Generating trees and pattern-avoiding inversion sequences

Gökhan Yıldırım

Bilkent University

Permutation Patterns 2023

July 3-7, 2023

based on joint work with Toufik Mansour

Overview of the talk

- Generating trees and kernel method
- Pattern-avoiding inversion sequences
- Some new enumerative results for inversion sequences avoiding 021 and a five-letter pattern.

Generating trees for combinatorial classes

Any set $\mathcal{C}$ of discrete objects with a notion of a size such that for each n there are finitely many objects of size n is called a combinatorial class.

Generating trees for combinatorial classes

Any set $\mathcal{C}$ of discrete objects with a notion of a size such that for each n there are finitely many objects of size n is called a combinatorial class.

A generating tree for $\mathcal{C}$ is a rooted, labeled, ordered tree whose vertices are the objects of $\mathcal{C}$ with the following properties:

(i) each object of $\mathcal{C}$ appears exactly once in the tree;

(ii) objects of size n appears at the level n in the tree;

(iii) each object's children are obtained by a set of succession rules which determines the number of children and their labels.

Examples

- Fibonacci Tree:

$$\begin{aligned} \text{Root: } & (1), \\ \text{Rules: } & (1) \rightsquigarrow (2) \\ & (2) \rightsquigarrow (1)(2) \end{aligned}$$

- Catalan Tree:

$$\begin{aligned} \text{Root: } & (1), \\ \text{Rules: } & (m) \rightsquigarrow (2)(3) \cdots (m+1) \end{aligned}$$

From generating trees to generating functions

Fibonacci Tree:

$$\text{Root: } (1),$$
$$\text{Rules: } (1) \rightsquigarrow (2)$$
$$(2) \rightsquigarrow (1)(2)$$

From generating trees to generating functions

Fibonacci Tree:

$$\begin{aligned} \text{Root: } & (1), \\ \text{Rules: } & (1) \rightsquigarrow (2) \\ & (2) \rightsquigarrow (1)(2) \end{aligned}$$

Let $a_n =$ the number of vertices at the level n, $n \geq 1$. The root is at the level 1.

From generating trees to generating functions

Fibonacci Tree:

$$\begin{aligned} &\text{Root: } (1), \\ &\text{Rules: } (1) \rightsquigarrow (2) \\ &\qquad\quad\; (2) \rightsquigarrow (1)(2) \end{aligned}$$

Let $a_n =$ the number of vertices at the level n, $n \geq 1$. The root is at the level 1.

We want to compute the generating function $A(x) = \sum_{n=1}^{\infty} a_n x^n$.

From generating trees to generating functions

Fibonacci Tree:

$$\text{Root: } (1),$$
$$\text{Rules: } (1) \rightsquigarrow (2)$$
$$(2) \rightsquigarrow (1)(2)$$

Let $a_n =$ the number of vertices at the level n, $n \geq 1$. The root is at the level 1.

We want to compute the generating function $A(x) = \sum_{n=1}^{\infty} a_n x^n$.

Let $A_i(x)$ denote the generating function corresponding to the (sub)-tree with root label (i). Then, we get

$$\begin{aligned} A_1(x) &= x + xA_2(x) \\ A_2(x) &= x + xA_1(x) + xA_2(x) \end{aligned}$$

From generating trees to generating functions

Fibonacci Tree:

$$\begin{aligned} &\text{Root: } (1), \\ &\text{Rules: } (1) \rightsquigarrow (2) \\ &\qquad\quad\ (2) \rightsquigarrow (1)(2) \end{aligned}$$

Let $a_n =$ the number of vertices at the level n, $n \geq 1$. The root is at the level 1.

We want to compute the generating function $A(x) = \sum_{n=1}^{\infty} a_n x^n$.

Let $A_i(x)$ denote the generating function corresponding to the (sub)-tree with root label (i). Then, we get

$$\begin{aligned} A_1(x) &= x + xA_2(x) \\ A_2(x) &= x + xA_1(x) + xA_2(x) \end{aligned}$$

Solving these equations yields $A_1(x) = \frac{x}{1-x-x^2}$.

The number of vertices at each level are $1, 1, 2, 3, 5, 8, \ldots$.

