Counting permutations by inversions

Permutation Patterns 2023
Dijon, France

Henning Ulfarsson, Reykjavik University

w/ Christian Bean, Keele University
Anders Claesson, University of Iceland
Atli Franklin, University of Iceland
Jay Pantone, Marquette University

Motivation: $\underbrace{\text{Counting } Av(1324)}$ is hard

?

$\lvert Av_0(1324)\rvert$	$\lvert Av_1(1324)\rvert$	$\lvert Av_2(1324)\rvert$	$\lvert Av_3(1324)\rvert$	
1	1	2	6	...

We want a formula, a generating function, fast algorithm to find these numbers.

But what if we change the question a bit

$$|Av_0(1324)| \quad |Av_1(1324)| \quad |Av_2(1324)| \quad |Av_3(1324)| \quad \cdots$$

Forget about length, use number of inversions instead!

We now get the sequence

$$\infty \quad \infty \quad \infty \quad \infty \quad \cdots$$

Much easier!

Why?

2 3 4 1 5 6 7 8 9 ⋯ 25

There is a lemma in Claesson - Jelinek - Steingrímsson 2012 that states: For a permutation $\pi \in S_n$

$$\mathrm{inv}(\pi) + \#\mathrm{comp} \geq n$$

The special case of sum indecomposable permutations is

There is a lemma in Claesson-Jelinek-Steingrímsson 2012 that states: For a permutation $\pi \in S_n$

$$\mathrm{inv}(\pi) + \#\mathrm{comp} \geq n$$

The special case of $\boxed{\text{sum indecomposable}}$ permutations is

$$\mathrm{inv}(\pi) + 1 \geq n$$

and implies that the set $\{\pi \mid \pi \text{ indecomp and } \mathrm{inv}(\pi) = k\}$ is finite.

Note also that if we learn something about the indecomposable permutations in Av(1324) then that would tell us something about all permutations in Av(1324).

So did we discover something about

$$I_k(1324) = \{\pi \in Av(1324) \mid \pi \text{ indecomp and } inv(\pi) = k\} \text{ ?}$$

So did we discover something about

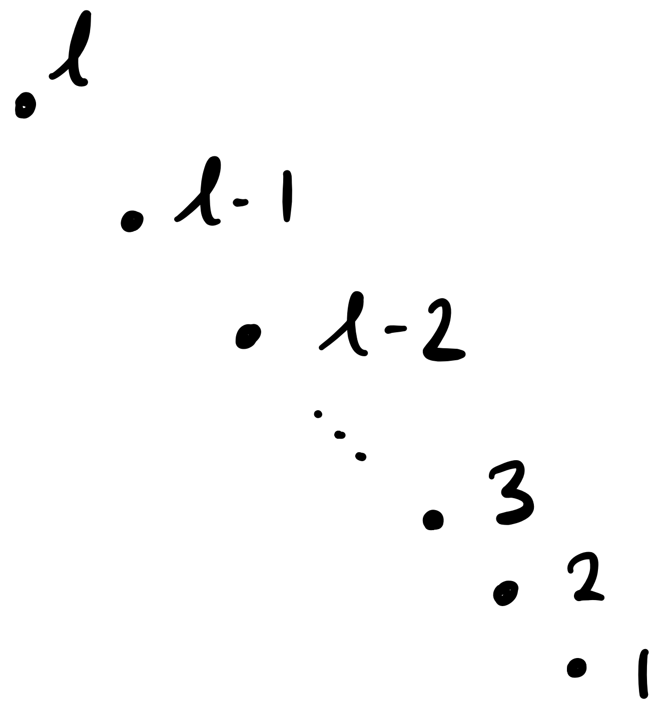
$$I_k(1324) = \{\pi \in Av(1324) \mid \pi \text{ indecomp and } inv(\pi) = k\} ?$$

No.

We did however find the counting sequences for easier cases, and hope that this might inspire others to solve $I_k(1324)$, or do similar work for another statistic than inv.

