$ and $b = \frac{n-1-a}{2}$.

- The proof is a fun application of Stirling's approximation.
- Differs from our result that every L is a growth rate of some infinite word w, because $0^a(10)^b$ is not of the form $w[n-1]$ for one infinite word w — both a and b grow.

III. Other results and further questions

Similar results hold for peak words, using words of the form $0^a(100)^b$.

III. Other results and further questions

Now suppose w is chosen uniformly at random.

Theorem (Elizalde & T. 2019): $\lim_{n\to\infty} \frac{d_n(w)}{(n/2)\lfloor n/2 \rfloor!} = \infty$ almost surely.

III. Other results and further questions

Now suppose w is chosen uniformly at random.

Theorem (Elizalde & T. 2019): $\lim_{n\to\infty} \frac{d_n(w)}{(n/2)\lfloor n/2\rfloor!} = \infty$ almost surely.

Conjecture (Omar & T. 2023+): $\operatorname{gr} d_n(w) > 0$ almost surely.

- A stronger statement would be that $\operatorname{gr} d_n(w) > 0$ for every w with infinitely many 0's and infinitely many 1's, but that is false: we can construct w to have sparse enough 0's that $\operatorname{gr} d_n(w) = 0$.

III. Other results and further questions

Now suppose w is chosen uniformly at random.

Theorem (Elizalde & T. 2019): $\lim_{n\to\infty} \frac{d_n(w)}{(n/2)\lfloor n/2\rfloor!} = \infty$ almost surely.

Conjecture (Omar & T. 2023+): $\operatorname{gr} d_n(w) > 0$ almost surely.

- A stronger statement would be that $\operatorname{gr} d_n(w) > 0$ for every w with infinitely many 0's and infinitely many 1's, but that is false: we can construct w to have sparse enough 0's that $\operatorname{gr} d_n(w) = 0$.

Conjecture (me while I was preparing this talk): $\operatorname{gr} d_n(w) = 1/2$ almost surely.