Catalan Tree

Root: (1),

Rules: $(m) \rightsquigarrow (2)(3)\cdots(m+1)$

Catalan Tree

Root: (1),

Rules: $(m) \rightsquigarrow (2)(3)\cdots(m+1)$

$$A_m(x) = x + x[A_2(x) + A_3(x) + \cdots + A_{m+1}(x)] \quad \text{for any } m \geq 1.$$

Catalan Tree

$$\text{Root: } (1),$$
$$\text{Rules: } (m) \rightsquigarrow (2)(3)\cdots(m+1)$$

$$A_m(x) = x + x[A_2(x) + A_3(x) + \cdots + A_{m+1}(x)] \quad \text{for any } m \geq 1.$$

Hence

$$A_{m+1}(x) - A_m(x) = xA_{m+2}(x).$$

Let $A(x,u) = \sum_{m=1}^{\infty} A_m(x) u^{m-1}$.

We want to determine

$$A(x,0) = A_1(x).$$

Kernel Method

Multiplying $A_{m+1}(x) - A_m(x) = xA_{m+2}(x)$ by u^{m-1} and summing over $m \geq 1$, we get

$$\frac{1}{u}(A(x,u) - A_1(x)) - A(x,u) = \frac{x}{u^2}(A(x,u) - A_1(x) - uA_2(x))$$

$$(\frac{1}{u} - 1 - \frac{x}{u^2})A(x,u) = -\frac{x}{u^2}A_1(x) + \frac{1}{u}$$

Kernel Method

Multiplying $A_{m+1}(x) - A_m(x) = xA_{m+2}(x)$ by u^{m-1} and summing over $m \geq 1$, we get

$$\frac{1}{u}(A(x,u) - A_1(x)) - A(x,u) = \frac{x}{u^2}(A(x,u) - A_1(x) - uA_2(x))$$

$$(\frac{1}{u} - 1 - \frac{x}{u^2})A(x,u) = -\frac{x}{u^2}A_1(x) + \frac{1}{u}$$

If we choose $u = \frac{1-\sqrt{1-4x}}{2}$, then we get $A_1(x) = \frac{1-\sqrt{1-4x}}{2x}$, the generating function of the Catalan numbers.

The number of vertices at each level are $1, 1, 2, 5, 14, 42, \ldots$

Inversion Sequences

An inversion sequence of length n is an integer sequence $e = e_1 \cdots e_n$ such that $0 \leq e_i < i$ for each $0 \leq i \leq n$.

Inversion Sequences

An inversion sequence of length n is an integer sequence $e = e_1 \cdots e_n$ such that $0 \leq e_i < i$ for each $0 \leq i \leq n$.

We use I_n to denote the set of inversion sequences of length n.

$$\begin{aligned} I_2 &= \{00, 01\} \\ I_3 &= \{000, 001, 010, 011, 002, 012\} \end{aligned}$$

There is a bijection between I_n and S_n, the set of permutations of length n.

Patterns: words over the alphabet $[k] := \{0, 1, \cdots, k-1\}$.

A pattern τ is a word of length k over the alphabet $[k]$.

There are basically thirteen patterns of length three up to order isomorphism.

$\mathcal{P}_3 = \{000, 001, 010, 100, 011, 101, 110, 021, 012, 102, 120, 201, 210\}$

Patterns: words over the alphabet $[k] := \{0, 1, \cdots, k-1\}$.

A pattern τ is a word of length k over the alphabet $[k]$.

There are basically thirteen patterns of length three up to order isomorphism.

$\mathcal{P}_3 = \{000, 001, 010, 100, 011, 101, 110, 021, 012, 102, 120, 201, 210\}$

An inversion sequence $e \in I_n$ contains the pattern τ if there is a subsequence of length k in e that is order isomorphic to τ; otherwise, e avoids the pattern τ.

$I_n(\tau)$ denotes the set of all τ-avoiding inversion sequences of length n.

For a given set of patterns B, we set $I_n(B) = \cap_{\tau \in B} I_n(\tau)$.

Examples for $I_n(\tau)$

$e = 01102321 \in I_8$ avoids the pattern 0000 because there is no subsequence $e_i e_j e_k e_l$ of length four in e with $i < j < k < l$ and $e_i = e_j = e_k = e_l$.