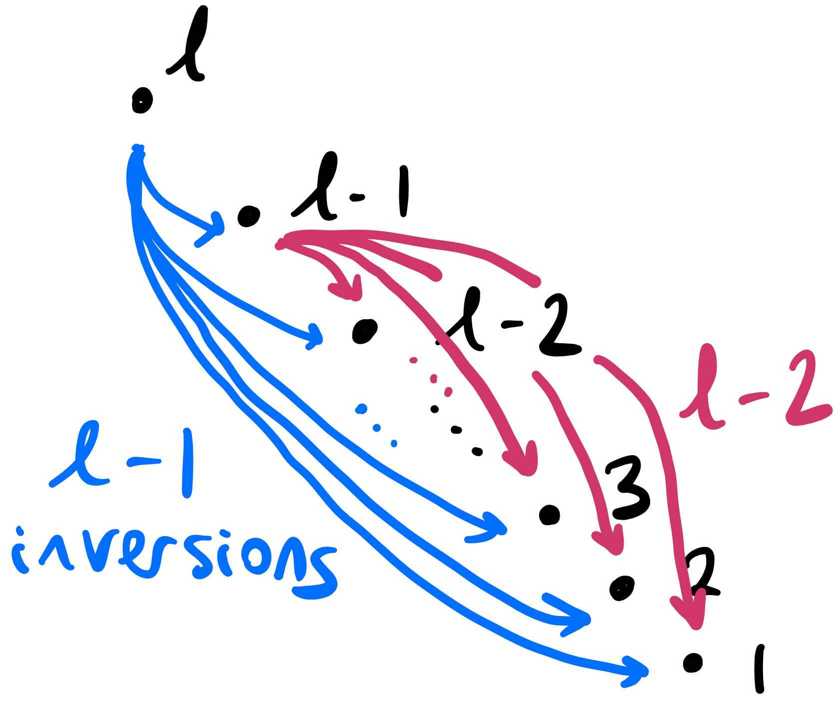
$$|I_k(12)| = \begin{cases} 1 & \text{if } k \text{ is a triangular number} \\ 0 & \text{otherwise} \end{cases}$$

First note that $I(12) = Av(12)$ and each $I_k(12)$ has at most one element of the form $\ell(\ell-1)\cdots 321$.



$$|I_k(12)| = 1 \quad \text{if } k \text{ is a triangular number, otherwise } 0$$

First note that $I(12) = Av(12)$ and each $I_k(12)$ has at most one element of the form $\ell(\ell-1)\cdots 321$. But such a permutation has exactly $\underbrace{(\ell-1)+(\ell-2)+\cdots+1}_{\frac{\ell\cdot(\ell-1)}{2}}$ inversions



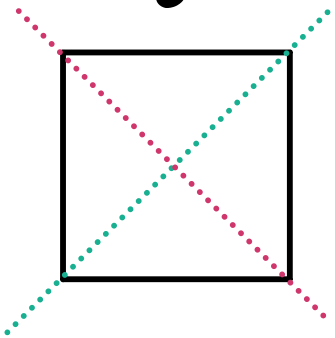
You have to be careful with symmetries here

$$|I_k(21)| \neq |I_k(12)|$$

note that ε is "decomposable"

because $I_k(21)$ is empty for all $k>0$ because $I(21) = \{1\}$.

The symmetries you can use are the reflections in the diagonals of the permutation diagram



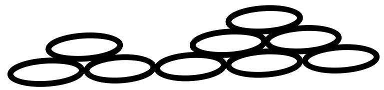
(inverse and reverse ∘ complement)

We now turn our attention to $I_k(p)$ with $p \in S_3$, and after taking symmetries into account we need to look at

$$I_k(123),\ I_k(132),\ I_k(231),\ I_k(321)$$

$I_k(123)$: Recurrence relation

$I_k(132)$: Number of partitions of k

$I_k(231)$: Number of fountains of k coins 

$I_k(321)$: Number of parallelogram polyominoes with k cells.

Note 1: No Wilf-equivalences like for $Av_n(123)$ and $Av_n(132)$.

Note 2: $I_k(123)$ and $I_k(321)$ are the hard cases.

$\pi = 546213$
433100

Proof of $|I_k(132)| =$ number of partitions of k.

Given $\pi \in I_k(132)$ we create the inversion table, which will have at least one zero at the end.

$\pi = 546213$

$433100 \rightsquigarrow 4+3+3+1 = 11$

Proof of $|I_k(132)|$ = number of partitions of k.

