A positional statistic for 1324-avoiding permutations

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VALPO

Outline

$\mathrm{Av}_{n,1}^{1\prec n}(1324)$

$\mathrm{Av}_{n}^{1\prec n}(1324)$

$\mathrm{Av}_{n}^{2\prec n}(1324)$

$\mathrm{Av}_{n}^{m\prec n}(1324)$?

Let

$$\mathrm{Av}_{n,k}^{1\prec n}(1324) = \left\{\sigma \in \mathrm{Av}_{n}^{1\prec n}(1324) : \ \sigma^{-1}(n) - \sigma^{-1}(1) = k\right\}.$$

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We start by looking at $\mathrm{Av}_{n,1}^{1\prec n}(1324)$.

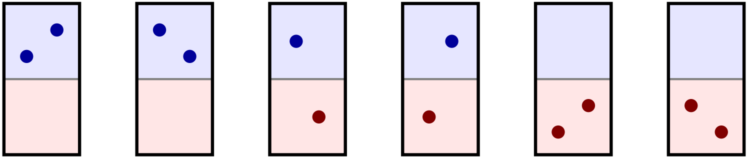
As studied by Bevan, Brignall, Elvey Price, and Pantone,[1] a 1324-avoiding vertical *domino* is a two-cell gridded permutation in $\mathrm{Grid}^{\#}\binom{\mathrm{Av}(213)}{\mathrm{Av}(132)}$ whose underlying permutation avoids 1324.

They are counted by [3, A000139], $1, 2, 6, 22, 91, 408, 1938, \ldots$

[1] A structural characterisation of Av(1324) and new bounds on its growth rate, *European J. Combin.* **88** (2020), 103115, 29 pp.

Example

For example, 1324-avoiding dominoes with two points:



Proposition

For $n \geq 2$, there is a one-to-one correspondence between $\mathrm{Av}_{n,1}^{1\prec n}(1324)$ and the set of 1324-avoiding dominoes with $n-2$ points.

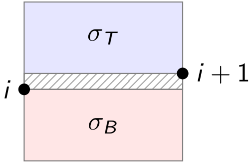
Proof: Every $\sigma \in \mathrm{Av}_{n,1}^{1\prec n}(1324)$ is of the form

$$\sigma = \sigma_L \, 1 n \, \sigma_R,$$

where σ_L and σ_R are words (possibly empty) such that $|\sigma_L + \sigma_R| = n-2$, and their reduced permutations $\mathrm{red}(\sigma_L)$ and $\mathrm{red}(\sigma_R)$ avoid 132 and 213, respectively.

proof continued

σ^{-1} also avoids 1324, and if $i = \sigma^{-1}(1)$, then σ^{-1} is of the form



where $\mathrm{red}(\sigma_T) = \mathrm{red}(\sigma_R)^{-1}$ avoids 213 and $\mathrm{red}(\sigma_B) = \mathrm{red}(\sigma_L)^{-1}$ avoids 132. Merging the lines through i and $i+1$ as a separator, we get a 1324-avoiding domino.

As a consequence,

$$|\mathrm{Av}^{1\prec n}_{n,1}(1324)| = \frac{2\,(3n-3)!}{(2n-1)!\,n!} \quad \text{for } n \geq 2.$$

We will use the elements of $\mathrm{Av}^{1\prec n}_{n,1}(1324)$ as primitives to construct the elements of $\mathrm{Av}^{1\prec n}_{n,k}(1324)$.

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For convenience, let

$$A139(x) = x + 2x^2 + 6x^3 + 22x^4 + 91x^5 + \cdots$$

Definition

A permutation $\sigma \in \mathrm{Av}_n^{1\prec n}(1324)$ is *primitive* if it is of the form

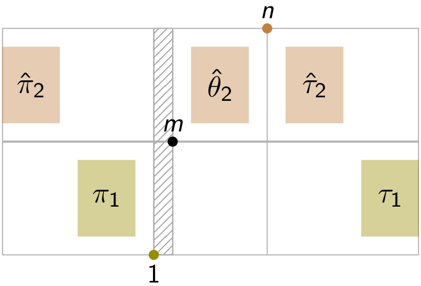
$$\sigma = \pi\, 1 n\, \tau$$

with $\mathrm{red}(\pi) \in \mathrm{Av}_k(132)$, $\mathrm{red}(\tau) \in \mathrm{Av}_\ell(213)$, and $k + \ell = n - 2$.
For a primitive $\sigma_1 = \pi_1\, 1m\, \tau_1$ and a permutation $\sigma_2 \in \mathrm{Av}_\ell(1324)$ of the form $\sigma_2 = \pi_2\, 1\, \theta_2\, \ell\, \tau_2$ with $|\theta_2| \geq 0$, we define the product

$$\sigma_1 \odot \sigma_2 = \widehat{\pi}_2 \pi_1\, 1m\, \widehat{\theta}_2\, n\, \widehat{\tau}_2 \tau_1 \in \mathrm{Av}_n(1324),$$

where $n = \ell + m - 1$, and $\widehat{\pi}_2$, $\widehat{\theta}_2$, and $\widehat{\tau}_2$ are obtained from π_2, θ_2, and τ_2, by increasing all of their entries by $m - 1$.

Given $\sigma_1 = \pi_1\, 1 m\, \tau_1$ and $\sigma_2 = \pi_2\, 1\, \theta_2\, \ell\, \tau_2$ with $|\theta_2| \geq 0$,



For example, $2143 \odot 41253$

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Also,

$$213 \odot 3142 = 5\,21\,364 \quad \text{and} \quad 3142 \odot 213 = 5\,31\,46\,2,$$

thus $\odot$ is not commutative.