Examples for $I_n(\tau)$

$e = 01102321 \in I_8$ avoids the pattern 0000 because there is no subsequence $e_i e_j e_k e_l$ of length four in e with $i < j < k < l$ and $e_i = e_j = e_k = e_l$.

On the other hand, $e = 01102321$ contains the patterns

- 010 because it has subsequences $-1---3-1$ or $----232-$ order isomorphic to 010,
- 000 because it has subsequence $-11----1$ order isomorphic to 000.

Pattern-avoiding inversion sequences

They provide a unifying interpretation that relates a vast array of combinatorial structures.

- Fibonacci numbers

- Catalan numbers

- Schröder numbers

- Euler up/down numbers

- Bell numbers

- ....

appear as enumerating sequences for pattern-restricted inversion sequences.

Avoiding a single pattern of length three

Mansour-Shattuck(2015) and Corteel et.al. (2015)

There are basically thirteen patterns of length three up to order isomorphism

$\mathcal{P}_3 = \{000, 001, 010, 100, 011, 101, 110, 021, 012, 102, 120, 201, 210\}$

- $|I_n(012)| = F_{2n-1}$, odd Fibonacci numbers.
  $F_0 = 0, F_1 = 1, F_n = F_{n-1} + F_{n-2}$ for $n \geq 2$.

- $|I_n(021)| = r_{n-1}$, r_n is the n^{th} large Schröder numbers.
  r_n is the number of paths in the first quadrant from $(0,0)$ to $(2n,0)$ with steps $(1,1),(1,-1)$ and $(2,0)$.

- $|I_n(000)| = E_{n+1}$ is the Euler up/down numbers.
  E_n is the number of permutations of length n such that $\sigma_1 < \sigma_2 > \sigma_3 < \cdots$.
  $\tan x + \sec x = \sum_{n=0}^{\infty} E_n \frac{x^n}{n!}$

Avoiding a single pattern of length three

- $|I_n(001)| = 2^{n-1}$
- $|I_n(011)| = B_n$, the n^{th} Bell number.
  B_n is the number of ways to partition an $n-$element set into non-empty subsets called blocks.
- $|I_n(101)| = |I_n(110)| =$ powered Catalan numbers.
- $|I_n(201)| = |I_n(210)|$ corresponds to A263777 in OEIS.
- $|I_n(102)|$, $|I_n(010)|, |I_n(100)|, |I_n(120)|$ corresponds to A200753, A263779, A263780, A263778, respectively.

The case 010 is solved by B. Testart(2022).

Avoiding pair of patterns of length three

Yan-Lin (2020)

Consider inversion sequences avoiding two element subsets of

$\mathcal{P}_3 = \{000, 001, 010, 100, 011, 101, 110, 021, 012, 102, 120, 201, 210\}$

There are 78 pairs but Yan and Lin (2021) showed that there 48 Wilf classes and enumerated 17 of them.

- $|I_n(001, 210)| = \binom{n}{3} + n$

- For $p \in \{(012, 201), (012, 210)\}$, $|I_n(p)| = 2^{n+1} - \binom{n+1}{3} - 2n - 1$

- For $p \in \{(012, 021), (110, 012)\}$, $|I_n(p)| = 2^n - n$

Enumeration of $I_n(B)$

algorithm+generating tree+kernel method $\rightarrow$ enumeration

Generating function $R(x) = \sum_{n\geq 1} |I_n(B)|x^n$.

We have three main steps:

- determine the succession rules for the corresponding generating tree.
- determine the equations satisfied by the generating functions from the tree rules.
- solve the equations and determine the enumerating sequence for $I_n(B)$.