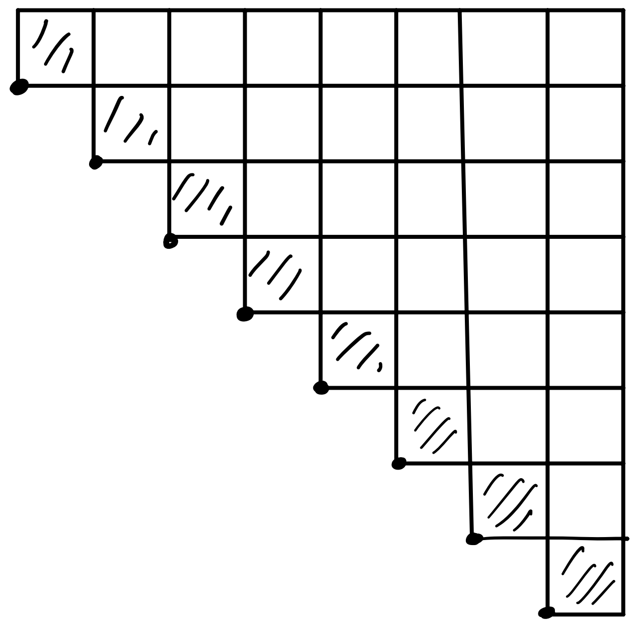
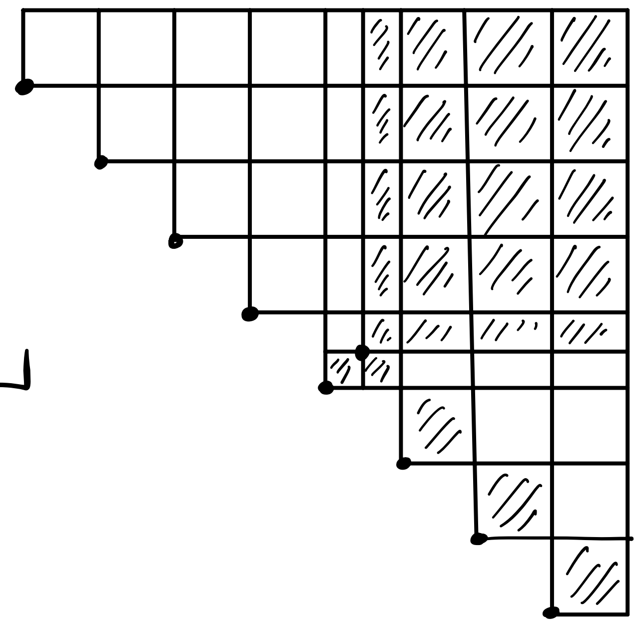
Given $\pi \in I_k(132)$ we create the inversion table, which will have at least one zero at the end.

Remove all the zeros. You now have a partition of k, because π avoids 132 if and only if its inversion table is weakly decreasing (CJS 2012).

This is a bijection.