Proposition

If $\sigma_1 \in \mathrm{Av}^{1\prec m}_{m}(1324)$ *is primitive and* $\sigma_2 \in \mathrm{Av}^{1\prec \ell}_{\ell,k}(1324)$*, then*

$$\sigma_1 \odot \sigma_2 \in \mathrm{Av}^{1\prec n}_{n,k+1}(1324) \text{ with } n = \ell + m - 1.$$

Proposition

Every non-primitive $\sigma \in \mathrm{Av}^{1\prec n}_{n}(1324)$ *admits a unique decomposition*

$$\sigma = \sigma_1 \odot \sigma_2,$$

where σ_1 *is a primitive element of* $\mathrm{Av}^{1\prec m}_{m}(1324)$ *and* $\sigma_2 \in \mathrm{Av}^{1\prec \ell}_{\ell}(1324)$ *with* $\ell = n - m + 1$*.*

Corollary

Every permutation in $\mathrm{Av}_{n,k}^{1\prec n}(1324)$ *can be uniquely decomposed as a product of* k *primitive permutations.*

For example, the five non-primitive permutations in $\mathrm{Av}_{4}^{1\prec 4}(1324)$ are:

$$1234 = 12 \odot (12 \odot 12),$$
$$1243 = 12 \odot 132, \quad 1342 = 132 \odot 12,$$
$$2134 = 213 \odot 12, \quad 3124 = 12 \odot 213.$$

Theorem

Let $a_{n,k} = |\text{Av}_{n,k}^{1\prec n}(1324)|$*. For* $n \geq 3$ *and* $2 \leq k \leq n-1$*, we have*

$$a_{n,k} = \sum_{m=2}^{n-k+1} a_{m,1} \cdot a_{n-m+1,k-1}.$$

Thus, for $g_1(x,t) = \sum_{n=2}^{\infty} \sum_{k=1}^{n-1} a_{n,k} t^k x^n$*, we have*

$$g_1(x,t) = \frac{t\ x\ A139(x)}{1 - t\ A139(x)},$$

where $A139(x) = \sum_{n=1}^{\infty} a_{n+1,1} x^n = x \cdot hypergeom([1, \frac{4}{3}, \frac{5}{3}], [\frac{5}{2}, 3], \frac{27}{4} x)$*.*

In this section, we let $a_{n,k} = |\mathrm{Av}_{n,k}^{1\prec n}(1324)|$.

Proposition

For $n \geq 3$ and $2 \leq k$[2] $\leq n-1$, we have

$$|\mathrm{Av}_{n,k}^{2\prec n}(1324)| \equiv |\mathrm{Av}_{n,k}^{2\prec n\prec 1}(1324)| = \tfrac{n-k}{2} a_{n-1,k}.$$

Moreover, the generating function

$g_2(x,t) = \sum_{n=3}^{\infty}\sum_{k=1}^{n-1} |\mathrm{Av}_{n,k}^{2\prec n}(1324)| t^k x^n$ *satisfies*

$$g_2(x,t) = \frac{1}{2}\left(x^2 \frac{\partial g_1}{\partial x}(x,t) - g_1(x,t)^2\right),$$

where $g_1(x,t)$ is the generating function for $a_{n,k}$.

[2] where k is now tracking the distance between the 2 and the n.

$$2|\mathrm{Av}_{n,k}^{2\prec n}(1324)| = (n-k)a_{n-1,k}.$$

Example $(n=9, k=2)$: Let $\sigma = 76513842$ in $Av_{8,2}^{1\prec 8}(1324)$.

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Example ($n = 9, k = 2$): Let $\sigma = 76513842$ in $Av_{8,2}^{1\prec 8}(1324)$.

We can look at the pair, $(\sigma, \sigma^{rc}) = (76513842, 75168432)$.

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$$876259153, 876259513, 876259531$$

and

$$862791543, 862795143, 862795413, 862795431$$

This gives us 7 elements in $Av_{9,2}^{2\prec 9\prec 1}(1324)$.

So,

$$g_1(x,t) = x\left(\frac{1}{1 - t\ P(x)} - 1\right)$$

where $P(x) = A139(x)$ is the generating function for the primitives (shifted).

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$$g_2(x,t) = x^2\left(\frac{1-t\ P(x)(1-2P(x))}{[1-t\ P(x)(1-P(x))]^2} - 1\right)$$

where $P(x) = A1764(x) = x + 3x^2 + 12x^3 + 55x^4 + 273x^5 + \ldots$ is the generating function for the primitives (shifted).

Conjecture

Let $g_m(x,t) = \sum_{n=m+1}^{\infty} \sum_{k=1}^{n-m+1} |\text{Av}_{n,k}^{m\prec n}(1324)| t^k x^n$ *and,*

for $1 \leq k \leq m-1$, $T_{m,k}(x) = \sum_{n=1}^{\infty} |\text{Av}_{n,k}^{m\prec n}| x^n$. *Then*

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for $1 \le k \le m-1$, $T_{m,k}(x) = \sum_{n=1}^{\infty} |\mathrm{Av}_{n,k}^{m\prec n}| x^n$. *Then*

$$0 = g_m(x,t) + (-1)^m a_{m-1} g_1(x,t)^m - \sum_{j=1}^{m-1} \sum_{i=0}^{m-j-1} (-1)^i \binom{i+j}{i} t^j \left(1 + \frac{g_1(x,t)}{x}\right)^{j+1} \left(\frac{g_1(x,t)}{x}\right)^i T_{m,j}(x)$$

where $a_n = |S_n(1324)|$.

Merci!

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D. Bevan, R. Brignall, A. Elvey Price, J. Pantone, A structural characterisation of Av(1324) and new bounds on its growth rate, *European J. Combin.* **88** (2020), 103115, 29 pp.

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