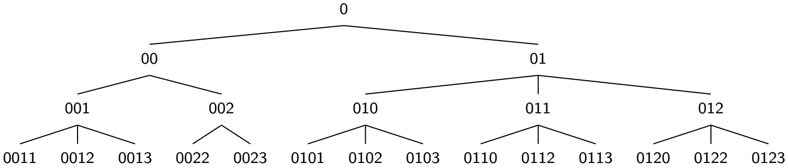
Tree representation of $\mathcal{I}(B) = \cup_{n=1}^{\infty} I_n(B)$

We will construct a pattern-avoidance tree $\mathcal{T}(B)$ corresponding to the class $\mathcal{I}(B)$ as follows:

- the root is 0 and stays at level 1.
- the n^{th} level of the tree consist exactly the elements of $I_n(B)$.
- the children of $e_1 e_2 \cdots e_{n-1} \in I_{n-1}(B)$ are obtained from the set $\{e_1 e_2 \cdots e_{n-1} e_n | e_n = 0, 1, \ldots, n-1\}$ by obeying the pattern avoidance restrictions of B.
- we arrange the nodes from the left to the right so that $e_1 e_2 \cdots e_{n-1} i$ appears on the left of $e_1 e_2 \cdots e_{n-1} j$ if $i < j$.

Tree representation of $I_n(000, 021)$

First four levels of $\mathcal{T}(000, 021)$



An equivalence relation on $\mathcal{T}(B)$

Let $\mathcal{T}(B; e)$ denote the subtree in $\mathcal{T}(B)$ which has the root e.

We define an equivalence relation on $\mathcal{T}(B)$ as follows: let $e, e' \in \mathcal{T}(B)$

$$e \sim e' \text{ if and only if } \mathcal{T}(B; e) \cong \mathcal{T}(B; e')$$

An equivalence relation on $\mathcal{T}(B)$

Let $\mathcal{T}(B; e)$ denote the subtree in $\mathcal{T}(B)$ which has the root e.

We define an equivalence relation on $\mathcal{T}(B)$ as follows: let $e, e' \in \mathcal{T}(B)$

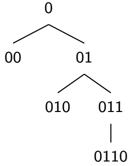
$$e \sim e' \text{ if and only if } \mathcal{T}(B; e) \cong \mathcal{T}(B; e')$$

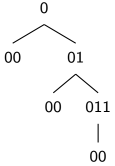
Lemma

Let t be the length of the longest pattern in B.

$$\mathcal{T}(B; e) \cong \mathcal{T}(B; e') \textit{ if and only if } \mathcal{T}^{2t}(B; e) \cong \mathcal{T}^{2t}(B; e')$$

a simple example: enumeration of $I_n(000, 001, 012)$





The generating tree succession rules are given by

$$\begin{aligned} \text{Root: } & 0, \\ \text{Rules: } & 0 \rightsquigarrow 00, 01 \\ & 01 \rightsquigarrow 00, 011 \\ & 011 \rightsquigarrow 00. \end{aligned}$$

a simple example: enumeration of $I_n(000, 001, 012)$

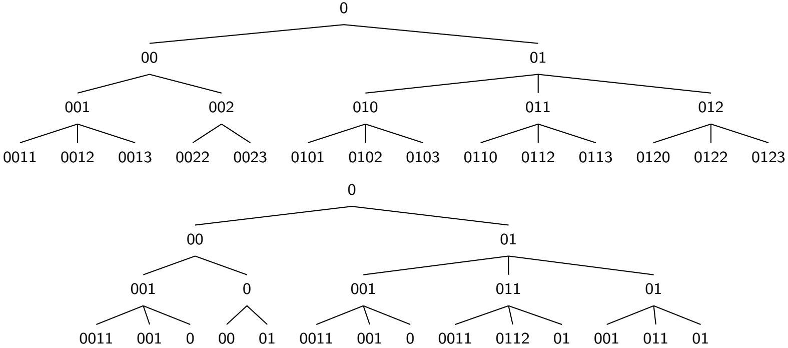
Let $R(x) = \sum_{n\geq 1} |I_n(000, 001, 012)|x^n$.

$$\begin{aligned} R(x) &= x + xA_{00}(x) + xA_{01}(x), \\ A_{01}(x) &= x + xA_{00}(x) + xA_{011}(x), \\ A_{011}(x) &= x + xA_{00}(x), \\ A_{00}(x) &= x. \end{aligned}$$

Then $R(x) = x^4 + 2x^3 + 2x^2 + x$.

the case $B = \{000, 021\}$

Based on the second tree, we try to figure out the succession rules of the generating tree for the class $\mathcal{I}(B)$.