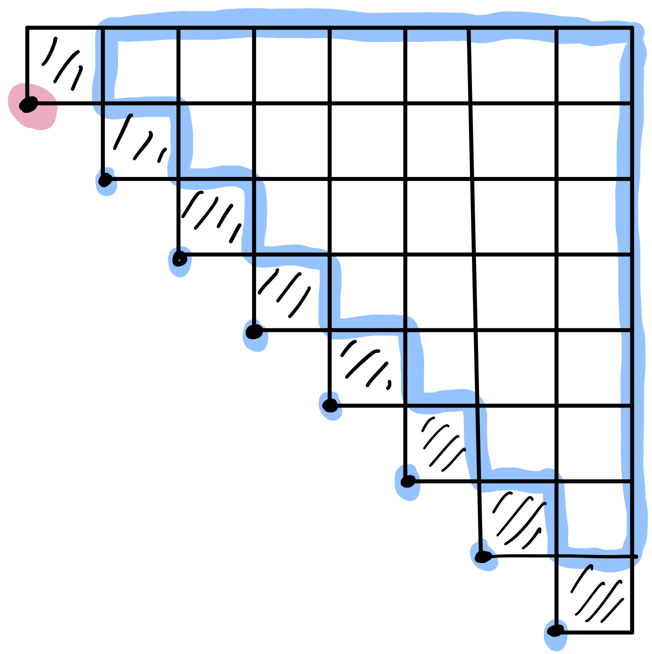
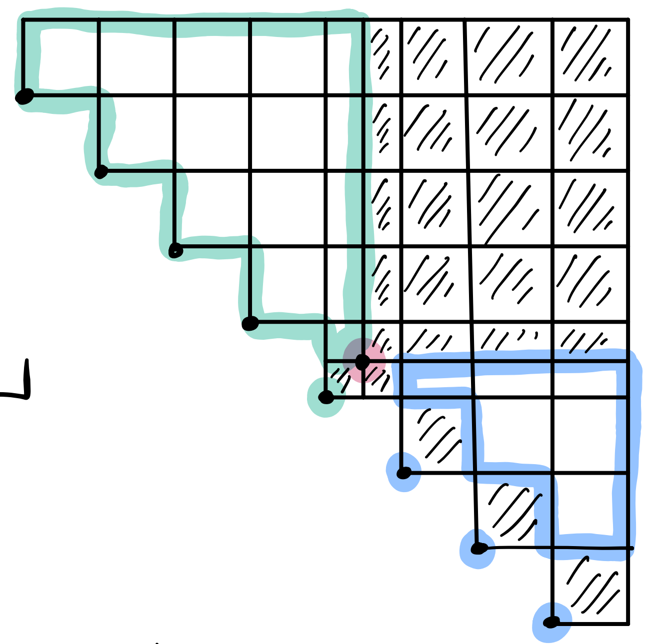
$I_k(231)$ and $I_k(321)$ are also done with bijections.

Let's look at $I_k(123)$, but first recall a nice way to enumerate $Av_n(123)$

$Av(123) = \varepsilon \sqcup$  $\sqcup$ 

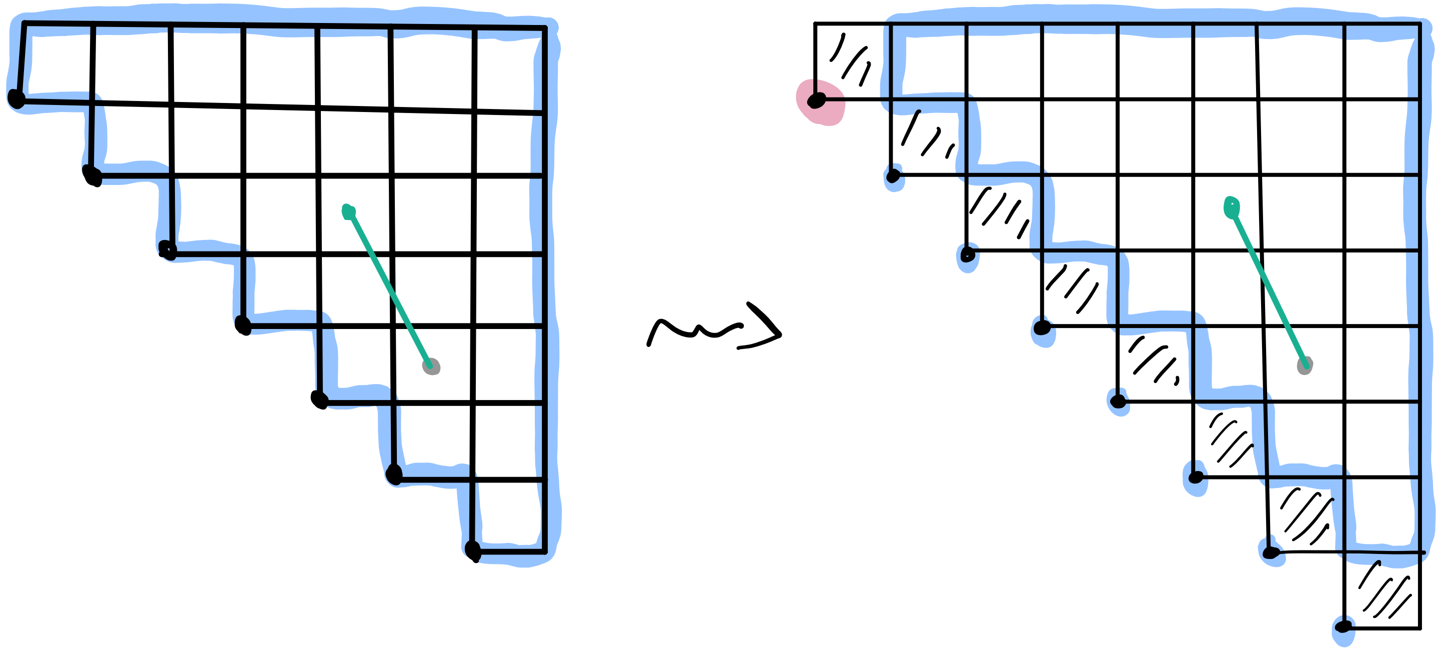
$I_k(231)$ and $I_k(321)$ are also done with bijections.

Let's look at $I_k(123)$, but first recall a nice way to enumerate $Av_n(123)$

$Av(123) = \varepsilon \sqcup$  $\sqcup$ 

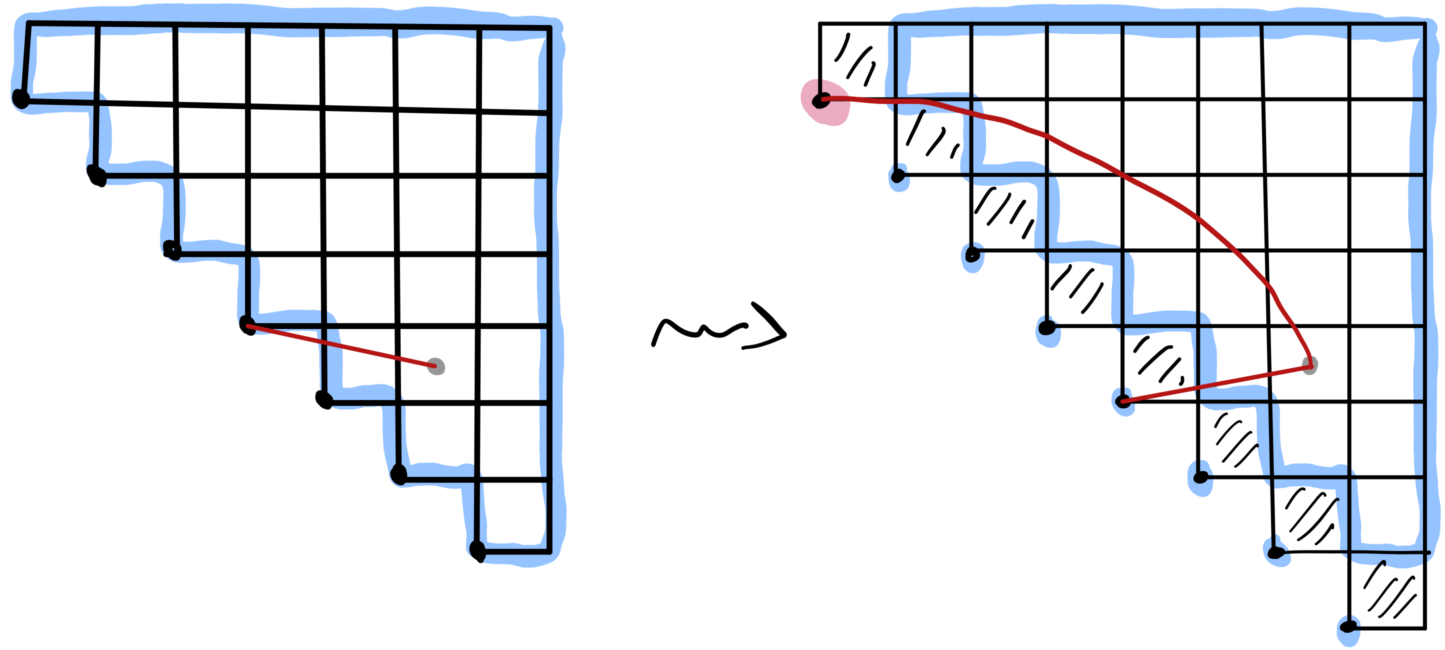
$$F(x) = 1 + x \cdot F(x) + (F(x)-1) \cdot x \cdot F(x)$$

... solve to get the Catalan generating function.