Succession rules for the generating tree of $I_n(000, 021)$

We define $r_0 = 0$, $b_0 = 0$, and for $m \geq 1$,

$$\begin{aligned} a_m &= 0011 \cdots mm \\ b_m &= 0011 \cdots (m-1)(m-1)m \\ c_m &= 01122 \cdots mm \\ d_m &= 01122 \cdots (m-1)(m-1)m. \end{aligned}$$

The generating tree $\mathcal{T}(000, 021)$ is given by

$$\begin{aligned} \text{Root: } & r_0, \\ \text{Rules: } & r_0 \rightsquigarrow a_0 d_1, \\ & a_m \rightsquigarrow b_{m+1} b_m \cdots b_0, \\ & b_m \rightsquigarrow a_m b_m b_{m-1} \cdots b_0, \\ & c_m \rightsquigarrow a_m d_{m+1} d_m \cdots d_1, \\ & d_m \rightsquigarrow b_m c_m d_m d_{m-1} \cdots d_1. \end{aligned}$$

Equations for the generating functions

Let $R(x) = \sum_{n\geq 1} |I_n(000, 021)|x^n$.

$$\begin{aligned}
R(x) &= x + xA_0(x) + xD_1(x),\\
A_m(x) &= x + x\sum_{j=0}^{m+1} B_j(x),\\
B_m(x) &= x + xA_m(x) + x\sum_{j=0}^{m} B_j(x),\\
C_m(x) &= x + xA_m(x) + x\sum_{j=1}^{m+1} D_j(x),\\
D_m(x) &= x + xB_m(x) + xC_m(x) + x\sum_{j=1}^{m} D_j(x).
\end{aligned}$$

Bivariate generating functions

We define

$$A(x,u) = \sum_{m\geq 0} A_m(x)u^m, B(x,u) = \sum_{m\geq 0} B_m(x)u^m$$

$$C(x,u) = \sum_{m\geq 1} C_m(x)u^{m-1}, D(x,u) = \sum_{m\geq 1} D_m(x)u^{m-1}.$$

Hence

$$\begin{aligned}
R(x) &= x + xA(x,0) + xD(x,0),\\
A(x,u) &= \frac{x}{1-u} + \frac{x}{u}(B(x,u) - B(x,0)) + \frac{x}{1-u}B(x,u),\\
B(x,u) &= \frac{x}{1-u} + xA(x,u) + \frac{x}{1-u}B(x,u),\\
C(x,u) &= \frac{x}{1-u} + \frac{x}{u}(A(x,u) + D(x,u) - A(x,0) - D(x,0)) + \frac{x}{1-u}D(x,u)\\
D(x,u) &= \frac{x}{1-u} + \frac{x}{u}(B(x,u) - B(x,0)) + xC(x,u) + \frac{x}{1-u}D(x,u).
\end{aligned}$$

Kernel Method

From the second and third equations, we get

$$\frac{(1-x)u - x^2 - u^2}{u(1-u-x)} A(x,u) = -\frac{x^2}{u(1-x)} A(x,0) + \frac{x}{(1-u-x)(1-x)}.$$

Kernel Method

From the second and third equations, we get

$$\frac{(1-x)u - x^2 - u^2}{u(1-u-x)} A(x,u) = -\frac{x^2}{u(1-x)} A(x,0) + \frac{x}{(1-u-x)(1-x)}.$$

By choosing $u = x^2 M(x)$, where $M(x) = \frac{1-x-\sqrt{1-2x-3x^2}}{2x^2}$ is the generating function for the Motzkin numbers, we obtain

$$A(x,0) = \frac{xM(x)}{1-x-x^2M(x)} = xM^2(x),$$

and then

Kernel Method

From the second and third equations, we get

$$\frac{(1-x)u-x^2-u^2}{u(1-u-x)}A(x,u)=-\frac{x^2}{u(1-x)}A(x,0)+\frac{x}{(1-u-x)(1-x)}.$$

By choosing $u=x^2M(x)$, where $M(x)=\frac{1-x-\sqrt{1-2x-3x^2}}{2x^2}$ is the generating function for the Motzkin numbers, we obtain

$$A(x,0)=\frac{xM(x)}{1-x-x^2M(x)}=xM^2(x),$$

and then

$$A(x,u)=\frac{xM(x)(x^2M(x)-u)}{u^2+(x-1)u+x^2}$$

and

$$B(x,u)=\frac{xM(x)((u+x)x^2M(x)-u+x^2)}{u^2+u(x-1)+x^2}.$$

Enumeration of $I_n(000, 021)$

The generating function $R(x) = \sum_{n\geq 1} |I_n(000,021)|x^n$ is given by

$$R(x) = \frac{3x^3 + x^2 - 3x + 1}{2x^2\sqrt{(1+x)(1-3x)}} - \frac{(1-x)^2}{2x^2}.$$

Hence

$$|I_n(000,021)| = \frac{1}{2}(3a_{n-1} + a_n - 3a_{n+1} + a_{n+2})$$

for all $n \geq 1$ where $a_n = \sum_{k=0}^{n}(-1)^{n-k}\binom{n}{k}\binom{2k}{k}$.

Enumeration of $I_n(021, 00011)$

the vertex labeling symbols:

$a_m = 0^m$, $b_m = 01^m$, $c_m = 001^m$, $d_m = 0^2 1^2 \ldots m^2$, $e_m = 0^2 1^2 \ldots (m-1)^2 m$, $f_m = 01^2 2^2 \ldots m^2$, and $g_m = 01^2 2^2 .. (m-1)^2 m$, for all $m \geq 1$; and $a_e = e$ for an inversion sequence e.

the generating tree succession rules:

$$
\begin{aligned}
&a_0 \rightsquigarrow a_{00} g_1, && a_{00} \rightsquigarrow a_3 e_1 a_{002}, \\
&a_{002} \rightsquigarrow a_3 a_{0022} a_{002}, && a_{0022} \rightsquigarrow a_4 a_{00222} e_1 a_{002}, \\
&a_{00222} \rightsquigarrow a_5 c_3 a_3 a_{0002} a_{0003}, && a_m \rightsquigarrow a_{m+1} \cdots a_3 a_{0002} a_{0003}, \\
&b_m \rightsquigarrow b_{m+1} c_m a_m \cdots a_3 a_{0002} a_{0003}, \quad m \geq 3 && d_m \rightsquigarrow a_{m+4} c_{m+2} e_{m+1} \cdots e_1 a_{002}, \quad m \geq 1 \\
&c_m \rightsquigarrow a_{m+3} c_{m+1} a_{m+1} \cdots a_3 a_{0002} a_{0003}, && f_m \rightsquigarrow d_m b_{m+2} g_{m+1} \cdots g_1, \quad m \geq 1 \\
&e_m \rightsquigarrow a_{m+3} d_m e_m \cdots e_1 a_{002}, && g_m \rightsquigarrow e_m f_m g_m \cdots g_1, \quad m \geq 1
\end{aligned}
$$

Then we have

$$
\begin{aligned}
F_{(021,00011)}(x) = &\frac{(1 - x - 3x^2)\sqrt{1-4x} - 9x^3 - 3x^2 + 4x - 1}{2x^3\sqrt{(1+x)(1-3x)}} \\
&- \frac{(2 - 2x + 3x^2)\sqrt{1-4x} + 2x^3 - 8x^2 + 7x - 2}{2x^3}.
\end{aligned}
$$

Enumeration of $I_n(021, 00012)$

the vertex labeling symbols:

$a_m = 0^m$, $b_m = 01^m$, $c_m = 001^m$, $d_m = 0^2 1^2 \ldots m^2$, $e_m = 0^2 1^2 \ldots (m-1)^2 m$, $f_m = 01^2 2^2 \ldots m^2$, and $g_m = 01^2 2^2 \ldots (m-1)^2 m$, for all $m \geq 1$; and $a_e = e$ for an inversion sequence e.

the generating tree succession rules:

$$a_0 \rightsquigarrow a_{00} g_1, \qquad a_{00} \rightsquigarrow a_3 e_1 a_{002},$$
$$a_{002} \rightsquigarrow a_3 a_{0022} a_{002}, \qquad a_{0022} \rightsquigarrow a_4 e_1 a_{002} a_{00222},$$
$$a_{00222} \rightsquigarrow a_5 c_3 a_{0001}^3, \qquad a_{0001} \rightsquigarrow a_{0001}^2,$$
$$a_m \rightsquigarrow a_{m+1} a_{0001}^m, \qquad b_m \rightsquigarrow b_{m+1} c_m a_{0001}^m,$$
$$c_m \rightsquigarrow a_{m+3} c_{m+1} a_{0001}^{m+1}, \qquad d_m \rightsquigarrow a_{m+4} c_{m+2} e_{m+1} \cdots e_1 a_{002},$$
$$e_m \rightsquigarrow a_{m+3} d_m e_m \cdots e_1 a_{002}, \qquad f_m \rightsquigarrow d_m b_{m+2} g_{m+1} \cdots g_1,$$
$$g_m \rightsquigarrow e_m f_m g_m \cdots g_1.$$