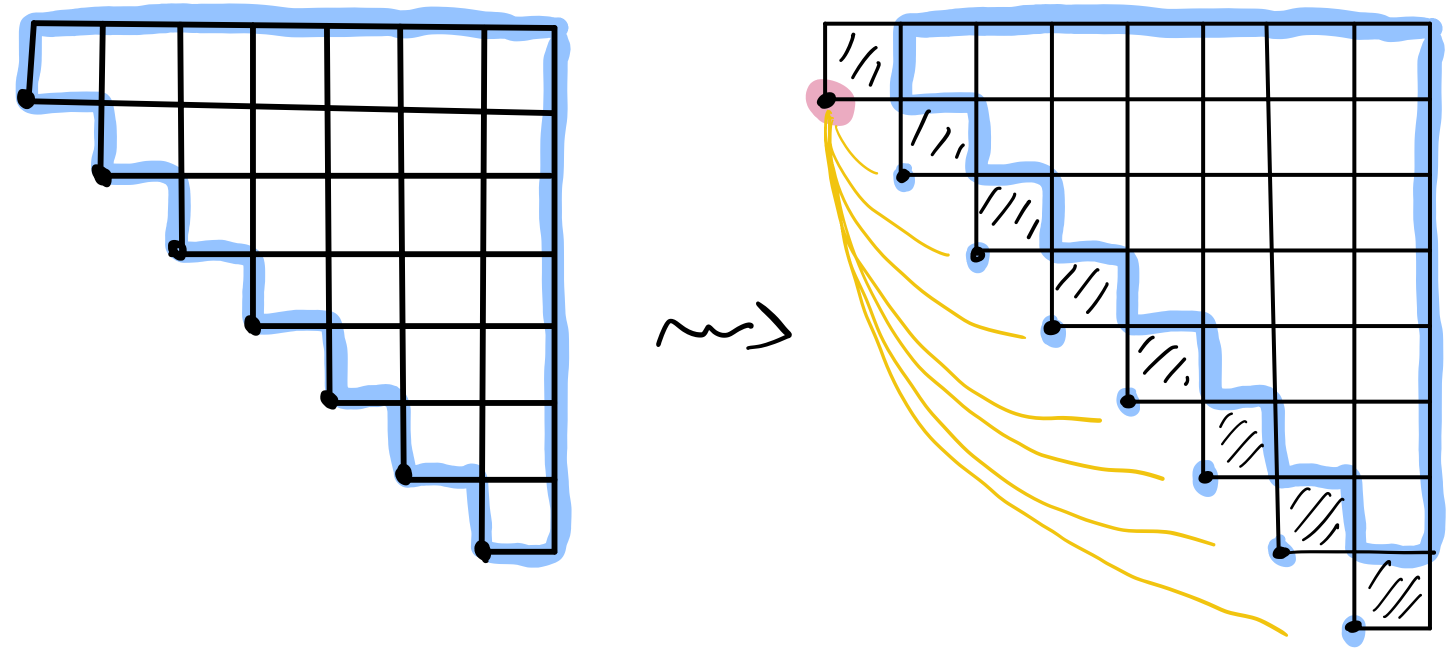
The blue permutation has
size n
7 lrmins
k inversions

The new one
size n+1,
8 lrmins



The blue permutation has
size n
7 lrmins
k inversions

The new one
size n+1,
8 lrmins



The blue permutation has
size n
7 lrmins
k inversions

The new one
size n+1,
8 lrmins
k+7 inversions

This allows us to write down a recurrence relation with 3 parameters.

What is next?

1) We have done $I_k(B)$ for $B \subseteq S_3$.

2) Anders Claesson and Atli Franklin have done several cases beyond that

3) We have done some preliminary work on other statistics and can share data.

   have to be careful not to get infinities. We added #comp = 1 for the inv statistic.

   For other statistics you might want to add different restrictions.

What is next ?

4) We have no clue about $I_k(1324)$.

5) What does $I_k(1324)$ really tell us about $Av_n(1324)$?

We can actually derive bounds :

$$I_k(1324) \subseteq \bigcup_{n=1}^{k+1} Av_n^{\text{ind.}}(1324)$$

$$Av_n^{\text{ind.}}(1324) \subseteq \bigcup_{k=0}^{\binom{n}{2}} I_k(1324)$$

Can use this to prove that $Av_n(132)$ is bounded by 6.13^n.

Fin!