Then we have

$$F_{(021,00012)}(x) = \frac{4x^9 - 8x^8 - 8x^7 + 10x^6 - 18x^5 + 6x^4 + 8x^3 - 9x^2 + 4x - 1}{2x^2(1+x)^2(1-x)^4(1-2x)} + \frac{18x^7 - 24x^6 + 24x^5 + 8x^4 - 26x^3 + 17x^2 - 6x + 1}{2x^2(1+x)(1-x)^4(1-2x)\sqrt{(1+x)(1-3x)}}.$$

Enumeration of $I_n(021, \tau)$

Mansour-Y. (2022)

We determined the generating trees and generating functions for the inversion sequences avoiding 021 and another pattern of length 4 or 5.

$$
\begin{aligned}
|I_n(\{021, 0001\})| &= \frac{(4n-25)(-1)^n}{32} - \frac{n(n+1)-1}{4} + \frac{1}{32}3^{n+4} \\
&\quad + \sum_{j=0}^{n+1} \left(\frac{(4j-39)(-1)^j}{32} + \frac{1}{4}j^2 - j + \frac{1}{2} - \frac{1}{32}3^{j+2} \right) M_{n+1-j}, \\
|I_n(\{021, 0010\})| &= \binom{2n}{n}, \\
|I_n(\{021, 0011\})| &= C_{n+2} + 1 - \sum_{j=0}^{n+1} C_j, \\
|I_n(\{021, 0012\})| &= 2^{n+3} - \frac{(n+1)(2n^2+7n+24)}{6} - 3, \\
|I_n(\{021, 0100\})| &= |I_n(\{021, 0110\})| = \frac{n^2+n+6}{8(2n+3)(2n+5)} \binom{2n+6}{n+3}, \\
|I_n(\{021, 0101\})| &= |I_n(\{021, 0111\})| = \sum_{i=1}^{n+1} \frac{1}{i} \binom{n}{i-1} \binom{2n+2-i}{i-1}
\end{aligned}
$$

$$|I_n(\{021,0102\})| = 2^{n+1} - \frac{(n+1)(n^2+2n+12)}{6} - 1 + \sum_{j=0}^{n+1} C_j$$

$$|I_n(\{021,0112\})| = C_{n+1} - 2^{n+1} + 1 + \sum_{j=0}^{n} 2^{n-j} C_j,$$

$$|I_n(\{021,0120\})| = |I_n(\{021,0122\})| = \frac{1}{2}\binom{2n+2}{n+1} - \frac{1}{2}\sum_{j=1}^{n}\binom{2j}{j},$$

$$|I_n(\{021,0123\})| = 2^{n-1}(n^2-3n+4) + \frac{n(n+1)}{2} - 1,$$

$$|I_n(\{021,1000\})| = |I_n(\{021,1100\})| = \frac{n^5+2n^4+23n^3+46n^2+120n+48}{2(n+1)(n+2)(n+3)(n+4)}\binom{2n}{n},$$

$$|I_n(\{021,1001\})| = |I_n(\{021,1011\})| = |I_n(\{021,1101\})| = \frac{1}{n+1}\sum_{j=0}^{\lfloor \frac{n}{2} \rfloor}\binom{n+1}{j}\binom{2n+2}{n-2j},$$

$$|I_n(\{021,1002\})| = \frac{1}{2}\binom{2n+6}{n+3} - \frac{5}{2}\binom{2n+4}{n+2} + \frac{5}{2}\binom{2n+2}{n+1} + \frac{1}{2}\sum_{j=0}^{n}\binom{2j}{j}$$
$$+ 2^{n+1} - \frac{1}{24}(n^4+2n^3+11n^2+34n+36),$$

$$|I_n(\{021,1020\})| = |I_n(\{021,1022\})|$$
$$= \binom{2n+8}{n+4} - \frac{13}{2}\binom{2n+6}{n+3} + \frac{21}{2}\binom{2n+3}{n+2} - \frac{1}{2}\sum_{j=0}^{n+1}\binom{2j}{j} - \frac{1}{2},$$

$$|I_n(\{021,1023\})| = \sum_{j=0}^{n+1}(2^{j+1}-j-1)C_{n+1-j} + \frac{n(3n^3+22n^2+129n+398)}{24} + 2^{n-1}(n^2-3n-52)+24,$$

$$|I_n(\{021,1102\})| = \frac{1}{2}\binom{2n+6}{n+3} - \frac{21}{4}\binom{2n+4}{n+2} + \binom{2n+2}{n+1} + \frac{(n+1)^2}{2} - 2^n + \frac{1}{2}\sum_{j=1}^{n+3}(2^{j-2}-3j+8)\binom{2n+6-2j}{n+3-j}$$

$$|I_n(\{021,1120\})| = 4^n - \frac{n}{2(2n+3)}\binom{2n+4}{n+2},$$

$$|I_n(\{021,1200\})| = |I_n(\{021,1220\})| = \frac{n+4}{2(n+2)}\binom{2n+2}{n+1} + \sum_{j=0}^{n-1}(2j+1)\binom{2j}{j} - 4^n,$$

$$|I_n(\{021,1203\})| = \frac{n+1}{24}(n^3+n^2-2*n-108) + 2^{n-1}(n^2-11n+28) - \frac{19}{2}$$
$$+\frac{1}{2}\binom{2n+2}{n+1} - \frac{1}{2}\sum_{j=2}^{n+1}(j-1)\binom{2n+2-2j}{n+1-j},$$

$$|I_n(\{021,1220\})| = \frac{n+4}{n+2}\binom{2n+1}{n} - 4^n + \sum_{j=0}^{n}(2j+1)\binom{2j}{j},$$

$$|I_n(\{021,1230\})| = \frac{1}{3}(2\cdot 4^n+1) + \sum_{j=1}^{n}(\frac{j}{2} - 2^{j-1})\binom{2n+2-2j}{n+1-j}.$$

Thank you!

References

- S. Corteel, M.A. Martinez, C.D. Savage, M. Weselcouch, Patterns in inversion sequences I, Discrete Math. Theor. Comput. Sci. 18 (2), 2016.
- T. Mansour, M. Shattuck, Pattern avoidance in inversion sequences, Pure Math. Appl. 25 (2), 157–176, 2015.
- M. Martinez, C. Savage, Patterns in inversion sequences II: inversion sequences avoiding triples of relations. J. Integer Seq. 21, no. 2, Art. 18.2.2, 44 pp, 2018.
- C. Yan and Z. Lin, Inversion sequences avoiding pairs of patterns. Discrete Math. Theor. Comput. Sci. 22, no. 1, Paper No. 23, 35 pp, [2020–2021].
- I. Kotsireas, T. Mansour, G. Yıldırım, An algorithmic approach based on generating trees for enumerating pattern-avoiding inversion sequences.
- T. Mansour, G. Yıldırım, Inversion sequences avoiding 021 and another pattern of length four.

succession rules

Consider the children of a node labeled by $a_m = 0011 \cdots mm$:

$$a_m j = 0011 \cdots mmj \text{ where } j = m+1, m+2, \ldots, 2m+2$$

otherwise, $a_m j$ does not avoid B.

- $a_m(m+1) = 0011 \cdots mm(m+1) = b_{m+1}$;
- for other j values, $a_m(m+j) = 0011 \cdots mm(m+j)$; note that $\mathcal{T}(B; a_m(m+j)) \cong \mathcal{T}(B; b_{m+2-j})$ by removing the letters $m+2-j, m+3-j, \ldots, m$ and decreasing each letter greater than m by $2j-1$.
- therefore the children of the node with label a_m are exactly the nodes labelled by $b_{m+1}, b_m, \ldots, b_0$, that is, $a_m \rightsquigarrow b_{m+1} b_m \cdots b_0$.